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# Stochastic Differential Systems with Memory: Theory, Examples and Applications (Sixth Workshop on Stochastic Analysis)

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**STOCHASTIC DIFFERENTIAL SYSTEMS  
WITH MEMORY**

**THEORY, EXAMPLES AND APPLICATIONS**

**Geilo, Norway : July 29-August 4, 1996**

**Salah-Eldin A. Mohammed**

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# Outline of Lectures

## Lecture I. Existence.

1. Simple examples: The noisy feedback loop:  $dx(t) = x(t-r) dW(t)$ . Solution process  $x(t)$  is not a Markov process in  $\mathbf{R}$ . No closed form solution when  $r > 0$ . Compare with the case  $r = 0$ . The logistic time-lag model with Gaussian noise:

$$dx(t) = [\alpha - \beta x(t-r)]x(t) dt + \sigma x(t) dW(t).$$

The classical “heat-bath” model of R. Kubo: Motion of a large molecule in a viscous fluid.

2. General Formulation. Choice of state space. Pathwise existence and uniqueness of solutions to sfde’s under local Lipschitz and linear growth hypotheses on the coefficients. Existence theorem allows for stochastic white-noise perturbations of the memory, e.g.

$$dx(t) = \left\{ \int_{[-r,0]} x(t+s) dW(s) \right\} dW(t) \quad t > 0$$

Above sfde is *not* covered by classical results of Protter, Metivier and Pellaumail, Doleans-Dade.

3. Mean Lipschitz, smooth and sublinear dependence of the trajectory random field.

## Lecture II. Markov Behavior and the Generator.

1. Markov (Feller) property holds for the trajectory random field. Time homogeneity.
2. Construction of the semigroup. Semigroup is not strongly continuous for positive delay. Domain of strong continuity does not contain tame (or cylinder) functions with evaluations away from 0, but contains “quasitame” functions. These are weakly dense in the underlying space of continuous functions and generate the Borel  $\sigma$ -algebra of the state space.
3. Derivation of a formula for the weak infinitesimal generator of the semigroup for sufficiently regular functions, and for a large class of quasitame functions.

### Lecture III. Regularity. Classification of SFDE's.

1. Pathwise regularity of the trajectory random field in the time variable.  $\alpha$ -Hölder continuity.
2. Almost sure (pathwise) dependence on the initial state. Non-existence of the stochastic flow for the singular sdde  $dx(t) = x(t-r)dW(t)$ . Breakdown of linearity and local boundedness. Classification of sfde's into regular and singular types.
3. Results on sufficient conditions for regularity of linear systems driven by white noise or semimartingales.
4. Sussman-Doss type nonlinear sfde's. Existence and compactness of semiflow.
5. Path regularity of general non-linear sfde's with "smooth memory".

## Lecture IV. Ergodic Theory of Linear SFDE's.

1. Existence of stochastic semiflows for certain classes of linear sfde's with smooth memory terms. The cocycle and its perfection.
2. Compactness of the semiflow in the finite memory case.
3. Ruelle-Oseledec multiplicative ergodic theorem in Hilbert space. Existence of a discrete Lyapunov spectrum. The Stable Manifold Theorem (viz. *random saddles*) for hyperbolic linear sfde's driven by white noise. The case of helix noise.

## Lecture V. Stability. Examples and Case Studies.

1. Estimates on the maximal exponential growth rate for the singular noisy feedback loop:  $dx(t) = \sigma x(t-r) dW(t)$ . Stability and instability for small  $\sigma$  (or large  $r$ ) using a Lyapunov functional argument. Comparison with the non-delay case for large  $\sigma$ .
2. Derivation of estimates on the top Lyapunov exponent  $\lambda_1$  for various examples of one-dimensional regular sfde's driven by white noise or a martingale with stationary ergodic increments.
3. Lyapunov spectrum for sdde  $dx(t) = x((t-1)-) dN(t)$  driven by a Poisson process  $N$ . Characterization of the Lyapunov spectrum.

## Lecture VI. Miscellany

1. Malliavin Calculus of SFDE's. Regularity of the solution  $x(t, \omega)$  in  $\omega$ . Malliavin smoothness and existence of smooth densities. Classical solution of a degenerate parabolic pde as an application.
2. Small delays. Applications to sode's. A proof of the classical existence theorem for solutions to sode's.
3. Affine systems of sfde's. Lyapunov spectrum. The hyperbolic splitting. Existence of stationary solutions in the hyperbolic case. Application to simple population model.
4. Random delays. Induced measure-valued process. Random families of Markov fields and random generators.
5. Infinite memory and stationary solutions.



## References

- [AKO] Arnold, L., Kliemann, W. and Oeljeklaus, E. Lyapunov exponents of linear stochastic systems, in *Lyapunov Exponents*, Springer Lecture Notes in Mathematics **1186** (1989), 85–125.
- [AOP] Arnold, L. Oeljeklaus, E. and Pardoux, E., Almost sure and moment stability for linear Itô equations, in *Lyapunov Exponents*, Springer Lecture Notes in Mathematics **1186** (ed. L. Arnold and V. Wihstutz) (1986), 129–159.
- [B] Baxendale, P.H., *Moment stability and large deviations for linear stochastic differential equations*, in Ikeda, N. (ed.) *Proceedings of the Taniguchi Symposium on Probabilistic Methods in Mathematical Physics*, Katata and Kyoto (1985), 31–54, Tokyo: Kinokuniya (1987).
- [CJPS] Cinlar, E., Jacod, J., Protter, P. and Sharpe, M. *Semimartingales and Markov processes*, *Z. Wahrsch. Verw. Gebiete* **54** (1980), 161–219.
- [FS] Flandoli, F. and Schaumlöffel, K.-U. *Stochastic parabolic equations in bounded domains: Random evolution operator and Lyapunov exponents*, *Stochastics and Stochastic Reports* **29**, 4 (1990), 461–485.
- [M1] Mohammed, S.-E.A. *Stochastic Functional Differential Equations*, *Research Notes in Mathematics* **99**, Pitman Advanced Publishing Program, Boston-London-Melbourne (1984).
- [M2] Mohammed, S.-E.A. *Non-linear flows for linear stochastic delay equations*, *Stochastics* **17**, 3 (1986), 207–212.
- [M3] Mohammed, S.-E.A. *The Lyapunov spectrum and stable manifolds for stochastic linear delay equations*, *Stochastics and Stochastic Reports* **29** (1990), 89–131.

- [M4] Mohammed, S.-E.A. *Lyapunov exponents and stochastic flows of linear and affine hereditary systems*, (1992) (Survey article), Birkhäuser (1992), 141–169.
- [MS] Mohammed, S.-E.A. and Scheutzow, M.K.R. *Lyapunov exponents of linear stochastic functional differential equation driven by semimartingales. Part I: The multiplicative ergodic theory*, (preprint, 1990), AIHP (to appear).
- [P] Protter, Ph.E. Semimartingales and measure-preserving flows, *Ann. Inst. Henri Poincaré, Probabilités et Statistiques*, vol. 22, (1986), 127-147.
- [R] Ruelle, D. Characteristic exponents and invariant manifolds in Hilbert space, *Annals of Mathematics* **115** (1982), 243–290.
- [S] Scheutzow, M.K.R. *Stationary and Periodic Stochastic Differential Systems: A study of Qualitative Changes with Respect to the Noise Level and Asymptotics*, Habilitationsschrift, University of Kaiserslautern, W. Germany (1988).
- [Sc] Schwartz, L., *Radon Measures on Arbitrary Topological Spaces and Cylindrical measures*, Tata Institute of Fundamental Research, Oxford University Press, (1973).
- [Sk] Skorohod, A. V., *Random Linear Operators*, D. Reidel Publishing Company (1984).
- [SM] de Sam Lazaro, J. and Meyer, P.A. Questions de théorie des flots, *Seminaire de Probab. IX*, Springer Lecture Notes in Mathematics **465**, (1975), 1–96.

- [E] Elworthy, K. D., Stochastic differential equations on manifolds, Cambridge (Cambridgeshire) ; New York : Cambridge University Press, (18), 326 p. ; 23 cm. 1982, London Mathematical Society lecture note series ; 70.
- [D] Dudley, R.M., The sizes of compact subsets of Hilbert space and continuity of Gaussian processes, *J. Functional Analysis*, 1, (1967), 290-330.
- [FS] Flandoli, F. and Schaumlöffel, K.-U. Stochastic parabolic equations in bounded domains: Random evolution operator and Lyapunov exponents, *Stochastics and Stochastic Reports* **29**, 4 (1990), 461–485.
- [H] Hale, J.K., *Theory of Functional Differential Equations*, Springer-Verlag, New York, Heidelberg, Berlin, (1977).
- [Ha] Has'minskii, R. Z., *Stochastic Stability of Differential Equations*, Sijthoff & Noordhoff (1980).
- [KN] Kolmanovskii, V.B. and Nosov, V.R., *Stability of Functional Differential Equations*, Academic Press, London, Orlando (1986) .
- [K] Kushner, H.J., On the stability of processes defined by stochastic differential-difference equations, *J. Differential equations*, 4, (1968), 424-443.
- [Ma] Mao, X.R., *Exponential Stability of Stochastic Differential Equations*, Pure and Applied Mathematics, Marcel Dekker, New York-Basel-Hong Kong (1994).
- [MT] Mizel, V.J. and Trutzer, V., Stochastic hereditary equations: existence and asymptotic stability, *J. Integral Equations*, (1984), 1-72

- [M1] Mohammed, S.-E.A., *Stochastic Functional Differential Equations*, Research Notes in Mathematics **99**, Pitman Advanced Publishing Program, Boston-London-Melbourne (1984).
- [M2] Mohammed, S.-E.A., Non-linear flows for linear stochastic delay equations, *Stochastics* **17**, 3 (1986), 207–212.
- [M3] Mohammed, S.-E.A., The Lyapunov spectrum and stable manifolds for stochastic linear delay equations, *Stochastics and Stochastic Reports* **29** (1990), 89–131.
- [M4] Mohammed, S.-E.A., Lyapunov exponents and stochastic flows of linear and affine hereditary systems, (1992) (Survey article), in *Diffusion Processes and Related Problems in Analysis, Volume II*, edited by Pinsky, M., and Wihstutz, V. Birkhäuser (1992), 141–169.
- [MS] Mohammed, S.-E.A. and Scheutzow, M.K.R., Lyapunov exponents of linear stochastic functional differential equation driven by semimartingales. Part I: The multiplicative ergodic theory, *Ann. Inst. Henri Poincaré, Probabilités et Statistiques*, Vol. **32**, no. 1 (1996), 69-105.
- [PW1] Pardoux, E. and Wihstutz, V., Lyapunov exponent and rotation number of two-dimensional stochastic systems with small diffusion, *SIAM J. Applied Math.*, **48**, (1988), 442-457.
- [PW2] Pinsky, M. and Wihstutz, V., Lyapunov exponents of nilpotent Itô systems, *Stochastics*, **25**, (1988), 43-57.
- [R] Ruelle, D. Characteristic exponents and invariant manifolds in Hilbert space, *Annals of Mathematics* **115** (1982), 243–290.
- [S] Scheutzow, M.K.R. *Stationary and Periodic Stochastic Differential Systems: A study of Qualitative Changes with Respect to the Noise Level and*

*Asymptotics*, Habilitationsschrift, University of Kaiserslautern, W. Germany (1988).

- [Sc] Schwartz, L., *Radon Measures on Arbitrary Topological Vector Spaces and Cylindrical Measures*, Tata Institute of Fundamental Research, Oxford University, Academic Press, London, Orlando (1986) .

## **I. EXISTENCE**

**Geilo, Norway**

**Monday, July 29, 1996**

**14:00-14:50**

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## I. EXISTENCE

### 1. Examples

**Example 1.** (*Noisy Feedbacks*)

*Box N:* Input =  $y(t)$ , output =  $x(t)$  at time  $t > 0$  related by

$$x(t) = x(0) + \int_0^t y(u) dZ(u) \quad (1)$$

where  $Z(u)$  is a semimartingale noise.

*Box D:* Delays signal  $x(t)$  by  $r (> 0)$  units of time. A proportion  $\sigma$  ( $0 \leq \sigma \leq 1$ ) is transmitted through  $D$  and the rest  $(1 - \sigma)$  is used for other purposes.

Therefore

$$y(t) = \sigma x(t - r)$$

Take  $\dot{Z}(u) :=$  white noise =  $\dot{W}(u)$

Then substituting in (1) gives the Itô integral equation

$$x(t) = x(0) + \sigma \int_0^t x(u-r) dW(u)$$

or the stochastic differential delay equation (sdde):

$$dx(t) = \sigma x(t-r) dW(t), \quad t > 0 \quad (I)$$

To solve (I), need an *initial process*  $\theta(t)$ ,  $-r \leq t \leq 0$ :

$$x(t) = \theta(t) \quad \text{a.s.}, \quad -r \leq t \leq 0$$

$r = 0$ : (I) becomes a linear stochastic ode and has closed form solution

$$x(t) = x(0)e^{\sigma W(t) - \frac{\sigma^2 t}{2}}, \quad t \geq 0.$$

$r > 0$ : Solve (I) by successive Itô integrations over steps of length  $r$ :

$$\begin{aligned} x(t) &= \theta(0) + \sigma \int_0^t \theta(u-r) dW(u), \quad 0 \leq t \leq r \\ x(t) &= x(r) + \sigma \int_r^t [\theta(0) + \sigma \int_0^{(v-r)} \theta(u-r) dW(u)] dW(v), \quad r < t \leq 2r, \\ \dots &= \dots \quad 2r < t \leq 3r, \end{aligned}$$

No closed form solution is known (even in deterministic case).

### Curious Fact!

In the sdde (I) the Itô differential  $dW$  may be replaced by the Stratonovich differential  $\circ dW$  *without changing the solution*  $x$ . Let  $x$  be the solution of (I) under an Itô differential  $dW$ . Then using finite partitions  $\{u_k\}$  of the interval  $[0, t]$  :

$$\int_0^t x(u-r) \circ dW(t) = \lim \sum_k \frac{1}{2} [x(u_k-r) + x(u_{k+1}-r)] [W(u_{k+1}) - W(u_k)]$$



where the limit in probability is taken as the mesh of the partition  $\{u_k\}$  goes to zero. Compare the Stratonovich and Itô integrals using the corresponding partial sums:

$$\begin{aligned}
& \lim E \left( \sum_k \frac{1}{2} [x(u_k - r) + x(u_{k+1} - r)] [W(u_{k+1}) - W(u_k)] \right. \\
& \quad \left. - \sum_k [x(u_k - r)] [W(u_{k+1}) - W(u_k)] \right)^2 \\
&= \lim E \left( \sum_k \frac{1}{2} [x(u_{k+1} - r) - x(u_k - r)] [W(u_{k+1}) - W(u_k)] \right)^2 \\
&= \lim \sum_k \frac{1}{4} E [x(u_{k+1} - r) - x(u_k - r)]^2 E [W(u_{k+1}) - W(u_k)]^2 \\
&= \lim \sum_k \frac{1}{4} E [x(u_{k+1} - r) - x(u_k - r)]^2 (u_{k+1} - u_k) \\
&= 0
\end{aligned}$$

because  $W$  has independent increments,  $x$  is adapted to the Brownian filtration,  $u \mapsto x(u) \in L^2(\Omega, \mathbf{R})$  is continuous, and the delay  $r$  is positive. Alternatively

$$\int_0^t x(u - r) \circ dW(u) = \int_0^t x(u - r) dW(u) + \frac{1}{2} \langle x(\cdot - r), W \rangle (t)$$

and  $\langle x(\cdot - r), W \rangle (t) = 0$  for all  $t > 0$ .

**Remark.**

When  $r > 0$ , the solution process  $\{x(t) : t \geq -r\}$  of (I) is a martingale but is *non-Markov*.

**Example 2.** (*Simple Population Growth*)

Consider a large population  $x(t)$  at time  $t$  evolving with a constant birth rate  $\beta > 0$  and a constant death rate  $\alpha$  per capita. Assume immediate removal of the dead from the population. Let  $r > 0$  (fixed,

non-random= 9, e.g.) be the development period of each individual and assume there is migration whose overall rate is distributed like white noise  $\sigma\dot{W}$  (mean zero and variance  $\sigma > 0$ ), where  $W$  is one-dimensional standard Brownian motion. The change in population  $\Delta x(t)$  over a small time interval  $(t, t + \Delta t)$  is

$$\Delta x(t) = -\alpha x(t)\Delta t + \beta x(t-r)\Delta t + \sigma\dot{W}\Delta t$$

Letting  $\Delta t \rightarrow 0$  and using Itô stochastic differentials,

$$dx(t) = \{-\alpha x(t) + \beta x(t-r)\} dt + \sigma dW(t), \quad t > 0. \quad (II)$$

Associate with the above affine sdde the initial condition  $(v, \eta) \in \mathbf{R} \times L^2([-r, 0], \mathbf{R})$

$$x(0) = v, \quad x(s) = \eta(s), \quad -r \leq s < 0.$$

Denote by  $M_2 = \mathbf{R} \times L^2([-r, 0], \mathbf{R})$  the Delfour-Mitter Hilbert space of all pairs  $(v, \eta)$ ,  $v \in \mathbf{R}$ ,  $\eta \in L^2([-r, 0], \mathbf{R})$  with norm

$$\|(v, \eta)\|_{M_2} = \left( |v|^2 + \int_{-r}^0 |\eta(s)|^2 ds \right)^{1/2}.$$

Let  $W : \mathbf{R}^+ \times \Omega \rightarrow \mathbf{R}$  be defined on the canonical filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}^+}, P)$  where

$$\Omega = C(\mathbf{R}^+, \mathbf{R}), \quad \mathcal{F} = \text{Borel } \Omega, \quad \mathcal{F}_t = \sigma\{\rho_u : u \leq t\}$$

$\rho_u : \Omega \rightarrow \mathbf{R}, u \in \mathbf{R}^+$ , are evaluation maps  $\omega \mapsto \omega(u)$ , and  $P =$  Wiener measure on  $\Omega$ .

**Example 3.** (*Logistic Population Growth*)

A single population  $x(t)$  at time  $t$  evolving logistically with *development (incubation) period*  $r > 0$  under Gaussian type noise (e.g. migration on a molecular level):

$$\dot{x}(t) = [\alpha - \beta x(t-r)]x(t) + \gamma x(t)\dot{W}(t), \quad t > 0$$

i.e.

$$dx(t) = [\alpha - \beta x(t-r)] x(t) dt + \gamma x(t) dW(t) \quad t > 0. \quad (III)$$

with *initial condition*

$$x(t) = \theta(t) \quad -r \leq t \leq 0.$$

For positive delay  $r$  the above sdde can be solved *implicitly* using forward steps of length  $r$ , i.e. for  $0 \leq t \leq r$ ,  $x(t)$  satisfies the *linear* sode (without delay)

$$dx(t) = [\alpha - \beta \theta(t-r)] x(t) dt + \gamma x(t) dW(t) \quad 0 < t \leq r. \quad (III')$$

$x(t)$  is a semimartingale and is non-Markov (Scheutzow [S], 1984).

**Example 4.** (*Heat bath*)

Model proposed by R. Kubo (1966) for physical Brownian motion. A molecule of mass  $m$  moving under random gas forces with position  $\xi(t)$  and velocity  $v(t)$  at time  $t$ ; cf classical work by Einstein and Ornstein and Uhlenbeck. Kubo proposed the following modification of the Ornstein-Uhlenbeck process

$$\left. \begin{aligned} d\xi(t) &= v(t) dt \\ m dv(t) &= -m \left[ \int_{t_0}^t \beta(t-t') v(t') dt' \right] dt + \gamma(\xi(t), v(t)) dW(t), \quad t > t_0. \end{aligned} \right\} \quad (IV)$$

$m$  = mass of molecule. No external forces.

$\beta$  = viscosity coefficient function with compact support.

$\gamma$  a function  $\mathbf{R}^3 \times \mathbf{R}^3 \rightarrow \mathbf{R}$  representing the random gas forces on the molecule.

$\xi(t)$  = position of molecule  $\in \mathbf{R}^3$ .

$v(t)$  = velocity of molecule  $\in \mathbf{R}^3$ .

$W$  = 3- dimensional Brownian motion.

([Mo], Pitman Books, RN # 99, 1984, pp. 223-226).

## Further Examples

Delay equation with Poisson noise:

$$\left. \begin{aligned} dx(t) &= x((t-r)-) dN(t) & t > 0 \\ x_0 &= \eta \in D([-r, 0], \mathbf{R}) \end{aligned} \right\} \quad (V)$$

$N :=$  Poisson process with iid interarrival times ([S], Hab. 1988).  
 $D([-r, 0], \mathbf{R}) =$  space of all cadlag paths  $[-r, 0] \rightarrow \mathbf{R}$ , with sup norm.

Simple model of dye circulation in the blood (or pollution) (cf. Bailey and Williams [B-W], JMAA, 1966, Lenhart and Travis ([L-T], PAMS, 1986).

$$\left. \begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r)\} dt + \sigma x(t) dW(t) & t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R} \times L^2([-r, 0], \mathbf{R}), \end{aligned} \right\} \quad (VI)$$

([Mo], Survey, 1992; [M-S], II, 1995.)

In above model:

$x(t) :=$  dye concentration (gm/cc)

$r =$  time taken by blood to traverse side tube (vessel)

Flow rate (cc/sec) is Gaussian with variance  $\sigma$ .

A fixed proportion of blood in main vessel is pumped into side vessel(s). Model will be analysed in Lecture V (Theorem V.5).

$$\left. \begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r)\} dt + \left\{ \int_{-r}^0 x(t+s) \sigma(s) ds \right\} dW(t), \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R} \times L^2([-r, 0], \mathbf{R}), t > 0. \end{aligned} \right\} \quad (VII)$$

([Mo], Survey, 1992; [M-S], II, 1995.)

Linear  $d$ -dimensional systems driven by  $m$ -dimensional Brownian motion  $W := (W_1, \dots, W_m)$  with constant coefficients.

$$\left. \begin{aligned} dx(t) &= H(x(t-d_1), \dots, x(t-d_N), x(t), x_t) dt \\ &\quad + \sum_{i=1}^m g_i x(t) dW_i(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (VIII)$$

$H := (\mathbf{R}^d)^N \times M_2 \rightarrow \mathbf{R}^d$  linear functional on  $(\mathbf{R}^d)^N \times M_2$ ;  $g_i$   $d \times d$ -matrices ([Mo], Stochastics, 1990).

Linear systems driven by (helix) semimartingale noise  $(N, L)$ , and memory driven by a (stationary) measure-valued process  $\nu$  and a (stationary) process  $K$  ([M-S], I, AIHP, 1996):

$$\left. \begin{aligned} dx(t) &= \left\{ \int_{[-r, 0]} \nu(t)(ds) x(t+s) \right\} dt \\ &\quad + dN(t) \int_{-r}^0 K(t)(s) x(t+s) ds + dL(t) x(t-), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (IX)$$

Multidimensional affine systems driven by (helix) noise  $Q$  ([M-S], Stochastics, 1990):

$$\left. \begin{aligned} dx(t) &= \left\{ \int_{[-r, 0]} \nu(t)(ds) x(t+s) \right\} dt + dQ(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (X)$$

Memory driven by white noise:

$$\left. \begin{aligned} dx(t) &= \left\{ \int_{[-r,0]} x(t+s) dW(s) \right\} dW(t) \quad t > 0 \\ x(0) &= v \in \mathbf{R}, \quad x(s) = \eta(s), \quad -r < s < 0, \quad r \geq 0 \end{aligned} \right\} \quad (XI)$$

([Mo], Survey, 1992).

## Formulation

Slice each solution path  $x$  over the interval  $[t - r, t]$  to get *segment*  $x_t$  as a process on  $[-r, 0]$ :

$$x_t(s) := x(t + s) \quad \text{a.s., } t \geq 0, s \in J := [-r, 0].$$

Therefore sdde's (I), (II), (III) and (XI) become

$$\left. \begin{aligned} dx(t) &= \sigma x_t(-r) dW(t), \quad t > 0 \\ x_0 &= \theta \in C([-r, 0], \mathbf{R}) \end{aligned} \right\} \quad (I)$$

$$\left. \begin{aligned} dx(t) &= \{-\alpha x(t) + \beta x_t(-r)\} dt + \sigma dW(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in \mathbf{R} \times L^2([-r, 0], \mathbf{R}) \end{aligned} \right\} \quad (II)$$

$$\left. \begin{aligned} dx(t) &= [\alpha - \beta x_t(-r)]x_t(0) dt + \gamma x_t(0) dW(t) \\ x_0 &= \theta \in C([-r, 0], \mathbf{R}) \end{aligned} \right\} \quad (III)$$

$$\left. \begin{aligned} dx(t) &= \left\{ \int_{[-r, 0]} x_t(s) dW(s) \right\} dW(t) \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in \mathbf{R} \times L^2([-r, 0], \mathbf{R}), \quad r \geq 0 \end{aligned} \right\} \quad (XI)$$

Think of R.H.S.'s of the above equations as functionals of  $x_t$  (and  $x(t)$ ) and generalize to *stochastic functional differential equation* (sfde)

$$\left. \begin{aligned} dx(t) &= h(t, x_t)dt + g(t, x_t)dW(t) \quad t > 0 \\ x_0 &= \theta \end{aligned} \right\} \quad (XII)$$

on filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  satisfying the usual conditions:

$(\mathcal{F}_t)_{t \geq 0}$  right-continuous and each  $\mathcal{F}_t$  contains all  $P$ -null sets in  $\mathcal{F}$ .

$C := C([-r, 0], \mathbf{R}^d)$  Banach space, sup norm.

$W(t) = m$ -dimensional Brownian motion.



$L^2(\Omega, C) :=$  Banach space of all  $(\mathcal{F}, \text{Borel } C)$ -measurable  $L^2$  (Bochner sense) maps  $\Omega \rightarrow C$  with the  $L^2$ -norm

$$\|\theta\|_{L^2(\Omega, C)} := \left[ \int_{\Omega} \|\theta(\omega)\|_C^2 dP(\omega) \right]^{1/2}$$

*Coefficients:*

$$h : [0, T] \times L^2(\Omega, C) \rightarrow L^2(\Omega, \mathbf{R}^d) \quad (\text{Drift})$$

$$g : [0, T] \times L^2(\Omega, C) \rightarrow L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d)) \quad (\text{Diffusion}).$$

*Initial data:*

$$\theta \in L^2(\Omega, C, \mathcal{F}_0).$$

*Solution:*

$x : [-r, T] \times \Omega \rightarrow \mathbf{R}^d$  measurable and sample-continuous,  $x|_{[0, T]}$   $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted and  $x(s)$  is  $\mathcal{F}_0$ -measurable for all  $s \in [-r, 0]$ .

*Exercise:*  $[0, T] \ni t \mapsto x_t \in C([-r, 0], \mathbf{R}^d)$  is  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted.

(*Hint:* *Borel*  $C$  is generated by all evaluations.)

**Hypotheses  $(E_1)$ .**

- (i)  $h, g$  are jointly continuous and uniformly Lipschitz in the second variable with respect to the first:

$$\|h(t, \psi_1) - h(t, \psi_2)\|_{L^2(\Omega, \mathbf{R}^d)} \leq L\|\psi_1 - \psi_2\|_{L^2(\Omega, C)}$$

for all  $t \in [0, T]$  and  $\psi_1, \psi_2 \in L^2(\Omega, C)$ . Similarly for the diffusion coefficient  $g$ .

- (ii) For each  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted process  $y : [0, T] \rightarrow L^2(\Omega, C)$ , the processes  $h(\cdot, y(\cdot)), g(\cdot, y(\cdot))$  are also  $(\mathcal{F}_t)_{0 \leq t \leq T}$ - adapted.

**Theorem I.1.** ([Mo], 1984) (Existence and Uniqueness).

Suppose  $h$  and  $g$  satisfy Hypotheses  $(E_1)$ . Let  $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ .

Then the sfde (XII) has a unique solution  ${}^\theta x : [-r, \infty) \times \Omega \rightarrow \mathbf{R}^d$  starting off at  $\theta \in L^2(\Omega, C; \mathcal{F}_0)$  with  $t \mapsto {}^\theta x_t$  continuous and  ${}^\theta x \in L^2(\Omega, C([-r, T], \mathbf{R}^d))$  for all  $T > 0$ . For a given  $\theta$ , uniqueness holds up to equivalence among all  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted processes in  $L^2(\Omega, C([-r, T], \mathbf{R}^d))$ .

**Proof.**

[Mo], Pitman Books, 1984, Theorem 2.1, pp. 36-39. □

Theorem I.1 covers equations (I), (II), (IV), (VI), (VII), (VIII), (XI) and a large class of sfde's driven by white noise. Note that (XI) *does not satisfy the hypotheses underlying the classical results* of Doleans-Dade [Dol], 1976, Metivier and Pellaumail [Met-P], 1980, Protter, Ann. Prob. 1987, Lipster and Shiriyayev [Lip-Sh], [Met], 1982. This is because the coefficient

$$\eta \rightarrow \int_{-r}^0 \eta(s) dW(s)$$

on the RHS of (XI) *does not admit almost surely Lipschitz (or even linear) versions  $C \rightarrow \mathbf{R}$ !* This will be shown later.

When the coefficients  $h, g$  factor through functionals

$$H : [0, T] \times C \rightarrow \mathbf{R}^d, \quad G : [0, T] \times C \rightarrow \mathbf{R}^{d \times m}$$

we can impose the following local Lipschitz and global linear growth conditions on the sfde

$$\left. \begin{aligned} dx(t) &= H(t, x_t) dt + G(t, x_t) dW(t) & t > 0 \\ x_0 &= \theta \end{aligned} \right\} \quad (XIII)$$

with  $W$   $m$ -dimensional Brownian motion:

**Hypotheses** ( $E_2$ )

- (i)  $H, G$  are Lipschitz on bounded sets in  $C$ : For each integer  $n \geq 1$  there exists  $L_n > 0$  such that

$$|H(t, \eta_1) - H(t, \eta_2)| \leq L_n \|\eta_1 - \eta_2\|_C$$

for all  $t \in [0, T]$  and  $\eta_1, \eta_2 \in C$  with  $\|\eta_1\|_C \leq n, \|\eta_2\|_C \leq n$ . Similarly for the diffusion coefficient  $G$ .

- (ii) There is a constant  $K > 0$  such that

$$|H(t, \eta)| + \|G(t, \eta)\| \leq K(1 + \|\eta\|_C)$$

for all  $t \in [0, T]$  and  $\eta \in C$ .

Note that the adaptability condition is not needed (explicitly) because  $H, G$  are deterministic and because the sample-continuity and adaptability of  $x$  imply that the segment  $[0, T] \ni t \mapsto x_t \in C$  is also adapted.

*Exercise:* Formulate the heat-bath model (IV) as a sfde of the form (XIII). ( $\beta$  has compact support in  $\mathbf{R}^+$ .)

**Theorem I.2.** ([Mo], 1984) (Existence and Uniqueness).

Suppose  $H$  and  $G$  satisfy Hypotheses  $(E_2)$  and let  $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ .

Then the sfde (XIII) has a unique  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted solution  ${}^\theta x : [-r, T] \times \Omega \rightarrow \mathbf{R}^d$  starting off at  $\theta \in L^2(\Omega, C; \mathcal{F}_0)$  with  $t \mapsto {}^\theta x_t$  continuous and  ${}^\theta x \in L^2(\Omega, C([-r, T], \mathbf{R}^d))$  for all  $T > 0$ . For a given  $\theta$ , uniqueness holds up to equivalence among all  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted processes in  $L^2(\Omega, C([-r, T], \mathbf{R}^d))$ .

Furthermore if  $\theta \in L^{2k}(\Omega, C; \mathcal{F}_0)$ , then  ${}^\theta x_t \in L^{2k}(\Omega, C; \mathcal{F}_t)$  and

$$E \| {}^\theta x_t \|_C^{2k} \leq C_k [1 + \|\theta\|_{L^{2k}(\Omega, C)}^{2k}]$$

for all  $t \in [0, T]$  and some positive constants  $C_k$ .

## Proofs of Theorems I.1, I.2.(Outline)

[Mo], pp. 150-152. Generalize some proofs in Gihman and Skorohod ([G-S], 1973) or Friedman ([Fr], 1975):

- (1) Truncate coefficients outside bounded sets in  $C$ . Reduce to globally Lipschitz case.
- (2) Successive approx. in globally Lipschitz situation.
- (3) Use local uniqueness ([Mo], Theorem 4.2, p. 151) to “patch up” solutions of the truncated sfde’s.

For (2) consider globally Lipschitz case and  $h \equiv 0$ .

We look for solutions of (XII) by successive approximation in  $L^2(\Omega, C([-r, a], \mathbf{R}^d))$ . Let  $J := [-r, 0]$ .

Suppose  $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$  is  $\mathcal{F}_0$ -measurable. Note that this is equivalent to saying that  $\theta(\cdot)(s)$  is  $\mathcal{F}_0$ -measurable for all  $s \in J$ , because  $\theta$  has a.a. sample paths continuous.

We prove by induction that there is a sequence of processes  ${}^k x : [-r, a] \times \Omega \rightarrow \mathbf{R}^d$ ,  $k = 1, 2, \dots$  having the

*Properties  $P(k)$ :*

(i)  ${}^k x \in L^2(\Omega, C([-r, a], \mathbf{R}^d))$  and is adapted to  $(\mathcal{F}_t)_{t \in [0, a]}$ .

(ii) For each  $t \in [0, a]$ ,  ${}^k x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$  and is  $\mathcal{F}_t$ -measurable.

(iii)

$$\left. \begin{aligned} \|{}^{k+1}x - {}^k x\|_{L^2(\Omega, C)} &\leq (ML^2)^{k-1} \frac{a^{k-1}}{(k-1)!} \|{}^2x - {}^1x\|_{L^2(\Omega, C)} \\ \|{}^{k+1}x_t - {}^k x_t\|_{L^2(\Omega, C)} &\leq (ML^2)^{k-1} \frac{t^{k-1}}{(k-1)!} \|{}^2x - {}^1x\|_{L^2(\Omega, C)} \end{aligned} \right\} \quad (1)$$

where  $M$  is a “martingale” constant and  $L$  is the Lipschitz constant of  $g$ .

Take  ${}^1x : [-r, a] \times \Omega \rightarrow \mathbf{R}^d$  to be

$${}^1x(t, \omega) = \begin{cases} \theta(\omega)(0) & t \in [0, a] \\ \theta(\omega)(t) & t \in J \end{cases}$$

a.s., and

$${}^{k+1}x(t, \omega) = \begin{cases} \theta(\omega)(0) + (\omega) \int_0^t g(u, {}^k x_u) dW(\cdot)(u) & t \in [0, a] \\ \theta(\omega)(t) & t \in J \end{cases} \quad (2)$$

a.s.

Since  $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$  and is  $\mathcal{F}_0$ -measurable, then  ${}^1x \in L^2(\Omega, C([-r, a], \mathbf{R}^d))$  and is trivially adapted to  $(\mathcal{F}_t)_{t \in [0, a]}$ . Hence  ${}^1x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$  and is  $\mathcal{F}_t$ -measurable for all  $t \in [0, a]$ .  $P(1)$  (iii) holds trivially.

Now suppose  $P(k)$  is satisfied for some  $k > 1$ . Then by Hypothesis  $(E_1)(i), (ii)$  and the continuity of the slicing map (*stochastic memory*), it follows from  $P(k)(ii)$  that the process

$$[0, a] \ni u \longmapsto g(u, {}^k x_u) \in L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d))$$

is continuous and adapted to  $(\mathcal{F}_t)_{t \in [0, a]}$ .  $P(k+1)(i)$  and  $P(k+1)(ii)$  follow from the continuity and adaptability of the stochastic integral. Check  $P(k+1)(iii)$ , by using Doob's inequality.

For each  $k > 1$ , write

$${}^k x = {}^1 x + \sum_{i=1}^{k-1} ({}^{i+1} x - {}^i x).$$

Now  $L^2_A(\Omega, C([-r, a], \mathbf{R}^d))$  is closed in  $L^2(\Omega, C([-r, a], \mathbf{R}^d))$ ; so the series

$$\sum_{i=1}^{\infty} ({}^{i+1} x - {}^i x)$$

converges in  $L^2_A(\Omega, C([-r, a], \mathbf{R}^d))$  because of (1) and the convergence of

$$\sum_{i=1}^{\infty} \left[ (ML^2)^{i-1} \frac{a^{i-1}}{(i-1)!} \right]^{1/2}.$$

Hence  $\{{}^k x\}_{k=1}^{\infty}$  converges to some  $x \in L^2_A(\Omega, C([-r, a], \mathbf{R}^d))$ .



Clearly  $x|J = \theta$  and is  $\mathcal{F}_0$ -measurable, so applying Doob's inequality to the Itô integral of the difference

$$u \mapsto g(u, {}^k x_u) - g(u, x_u)$$

gives

$$\begin{aligned} E \left( \sup_{t \in [0, a]} \left| \int_0^t g(u, {}^k x_u) dW(\cdot)(u) - \int_0^t g(u, x_u) dW(\cdot)(u) \right|^2 \right) \\ < ML^2 a \|x - x\|_{L^2(\Omega, C)}^2 \\ \longrightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

Thus viewing the right-hand side of (2) as a process in  $L^2(\Omega, C([-r, a], \mathbf{R}^d))$  and letting  $k \rightarrow \infty$ , it follows from the above that  $x$  must satisfy the sfde (XII) a.s. for all  $t \in [-r, a]$ .

For uniqueness, let  $\tilde{x} \in L^2_A(\Omega, ([-r, a], \mathbf{R}^d))$  be also a solution of (XII) with initial process  $\theta$ . Then by the Lipschitz condition:

$$\|x_t - \tilde{x}_t\|_{L^2(\Omega, C)}^2 < ML^2 \int_0^t \|x_u - \tilde{x}_u\|_{L^2(\Omega, C)}^2 du$$

for all  $t \in [0, a]$ . Therefore we must have  $x_t - \tilde{x}_t = 0$  for all  $t \in [0, a]$ ; so  $x = \tilde{x}$  in  $L^2(\Omega, C([-r, a], \mathbf{R}^d))$  a.s. □

## Remarks and Generalizations.

- (i) In Theorem I.2 replace the process  $(t, W(t))$  by a (square integrable) semimartingale  $Z(t)$  satisfying appropriate conditions. ([Mo], 1984, Chapter II).
- (ii) Results on existence of solutions of sfde's driven by white noise were first obtained by Itô and Nisio ([I-N], J. Math. Kyoto University, 1968) and then Kushner (JDE, 197).
- (iii) Extensions to sfde's with *infinite* memory. Fading memory case: work by Mizel and Trützer [M-T], JIE, 1984, Marcus and Mizel [M-M], Stochastics, 1988; general infinite memory: Itô and Nisio [I-N], J. Math. Kyoto University, 1968.
- (iii) Pathwise local uniqueness holds for sfde's of type (XIII) under a global Lipschitz condition: If coefficients of two sfde's agree on an open set in  $C$ , then the corresponding trajectories leave the open set at the same time and agree almost surely up to the time they leave the open set ([Mo], Pitman Books, 1984, Theorem 4.2, pp. 150-151.)

- (iv) Replace the state space  $C$  by the Delfour-Mitter Hilbert space  $M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$  with the Hilbert norm

$$\|(v, \eta)\|_{M_2} = \left( |v|^2 + \int_{-r}^0 |\eta(s)|^2 ds \right)^{1/2}$$

for  $(v, \eta) \in M_2$  (T. Ahmed, S. Elsanousi and S. Mohammed, 1983).

- (v) Have Lipschitz and smooth dependence of  $\theta_{x_t}$  on the initial process  $\theta \in L^2(\Omega, C)$  ([Mo], 1984, Theorems 3.1, 3.2, pp. 41-45).

**II. MARKOV BEHAVIOR  
AND THE WEAK GENERATOR**

**Geilo, Norway**

**Tuesday, July 30, 1996**

**14:00-14:50**

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## II. MARKOV BEHAVIOR AND THE GENERATOR

Consider the sfde

$$\left. \begin{aligned} dx(t) &= H(t, x_t) dt + G(t, x_t) dW(t), & t > 0 \\ x_0 &= \eta \in C := C([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (XIII)$$

with coefficients  $H : [0, T] \times C \rightarrow \mathbf{R}^d$ ,  $G : [0, T] \times C \rightarrow \mathbf{R}^{d \times m}$ ,  $m$ -dimensional Brownian motion  $W$  and trajectory field  $\{\eta x_t : t \geq 0, \eta \in C\}$ .

### 1. Questions

- (i) For the sfde (XIII) does the trajectory field  $x_t$  give a diffusion in  $C$  (or  $M_2$ )?
- (ii) How does the trajectory  $x_t$  transform under smooth non-linear functionals  $\phi : C \rightarrow \mathbf{R}$ ?
- (iii) What “diffusions” on  $C$  (or  $M_2$ ) correspond to sfde’s on  $\mathbf{R}^d$ ?

We will only answer the first two questions. More details in [Mo], Pitman Books, 1984, Chapter III, pp. 46-112. Third question is OPEN.

## Difficulties

- (i) Although the current state  $x(t)$  is a semimartingale, the trajectory  $x_t$  does *not* seem to possess any martingale properties when viewed as  $C$ -(or  $M_2$ )-valued process: e.g. for Brownian motion  $W$  ( $H \equiv 0, G \equiv 1$ ):

$$[E(W_t | \mathcal{F}_{t_1})](s) = W(t_1) = W_{t_1}(0), \quad s \in [-r, 0]$$

whenever  $t_1 \leq t - r$ .

- (ii) Lack of strong continuity leads to the use of weak limits in  $C$  which tend to live outside  $C$ .
- (iii) We will show that  $x_t$  is a Markov process in  $C$ . However almost all tame functions lie *outside* the domain of the (weak) generator.
- (iv) Lack of an Itô formula makes the computation of the generator hard.

## Hypotheses ( $M$ )

- (i)  $\mathcal{F}_t :=$  completion of  $\sigma\{W(u) : 0 \leq u \leq t\}$ ,  $t \geq 0$ .
- (ii)  $H, G$  are jointly continuous and globally Lipschitz in second variable uniformly wrt the first:

$$|H(t, \eta_1) - H(t, \eta_2)| + \|G(t, \eta_1) - G(t, \eta_2)\| \leq L\|\eta_1 - \eta_2\|_C$$

for all  $t \in [0, T]$  and  $\eta_1, \eta_2 \in C$ .

## 2. The Markov Property

$\eta_{x^{t_1}}$  := solution starting off at  $\theta \in L^2(\Omega, C; \mathcal{F}_{t_1})$  at  $t = t_1$  for the sfde:

$$\eta_{x^{t_1}}(t) = \begin{cases} \eta(0) + \int_{t_1}^t H(u, x_u^{t_1}) du + \int_{t_1}^t G(u, x_u^{t_1}) dW(u), & t > t_1 \\ \eta(t - t_1), & t_1 - r \leq t \leq t_1. \end{cases}$$

This gives a two-parameter family of mappings

$$T_{t_2}^{t_1} : L^2(\Omega, C; \mathcal{F}_{t_1}) \rightarrow L^2(\Omega, C; \mathcal{F}_{t_2}), \quad t_1 \leq t_2,$$

$$T_{t_2}^{t_1}(\theta) := {}^\theta x_{t_2}^{t_1}, \quad \theta \in L^2(\Omega, C; \mathcal{F}_{t_1}). \quad (1)$$

Uniqueness of solutions gives the *two-parameter* semigroup property:

$$T_{t_2}^{t_1} \circ T_{t_1}^0 = T_{t_2}^0, \quad t_1 \leq t_2. \quad (2)$$

([Mo], Pitman Books, 1984, Theorem II (2.2), p. 40.)

**Theorem II.1** (Markov Property)([Mo], 1984).

*In (XIII) suppose Hypotheses (M) hold. Then the trajectory field  $\{\eta x_t : t \geq 0, \eta \in C\}$  is a Feller process on  $C$  with transition probabilities*

$$p(t_1, \eta, t_2, B) := P({}^\eta x_{t_2}^{t_1} \in B) \quad t_1 \leq t_2, \quad B \in \text{Borel } C, \quad \eta \in C.$$

*i.e.*

$$P(x_{t_2} \in B | \mathcal{F}_{t_1}) = p(t_1, x_{t_1}(\cdot), t_2, B) = P(x_{t_2} \in B | x_{t_1}) \text{ a.s.}$$

*Further, if  $H$  and  $G$  do not depend on  $t$ , then the trajectory is time-homogeneous:*

$$p(t_1, \eta, t_2, \cdot) = p(0, \eta, t_2 - t_1, \cdot), \quad 0 \leq t_1 \leq t_2, \quad \eta \in C.$$

**Proof.**

[Mo], 1984, Theorem III.1.1, pp. 51-58. [Mo], 1984, Theorem III.2.1, pp. 64-65. □

### 3. The Semigroup

In the autonomous sfde

$$\left. \begin{aligned} dx(t) &= H(x_t) dt + G(x_t) dW(t) & t > 0 \\ x_0 &= \eta \in C \end{aligned} \right\} \quad (XIV)$$

suppose the coefficients  $H : C \rightarrow \mathbf{R}^d$ ,  $G : C \rightarrow \mathbf{R}^{d \times m}$  are *globally bounded* and globally Lipschitz.

$C_b :=$  Banach space of all bounded uniformly continuous functions  $\phi : C \rightarrow \mathbf{R}$ , with the sup norm

$$\|\phi\|_{C_b} := \sup_{\eta \in C} |\phi(\eta)|, \quad \phi \in C_b.$$

Define the operators  $P_t : C_b \hookrightarrow C_b, t \geq 0$ , on  $C_b$  by

$$P_t(\phi)(\eta) := E\phi({}^n x_t) \quad t \geq 0, \eta \in C.$$

A family  $\phi_t, t > 0$ , *converges weakly* to  $\phi \in C_b$  as  $t \rightarrow 0+$  if  $\lim_{t \rightarrow 0+} \langle \phi_t, \mu \rangle = \langle \phi, \mu \rangle$  for all finite regular Borel measures  $\mu$  on  $C$ . Write  $\phi := w - \lim_{t \rightarrow 0+} \phi_t$ . This is equivalent to

$$\left\{ \begin{array}{l} \phi_t(\eta) \rightarrow \phi(\eta) \text{ as } t \rightarrow 0+, \text{ for all } \eta \in C \\ \{\|\phi_t\|_{C_b} : t \geq 0\} \text{ is bounded.} \end{array} \right.$$

(Dynkin, [Dy], Vol. 1, p. 50). Proof uses uniform boundedness principle and dominated convergence theorem.

**Theorem II.2**([Mo], Pitman Books, 1984)

(i)  $\{P_t\}_{t \geq 0}$  is a one-parameter contraction semigroup on  $C_b$ .



(ii)  $\{P_t\}_{t \geq 0}$  is weakly continuous at  $t = 0$ :

$$\begin{cases} P_t(\phi)(\eta) \rightarrow \phi(\eta) \text{ as } t \rightarrow 0+ \\ \{|P_t(\phi)(\eta)| : t \geq 0, \eta \in C\} \text{ is bounded by } \|\phi\|_{C_b}. \end{cases}$$

(iii) If  $r > 0$ ,  $\{P_t\}_{t \geq 0}$  is never strongly continuous on  $C_b$  under the sup norm.

**Proof.**

(i) One parameter semigroup property

$$P_{t_2} \circ P_{t_1} = P_{t_1+t_2}, \quad t_1, t_2 \geq 0$$

follows from the continuation property (2) and time-homogeneity of the Feller process  $x_t$  (Theorem II.1).

(ii) Definition of  $P_t$ , continuity and boundedness of  $\phi$  and sample-continuity of trajectory  ${}^n x_t$  give weak continuity of  $\{P_t(\phi) : t > 0\}$  at  $t = 0$  in  $C_b$ .

(iii) Lack of strong continuity of semigroup:

Define the canonical shift (static) semigroup

$$S_t : C_b \rightarrow C_b, \quad t \geq 0,$$

by

$$S_t(\phi)(\eta) := \phi(\tilde{\eta}_t), \quad \phi \in C_b, \quad \eta \in C,$$

where  $\tilde{\eta} : [-r, \infty) \rightarrow \mathbf{R}^d$  is defined by

$$\tilde{\eta}(t) = \begin{cases} \eta(0) & t \geq 0 \\ \eta(t) & t \in [-r, 0). \end{cases}$$

Then  $P_t$  is strongly continuous iff  $S_t$  is strongly continuous.  $P_t$  and  $S_t$  have the same “domain of strong continuity” independently of  $H$ ,  $G$ , and  $W$ . This follows from the global boundedness of  $H$  and  $G$ . ([Mo], Theorem IV.2.1, pp. 72-73). Key relation is

$$\lim_{t \rightarrow 0+} E\|{}^n x_t - \tilde{\eta}_t\|_C^2 = 0$$

*uniformly* in  $\eta \in C$ . But  $\{S_t\}$  is strongly continuous on  $C_b$  iff  $C$  is locally compact iff  $r = 0$  (no memory) ! ([Mo], Theorems IV.2.1 and IV.2.2, pp.72-73). Main idea is to pick any  $s_0 \in [-r, 0)$  and consider the function  $\phi_0 : C \rightarrow \mathbf{R}$  defined by

$$\phi_0(\eta) := \begin{cases} \eta(s_0) & \|\eta\|_C \leq 1 \\ \frac{\eta(s_0)}{\|\eta\|_C} & \|\eta\|_C > 1 \end{cases}$$

Let  $C_b^0$  be the domain of strong continuity of  $P_t$ , viz.

$$C_b^0 := \{\phi \in C_b : P_t(\phi) \rightarrow \phi \text{ as } t \rightarrow 0+ \text{ in } C_b\}.$$

Then  $\phi_0 \in C_b$ , but  $\phi_0 \notin C_b^0$  because  $r > 0$ . □

#### 4. The Generator

Define the *weak generator*  $A : D(A) \subset C_b \rightarrow C_b$  by the weak limit

$$A(\phi)(\eta) := w - \lim_{t \rightarrow 0+} \frac{P_t(\phi)(\eta) - \phi(\eta)}{t}$$

where  $\phi \in D(A)$  iff the above weak limit exists. Hence  $D(A) \subset C_b^0$  (Dynkin [Dy], Vol. 1, Chapter I, pp. 36-43). Also  $D(A)$  is weakly dense in  $C_b$  and  $A$  is weakly closed. Further

$$\frac{d}{dt} P_t(\phi) = A(P_t(\phi)) = P_t(A(\phi)), \quad t > 0$$

for all  $\phi \in D(A)$  ([Dy], pp. 36-43).

Next objective is to derive a formula for the weak generator  $A$ . *We need to augment  $C$  by adjoining a canonical  $d$ -dimensional direction. The generator  $A$  will be equal to the weak generator of the shift semigroup  $\{S_t\}$  plus a second order linear partial differential operator along this new direction.* Computation requires the following lemmas.

Let

$$F_d = \{v\chi_{\{0\}} : v \in \mathbf{R}^d\}$$

$$C \oplus F_d = \{\eta + v\chi_{\{0\}} : \eta \in C, v \in \mathbf{R}^d\}, \quad \|\eta + v\chi_{\{0\}}\| = \|\eta\|_C + |v|$$

**Lemma II.1.**([Mo], Pitman Books, 1984)

Suppose  $\phi : C \rightarrow \mathbf{R}$  is  $C^2$  and  $\eta \in C$ . Then  $D\phi(\eta)$  and  $D^2\phi(\eta)$  have unique weakly continuous linear and bilinear extensions

$$\overline{D\phi(\eta)} : C \oplus F_d \rightarrow \mathbf{R}, \quad \overline{D^2\phi(\eta)} : (C \oplus F_d) \times (C \oplus F_d) \rightarrow \mathbf{R}$$

respectively.

**Proof.**

First reduce to the one-dimensional case  $d = 1$  by using coordinates.

Let  $\alpha \in C^* = [C([-r, 0], \mathbf{R})]^*$ . We will show that there is a weakly continuous linear extension  $\bar{\alpha} : C \oplus F_1 \rightarrow \mathbf{R}$  of  $\alpha$ ; viz. If  $\{\xi^k\}$  is a bounded sequence in  $C$  such that  $\xi^k(s) \rightarrow \xi(s)$  as  $k \rightarrow \infty$  for all  $s \in [-r, 0]$ , where  $\xi \in C \oplus F_1$ , then  $\alpha(\xi^k) \rightarrow \bar{\alpha}(\xi)$  as  $k \rightarrow \infty$ . By the Riesz representation theorem there is a unique finite regular Borel measure  $\mu$  on  $[-r, 0]$  such that

$$\alpha(\eta) = \int_{-r}^0 \eta(s) d\mu(s)$$

for all  $\eta \in C$ . Define  $\bar{\alpha} \in [C \oplus F_1]^*$  by

$$\bar{\alpha}(\eta + v\chi_{\{0\}}) = \alpha(\eta) + v\mu(\{0\}), \quad \eta \in C, \quad v \in \mathbf{R}.$$

Easy to check that  $\bar{\alpha}$  is weakly continuous. (*Exercise:* Use Lebesgue dominated convergence theorem.)

Weak extension  $\bar{\alpha}$  is unique because each function  $v\chi_{\{0\}}$  can be approximated weakly by a sequence of continuous functions  $\{\xi_0^k\}$ :

$$\xi_0^k(s) := \begin{cases} (ks + 1)v, & -\frac{1}{k} \leq s \leq 0 \\ 0 & -r \leq s < -\frac{1}{k}. \end{cases}$$

Put  $\alpha = D\phi(\eta)$  to get first assertion of lemma.

To construct a weakly continuous bilinear extension  $\bar{\beta} : (C \oplus F_1) \times (C \oplus F_1) \rightarrow \mathbf{R}$  for any continuous bilinear form  $\beta : C \times C \rightarrow \mathbf{R}$ , use classical theory of vector measures (Dunford and Schwartz, [D-S], Vol. I, Section 6.3). Think of  $\beta$  as a continuous *linear* map  $C \rightarrow C^*$ . Since  $C^*$  is weakly complete ([D-S], I.13.22, p. 341), then  $\beta$  is a weakly compact linear operator ([D-S], Theorem I.7.6, p. 494): i.e. it maps norm-bounded sets in  $C$  into weakly sequentially compact sets in  $C^*$ . By the Riesz representation theorem (for vector measures), there is a unique  $C^*$ -valued Borel measure  $\lambda$  on  $[-r, 0]$  (of finite semi-variation) such that

$$\beta(\xi) = \int_{-r}^0 \xi(s) d\lambda(s)$$

for all  $\xi \in C$ . ([D-S], Vol. I, Theorem VI.7.3, p. 493). By the dominated convergence theorem for vector measures ([D-S], Theorem IV.10.10, p. 328), one could reach elements in  $F_1$  using weakly convergent sequences of type  $\{\xi_0^k\}$ . This gives a unique weakly continuous extension  $\hat{\beta} : C \oplus F_1 \rightarrow C^*$ . Next for each  $\eta \in C$ ,  $v \in \mathbf{R}$ , extend  $\hat{\beta}(\eta + v\chi_{\{0\}}) : C \rightarrow \mathbf{R}$  to a weakly continuous linear map  $\hat{\beta}(\eta + v\chi_{\{0\}}) : C \oplus F_1 \rightarrow \mathbf{R}$ . Thus  $\bar{\beta}$  corresponds to the weakly continuous bilinear extension  $\hat{\beta}(\cdot)(\cdot) : [C \oplus F_1] \times [C \oplus F_1] \rightarrow \mathbf{R}$  of  $\beta$ . (Check this as exercise).

Finally use  $\beta = D^2\phi(\eta)$  for each fixed  $\eta \in C$  to get the required bilinear extension  $\overline{D^2\phi(\eta)}$ .  $\square$

**Lemma II.2.** ([Mo], Pitman Books, 1984)

For  $t > 0$  define  $W_t^* \in C$  by

$$W_t^*(s) := \begin{cases} \frac{1}{\sqrt{t}}[W(t+s) - W(0)], & -t \leq s < 0, \\ 0 & -r \leq s \leq -t. \end{cases}$$

Let  $\beta$  be a continuous bilinear form on  $C$ . Then

$$\lim_{t \rightarrow 0^+} \left[ \frac{1}{t} E\beta({}^n x_t - \tilde{\eta}_t, {}^n x_t - \tilde{\eta}_t) - E\beta(G(\eta) \circ W_t^*, G(\eta) \circ W_t^*) \right] = 0$$

**Proof.**

Use

$$\lim_{t \rightarrow 0^+} E \left\| \frac{1}{\sqrt{t}} ({}^n x_t - \tilde{\eta}_t - G(\eta) \circ W_t^*) \right\|_C^2 = 0.$$

The above limit follows from the Lipschitz continuity of  $H$  and  $G$  and the martingale properties of the Itô integral. Conclusion of lemma is obtained by a computation using the bilinearity of  $\beta$ , Hölder's inequality and the above limit. ([Mo], Pitman Books, 1984, pp. 86-87.)  $\square$

**Lemma II.3.** ([Mo], Pitman Books, 1984)

Let  $\beta$  be a continuous bilinear form on  $C$  and  $\{e_i\}_{i=1}^m$  be any basis for  $\mathbf{R}^m$ .

Then

$$\lim_{t \rightarrow 0^+} \frac{1}{t} E\beta({}^n x_t - \tilde{\eta}_t, {}^n x_t - \tilde{\eta}_t) = \sum_{i=1}^m \bar{\beta}(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}})$$

for each  $\eta \in C$ .

**Proof.**

By taking coordinates reduce to the one-dimensional case  $d = m = 1$ :

$$\lim_{t \rightarrow 0^+} E\beta(W_t^*, W_t^*) = \bar{\beta}(\chi_{\{0\}}, \chi_{\{0\}})$$

with  $W$  one-dimensional Brownian motion. The proof of the above relation is lengthy and difficult. A key idea is the use of the projective tensor product  $C \otimes_{\pi} C$  in order to view the continuous *bilinear* form  $\beta$  as a continuous *linear* functional on  $C \otimes_{\pi} C$ . At this level  $\beta$  commutes with the (Bochner) expectation. Rest of computation is effected using Mercer's theorem and some Fourier analysis. See [Mo], 1984, pp. 88-94.  $\square$

**Theorem II.3.** ([Mo], Pitman Books, 1984)

In (XIV) suppose  $H$  and  $G$  are globally bounded and Lipschitz. Let  $S : D(S) \subset C_b \rightarrow C_b$  be the weak generator of  $\{S_t\}$ . Suppose  $\phi \in D(S)$  is sufficiently smooth (e.g.  $\phi$  is  $C^2$ ,  $D\phi$ ,  $D^2\phi$  globally bounded and Lipschitz). Then  $\phi \in D(A)$  and

$$\begin{aligned} A(\phi)(\eta) &= S(\phi)(\eta) + \overline{D\phi(\eta)}(H(\eta)\chi_{\{0\}}) \\ &\quad + \frac{1}{2} \sum_{i=1}^m \overline{D^2\phi(\eta)}(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}). \end{aligned}$$

where  $\{e_i\}_{i=1}^m$  is any basis for  $\mathbf{R}^m$ .

**Proof.**

*Step 1.*

For fixed  $\eta \in C$ , use Taylor's theorem:

$$\phi({}^n x_t) - \phi(\eta) = \phi(\tilde{\eta}_t) - \phi(\eta) + D\phi(\tilde{\eta}_t)({}^n x_t - \tilde{\eta}_t) + R(t)$$

a.s. for  $t > 0$ ; where

$$R(t) := \int_0^1 (1-u) D^2\phi[\tilde{\eta}_t + u({}^n x_t - \tilde{\eta}_t)]({}^n x_t - \tilde{\eta}_t, {}^n x_t - \tilde{\eta}_t) du.$$

Take expectations and divide by  $t > 0$ :

$$\frac{1}{t}E[\phi({}^n x_t) - \phi(\eta)] = \frac{1}{t}[S_t(\phi(\eta) - \phi(\eta)) + D\phi(\tilde{\eta}_t)\left\{E\left[\frac{1}{t}({}^n x_t - \tilde{\eta}_t)\right]\right\} + \frac{1}{t}ER(t)] \quad (3)$$

for  $t > 0$ .

As  $t \rightarrow 0+$ , the first term on the RHS converges to  $S(\phi)(\eta)$ , because  $\phi \in D(S)$ .

*Step 2.*

Consider second term on the RHS of (3). Then

$$\begin{aligned} \lim_{t \rightarrow 0+} \left[ E\left\{ \frac{1}{t}({}^n x_t - \tilde{\eta}_t) \right\} \right](s) &= \begin{cases} \lim_{t \rightarrow 0+} \frac{1}{t} \int_0^t E[H({}^n x_u)] du, & s = 0 \\ 0 & -r \leq s < 0. \end{cases} \\ &= [H(\eta)\chi_{\{0\}}](s), \quad -r \leq s \leq 0. \end{aligned}$$

Since  $H$  is bounded, then  $\|E\{\frac{1}{t}({}^n x_t - \tilde{\eta}_t)\}\|_C$  is bounded in  $t > 0$  and  $\eta \in C$  (*Exercise*). Hence

$$w - \lim_{t \rightarrow 0+} \left[ E\left\{ \frac{1}{t}({}^n x_t - \tilde{\eta}_t) \right\} \right] = H(\eta)\chi_{\{0\}} \quad (\notin C).$$

Therefore by Lemma II.1 and the continuity of  $D\phi$  at  $\eta$ :

$$\begin{aligned} \lim_{t \rightarrow 0+} D\phi(\tilde{\eta}_t)\left\{E\left[\frac{1}{t}({}^n x_t - \tilde{\eta}_t)\right]\right\} &= \lim_{t \rightarrow 0+} D\phi(\eta)\left\{E\left[\frac{1}{t}({}^n x_t - \tilde{\eta}_t)\right]\right\} \\ &= \overline{D\phi(\eta)}(H(\eta)\chi_{\{0\}}) \end{aligned}$$

*Step 3.*

To compute limit of third term in RHS of (3), consider

$$\begin{aligned}
& \left| \frac{1}{t} ED^2\phi[\tilde{\eta}_t + u({}^n x_t - \tilde{\eta}_t)]({}^n x_t - \tilde{\eta}_t, {}^n x_t - \tilde{\eta}_t) \right. \\
& \quad \left. - \frac{1}{t} ED^2\phi(\eta)({}^n x_t - \tilde{\eta}_t, {}^n x_t - \tilde{\eta}_t) \right| \\
& \leq (E\|D^2\phi[\tilde{\eta}_t + u({}^n x_t - \tilde{\eta}_t)] - D^2\phi(\eta)\|^2)^{1/2} \left[ \frac{1}{t^2} E\|{}^n x_t - \tilde{\eta}_t\|^4 \right]^{1/2} \\
& \leq K(t^2 + 1)^{1/2} [E\|D^2\phi[\tilde{\eta}_t + u({}^n x_t - \tilde{\eta}_t)] - D^2\phi(\eta)\|^2]^{1/2} \\
& \rightarrow 0
\end{aligned}$$

as  $t \rightarrow 0+$ , uniformly for  $u \in [0, 1]$ , by martingale properties of the Itô integral and the Lipschitz continuity of  $D^2\phi$ . Therefore by Lemma II.3

$$\begin{aligned}
\lim_{t \rightarrow 0+} \frac{1}{t} ER(t) &= \int_0^1 (1-u) \lim_{t \rightarrow 0+} \frac{1}{t} ED^2\phi(\eta)({}^n x_t - \tilde{\eta}_t, {}^n x_t - \tilde{\eta}_t) du \\
&= \frac{1}{2} \sum_{i=1}^m \overline{D^2\phi(\eta)}(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}).
\end{aligned}$$

The above is a weak limit since  $\phi \in D(S)$  and has first and second derivatives globally bounded on  $C$ .  $\square$

## 5. Quasitame Functions

Recall that a function  $\phi : C \rightarrow \mathbf{R}$  is *tame* (or a *cylinder function*) if there is a finite set  $\{s_1 < s_2 < \dots < s_k\}$  in  $[-r, 0]$  and a  $C^\infty$ -bounded function  $f : (\mathbf{R}^d)^k \rightarrow \mathbf{R}$  such that

$$\phi(\eta) = f(\eta(s_1), \dots, \eta(s_k)), \quad \eta \in C.$$

The set of all tame functions is a weakly dense subalgebra of  $C_b$ , invariant under the static shift  $S_t$  and generates *Borel C*. For  $k \geq 2$  the tame function  $\phi$  *lies outside* the domain of strong continuity  $C_b^0$  of  $P_t$ , and hence *outside*  $D(A)$  ([Mo], Pitman Books, 1984, pp.98-103; see also proof of Theorem IV .2.2, pp. 73-76). To overcome this difficulty we introduce



**Definition.**

Say  $\phi : C \rightarrow \mathbf{R}$  is *quasitame* if there are  $C^\infty$ -bounded maps  $h : (\mathbf{R}^d)^k \rightarrow \mathbf{R}$ ,  $f_j : \mathbf{R}^d \rightarrow \mathbf{R}^d$ , and piecewise  $C^1$  functions  $g_j : [-r, 0] \rightarrow \mathbf{R}$ ,  $1 \leq j \leq k-1$ , such that

$$\phi(\eta) = h\left(\int_{-r}^0 f_1(\eta(s))g_1(s) ds, \dots, \int_{-r}^0 f_{k-1}(\eta(s))g_{k-1}(s) ds, \eta(0)\right) \quad (4)$$

for all  $\eta \in C$ .

**Theorem II.4.** ([Mo], Pitman Books, 1984)

The set of all quasitame functions is a weakly dense subalgebra of  $C_b^0$ , invariant under  $S_t$ , generates Borel  $C$  and belongs to  $D(A)$ . In particular, if  $\phi$  is the quasitame function given by (4), then

$$\begin{aligned} A(\phi)(\eta) &= \sum_{j=1}^{k-1} D_j h(m(\eta)) \{f_j(\eta(0))g_j(0) - f_j(\eta(-r))g_j(-r) \\ &\quad - \int_{-r}^0 f_j(\eta(s))g'_j(s) ds\} \\ &\quad + D_k h(m(\eta))(H(\eta)) + \frac{1}{2} \text{trace}[D_k^2 h(m(\eta)) \circ (G(\eta) \times G(\eta))]. \end{aligned}$$

for all  $\eta \in C$ , where

$$m(\eta) := \left(\int_{-r}^0 f_1(\eta(s))g_1(s) ds, \dots, \int_{-r}^0 f_{k-1}(\eta(s))g_{k-1}(s) ds, \eta(0)\right).$$

**Remarks.**

- (i) Replace  $C$  by the Hilbert space  $M_2$ . No need for the weak extensions because  $M_2$  is weakly complete. Extensions of  $D\phi(v, \eta)$  and  $D^2\phi(v, \eta)$  correspond to partial derivatives in the  $\mathbf{R}^d$ -variable. *Tame functions do not exist on  $M_2$  but quasitame functions do!* (with  $\eta(0)$  replaced by  $v \in \mathbf{R}^d$ ).

Analysis of supermartingale behavior and stability of  $\phi({}^\eta x_t)$  given in Kushner ([Ku], JDE, 1968). Infinite fading memory setting by Mizel and Trützer ([M-T], JIE, 1984) in the weighted state space  $\mathbf{R}^d \times L^2((-\infty, 0], \mathbf{R}^d; \rho)$ .

- (ii) For each quasitame  $\phi$  on  $C$ ,  $\phi({}^\eta x_t)$  is a semimartingale, and the Itô formula holds:

$$d[\phi({}^\eta x_t)] = A(\phi)({}^\eta x_t) dt + \overline{D\phi(\eta)}(H(\eta)\chi_{\{0\}}) dW(t).$$

**III. REGULARITY**  
**CLASSIFICATION OF SFDE'S**

**Geilo, Norway**

**Wednesday, July 31, 1996**

**10:30-11:20**

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### III. REGULARITY. CLASSIFICATION OF SFDE'S

Denote the state space by  $E$  where  $E = C$  or  $M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$ . Most results hold for either choice of state space.

#### Objectives

To study regularity properties of the trajectory of a sfde as a random field  $X := \{\eta x_t : t \geq 0, \eta \in C\}$  in the variables  $(t, \eta, \omega)$  ( $E = C$ ) or  $(t, (v, \eta), \omega)$  ( $E = M_2$ ):

- (i) Pathwise regularity of trajectories in the time variable.
- (ii) Regularity of trajectories (in probability or pathwise) in the initial state  $\eta \in C$  or  $(v, \eta) \in M_2$ .
- (iii) Classification of sfde's into regular and singular types.

Denote by  $C^\alpha := C^\alpha([-r, 0], \mathbf{R}^d)$  the (separable) Banach space of  $\alpha$ -Hölder continuous paths  $\eta : [-r, 0] \rightarrow \mathbf{R}^d$  with the Hölder norm

$$\|\eta\|_\alpha := \|\eta\|_C + \sup \left\{ \frac{|\eta(s_1) - \eta(s_2)|}{|s_1 - s_2|^\alpha} : s_1, s_2 \in [-r, 0], s_1 \neq s_2 \right\}.$$

$C^\alpha$  can be constructed in a *separable manner* by completing the space of smooth paths  $[-r, 0] \rightarrow \mathbf{R}^d$  with respect to the above norm (Tromba [Tr], JFA, 1972). First step is to think of  $\eta x_t(\omega)$  as a measurable mapping  $X : \mathbf{R}^+ \times C \times \Omega \rightarrow C$  in the three variables  $(t, \eta, \omega)$  simultaneously:

**Theorem III.1** ([Mo], Pitman Books, 1984)

*In the sfde*

$$\left. \begin{aligned} dx(t) &= H(t, x_t) dt + G(t, x_t) dW(t) & t > 0 \\ x_0 &= \eta \in C \end{aligned} \right\} \quad (XIII)$$

assume that the coefficients  $H, G$  are (jointly) continuous and globally Lipschitz in the second variable uniformly wrt the first. Then

(i) For any  $0 < \alpha < \frac{1}{2}$ , and each initial path  $\eta \in C$ ,

$$P(\eta x_t \in C^\alpha, \text{ for all } t \geq r) = 1.$$

(ii) the trajectory field has a measurable version

$$X : \mathbf{R}^+ \times C \times \Omega \rightarrow C.$$

(iii) The trajectory field  $\eta x_t, t \geq r, \eta \in C$ , admits a measurable version

$$[r, \infty) \times C \times \Omega \rightarrow C^\alpha.$$

**Remark.**

Similar statements hold for  $E = M_2$ .

Give  $L^0(\Omega, E)$  the complete (psuedo)metric

$$d_E(\theta_1, \theta_2) := \inf_{\epsilon > 0} [\epsilon + P(\|\theta_1 - \theta_2\|_E \geq \epsilon)], \quad \theta_1, \theta_2 \in L^0(\Omega, E),$$

(which corresponds to convergence in probability, Dunford and Schwartz [D-S], Lemma III.2.7, p. 104).

**Proof of Theorem III.1.**

(i) Sufficient to show that

$$P(\eta x|[0, a] \in C^\alpha([0, a], \mathbf{R}^d)) = 1$$

by using the estimate

$$P\left(\sup_{0 \leq t_1, t_2 \leq a, t_1 \neq t_2} \frac{|\eta x(t_1) - \eta x(t_2)|}{|t_1 - t_2|^\alpha} \geq N\right) \leq C_k^1 (1 + \|\eta\|_C^{2k}) \frac{1}{N^{2k}},$$

for all integers  $k > (1 - 2\alpha)^{-1}$ , and the Borel-Cantelli lemma. Above estimate is proved using Gronwall's lemma, Chebyshev's inequality, and Garsia-Rodemick-Rumsey lemma ([Mo], Pitman

Books, 1984, Theorem 4.1, p. 150; [Mo], Pitman Books, 1984, Theorem 4.4, pp.152-154.)

- (ii) By mean-square Lipschitz dependence ([Mo], Pitman Books, 1984, Theorem 3.1, p. 41), the trajectory

$$\begin{aligned} [0, a] \times C &\rightarrow L^2(\Omega, C) \subset L^0(\Omega, C) \\ (t, \eta) &\mapsto {}^\eta x_t \end{aligned}$$

is globally Lipschitz in  $\eta$  uniformly wrt  $t$  in compact sets, and is continuous in  $t$  for fixed  $\eta$ . Therefore it is jointly continuous in  $(t, \eta)$  as a map

$$[0, a] \times C \ni (t, \eta) \mapsto {}^\eta x_t \in L^0(\Omega, C).$$

Then apply the Cohn-Hoffman-Jørgensen theorem:

*If  $T, E$  are complete separable metric spaces, then each Borel map  $X : T \rightarrow L^0(\Omega, E; \mathcal{F})$  admits a measurable version*

$$T \times \Omega \rightarrow E$$

to the trajectory field to get measurability in  $(t, \eta)$ . (Take  $T = [0, a] \times C$ ,  $E = C$  ([Mo], Pitman Books, 1984, p. 16).)

- (iii) Use the estimate

$$P(\|{}^{\eta_1} x_t - {}^{\eta_2} x_t\|_{C^\alpha} \geq N) \leq \frac{C_k^2}{N^{2k}} \|\eta_1 - \eta_2\|_C^{2k}$$

for  $t \in [r, a], N > 0$ , ([Mo], 1984, Theorem 4.7, pp.158-162) to prove joint continuity of the trajectory

$$\begin{aligned} [r, a] \times C &\rightarrow L^0(\Omega, C^\alpha) \\ (t, \eta) &\mapsto {}^\eta x_t \end{aligned}$$

([Mo], Theorem 4.7, pp. 158-162) viewed as a process with values in the separable Banach space  $C^\alpha$ . Again apply the Cohn-Hoffman-Jørgensen theorem.  $\square$

As we have seen in Lecture I, the trajectory of a sfde possesses good regularity properties *in the mean-square*. The following theorem shows good behavior in distribution.

**Theorem III.2.** ([Mo], Pitman Books, 1984)

Suppose the coefficients  $H, G$  are globally Lipschitz in the second variable uniformly with respect to the first. Let  $\alpha \in (0, 1/2)$  and  $k$  be any integer such that  $k > (1 - 2\alpha)^{-1}$ . Then there are positive constants  $C_k^3, C_k^4, C_k^5$  such that

$$d_C(\eta^1 x_t, \eta^2 x_t) \leq C_k^3 \|\eta_1 - \eta_2\|_C^{2k/(2k+1)} \quad t \in [0, a]$$

$$d_{C^\alpha}(\eta^1 x_t, \eta^2 x_t) \leq C_k^4 \|\eta_1 - \eta_2\|_C^{2k/(2k+1)} \quad t \in [r, a]$$

$$P(\|\eta x_t\|_{C^\alpha} \geq N) \leq C_k^5 (1 + \|\eta\|_C^{2k}) \frac{1}{N^{2k}}, \quad t \in [r, a], \quad N > 0.$$

In particular the transition probabilities

$$[r, a] \times C \rightarrow \mathcal{M}_p(C)$$

$$(t, \eta) \mapsto p(0, \eta, t, \cdot)$$

take bounded sets into relatively weak\* compact sets in the space  $\mathcal{M}_p(C)$  of probability measures on  $C$ .

**Proof of Theorem III.2.**

Proofs of the estimates use Gronwall's lemma, Chebyshev's inequality, and Garsia-Rodemick-Rumsey lemma ([Mo], 1984, Theorem 4.1, p. 150; [Mo], 1984, Theorem 4.7, pp.159-162.) The weak\* compactness assertion follows from the last estimate, Prohorov's theorem

and the compactness of the embedding  $C^\alpha \hookrightarrow C$  ([Mo], 1984, Theorem 4.6, pp. 156-158).  $\square$

## Erratic Behavior. The Noisy Loop Revisited

### Definition.

A sfde is *regular* with respect to  $M_2$  if its trajectory random field  $\{(x(t), x_t) : (x(0), x_0) = (v, \eta) \in M_2, t \geq 0\}$  admits a  $(\text{Borel } \mathbf{R}^+ \otimes \text{Borel } M_2 \otimes \mathcal{F}, \text{Borel } M_2)$ -measurable version  $X : \mathbf{R}^+ \times M_2 \times \Omega \rightarrow M_2$  with a.a. sample functions continuous on  $\mathbf{R}^+ \times M_2$ . The sfde is said to be *singular* otherwise. Similarly for regularity with respect to  $C$ .

Consider the one-dimensional linear sdde with a *positive delay*  $r$

$$\left. \begin{aligned} dx(t) &= \sigma x(t-r) dW(t), & t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R} \times L^2([-r, 0], \mathbf{R}), \end{aligned} \right\} \quad (I)$$

driven by a Wiener process  $W$ .

Theorem III.3 below implies that (I) is singular with respect to  $M_2$  (and  $C$ ). (See also [Mo], Stochastics, 1986).

Consider the regularity of the more general one-dimensional linear sfde:

$$\left. \begin{aligned} dx(t) &= \int_{-r}^0 x(t+s) d\nu(s) dW(t), & t > 0 \\ (x(0), x_0) &\in M_2 := \mathbf{R} \times L^2([-r, 0], \mathbf{R}) \end{aligned} \right\} \quad (II')$$

where  $W$  is a Wiener process and  $\nu$  is a fixed finite real-valued Borel measure on  $[-r, 0]$ .

*Exercise:*

(II') is regular if  $\nu$  has a  $C^1$  (or even  $L^2_1$ ) density with respect to Lebesgue measure on  $[-r, 0]$ . (Hint: Use integration by parts to eliminate the Itô integral!)



The following theorem gives conditions on the measure  $\nu$  under which (II') is singular.

**Theorem III.3** ([M-S], II, 1996)

Let  $r > 0$ , and suppose that there exists  $\epsilon \in (0, r)$  such that  $\text{supp } \nu \subset [-r, -\epsilon]$ . Suppose  $0 < t_0 \leq \epsilon$ . For each  $k \geq 1$ , set

$$\nu_k := \sqrt{t_0} \left| \int_{[-r, 0]} e^{2\pi i k s / t_0} d\nu(s) \right|.$$

Assume that

$$\sum_{k=1}^{\infty} \nu_k x^{1/\nu_k^2} = \infty \tag{1}$$

for all  $x \in (0, 1)$ . Let  $Y : [0, \epsilon] \times M_2 \times \Omega \rightarrow \mathbf{R}$  be any Borel-measurable version of the solution field  $\{x(t) : 0 \leq t \leq \epsilon, (x(0), x_0) = (v, \eta) \in M_2\}$  of (II'). Then for a.a.  $\omega \in \Omega$ , the map  $Y(t_0, \cdot, \omega) : M_2 \rightarrow \mathbf{R}$  is unbounded in every neighborhood of every point in  $M_2$ , and (hence) non-linear.

**Corollary.** ([Mo], Pitman Books, 1984 )

Suppose  $r > 0, \sigma \neq 0$  in (I). Then the trajectory  $\{\eta x_t : 0 \leq t \leq r, \eta \in C\}$  of (I) has a measurable version  $X : \mathbf{R}^+ \times C \times \Omega \rightarrow C$  s.t. for every  $t \in (0, r]$

$$P\left(X(t, \eta_1 + \lambda\eta_2, \cdot) = X(t, \eta_1, \cdot) + \lambda X(t, \eta_2, \cdot)\right. \\ \left. \text{for all } \lambda \in \mathbf{R}, \eta_1, \eta_2 \in C\right) = 0.$$

But

$$P\left(X(t, \eta_1 + \lambda\eta_2, \cdot) = X(t, \eta_1, \cdot) + \lambda X(t, \eta_2, \cdot)\right) = 1.$$

for all  $\lambda \in \mathbf{R}, \eta_1, \eta_2 \in C$ .

**Remark.**

(i) Condition (1) of the theorem is implied by

$$\lim_{k \rightarrow \infty} \nu_k \sqrt{\log k} = \infty.$$

- (ii) For the delay equation (I),  $\nu = \sigma\delta_{-r}$ ,  $\epsilon = r$ . In this case condition (1) is satisfied for *every*  $t_0 \in (0, r]$ .
- (iii) Theorem III.3 also holds for state space  $C$  since every bound-ed set in  $C$  is also bounded in  $L^2([-r, 0], \mathbf{R})$ .

### Proof of Theorem III.3.

Joint work with V. Mizel.

Main idea is to track the solution random field of (a complexified version of) (II') along the classical Fourier basis

$$\eta_k(s) = e^{2\pi i k s / t_0}, \quad -r \leq s \leq 0, \quad k \geq 1 \quad (2)$$

in  $L^2([-r, 0], \mathbf{C})$ . On this basis, the solution field gives an infinite family of independent Gaussian random variables. This allows us to show that no Borel measurable version of the solution field can be bounded with positive probability on an arbitrarily small neighborhood of 0 in  $M_2$ , and hence on any neighborhood of any point in  $M_2$  (cf. [Mo], Pitman Books, 1984; [Mo], Stochastics, 1986). For simplicity of computations, complexify the state space in (II') by allowing  $(v, \eta)$  to belong to  $M_2^{\mathbf{C}} := \mathbf{C} \times L^2([-r, 0], \mathbf{C})$ . Thus consider the sfde

$$\left. \begin{aligned} dx(t) &= \int_{[-r, 0]} x(t+s) d\nu(s) dW(t), t > 0, \\ (x(0), x_0) &= (v, \eta) \in M_2^{\mathbf{C}} \end{aligned} \right\} \quad (II' - C)$$

where  $x(t) \in \mathbf{C}$ ,  $t \geq -r$ , and  $\nu, W$  are real-valued.

Use contradiction. Let  $Y : [0, \epsilon] \times M_2 \times \Omega \rightarrow \mathbf{R}$  be any Borel-measurable version of the solution field  $\{x(t) : 0 \leq t \leq \epsilon, (x(0), x_0) = (v, \eta) \in M_2\}$  of (II'). Suppose, if possible, that there exists a set  $\Omega_0 \in \mathcal{F}$  of positive  $P$ -measure,  $(v_0, \eta_0) \in M_2$  and a positive  $\delta$  such that for all  $\omega \in \Omega_0$ ,  $Y(t_0, \cdot, \omega)$  is bounded on the open ball  $B((v_0, \eta_0), \delta)$  in  $M_2$  of center  $(v_0, \eta_0)$  and radius  $\delta$ . Define the complexification  $Z(\cdot, \omega) : M_2^{\mathbf{C}} \rightarrow \mathbf{C}$  of  $Y(t_0, \cdot, \omega) : M_2 \rightarrow \mathbf{R}$  by

$$Z(\xi_1 + i\xi_2, \omega) := Y(t_0, \xi_1, \omega) + iY(t_0, \xi_2, \omega), \quad i = \sqrt{-1},$$

for all  $\xi_1, \xi_2 \in M_2$ ,  $\omega \in \Omega$ . Let  $(v_0, \eta_0)^{\mathbf{C}}$  denote the complexification  $(v_0, \eta_0)^{\mathbf{C}} := (v_0, \eta_0) + i(v_0, \eta_0)$ . Clearly  $Z(\cdot, \omega)$  is bounded on the complex

ball  $B((v_0, \eta_0)^C, \delta)$  in  $M_2^C$  for all  $\omega \in \Omega_0$ . Define the sequence of complex random variables  $\{Z_k\}_{k=1}^\infty$  by

$$Z_k(\omega) := Z((\eta_k(0), \eta_k), \omega) - \eta_k(0), \quad \omega \in \Omega, \quad k \geq 1.$$

Then

$$Z_k = \int_0^{t_0} \int_{[-r, -\epsilon]} \eta_k(u+s) d\nu(s) dW(u), \quad k \geq 1.$$

By standard properties of the Itô integral, and Fubini's theorem,

$$EZ_k \overline{Z_l} = \int_{[-r, -\epsilon]} \int_{[-r, -\epsilon]} \int_0^{t_0} \eta_k(u+s) \overline{\eta_l(u+s')} du d\nu(s) d\nu(s') = 0$$

for  $k \neq l$ , because

$$\int_0^{t_0} \eta_k(u+s) \overline{\eta_l(u+s')} du = 0$$

whenever  $k \neq l$ , for all  $s, s' \in [-r, 0]$ . Furthermore

$$\int_0^{t_0} \eta_k(u+s) \overline{\eta_k(u+s')} du = t_0 e^{2\pi i k(s-s')/t_0}$$

for all  $s, s' \in [-r, 0]$ . Hence

$$\begin{aligned} E|Z_k|^2 &= \int_{[-r, -\epsilon]} \int_{[-r, -\epsilon]} t_0 e^{2\pi i k(s-s')/t_0} d\nu(s) d\nu(s') \\ &= t_0 \left| \int_{[-r, 0]} e^{2\pi i ks/t_0} d\nu(s) \right|^2 \\ &= \nu_k^2. \end{aligned}$$

$Z(\cdot, \omega) : M_2^C \rightarrow \mathbf{C}$  is bounded on  $B((v_0, \eta_0)^C, \delta)$  for all  $\omega \in \Omega_0$ , and  $\|(\eta_k(0), \eta_k)\| = \sqrt{r+1}$  for all  $k \geq 1$ . By the linearity property

$$\begin{aligned} &Z\left((v_0, \eta_0)^C + \frac{\delta}{2\sqrt{r+1}}(\eta_k(0), \eta_k), \cdot\right) \\ &= Z((v_0, \eta_0)^C, \cdot) + \frac{\delta}{2\sqrt{r+1}} Z((\eta_k(0), \eta_k), \cdot), \quad k \geq 1, \end{aligned}$$

a.s., it follows that

$$P\left(\sup_{k \geq 1} |Z_k| < \infty\right) > 0. \quad (3)$$

It is easy to check that  $\{ReZ_k, ImZ_k : k \geq 1\}$  are independent  $\mathcal{N}(0, \nu_k^2/2)$ -distributed Gaussian random variables. Get a contradiction to (3):

For each integer  $N \geq 1$ ,

$$\begin{aligned} P\left(\sup_{k \geq 1} |Z_k| < N\right) &\leq \prod_{k \geq 1} P\left(|ReZ_k| < N\right) \\ &= \prod_{k \geq 1} \left[1 - \frac{2}{\sqrt{2\pi}} \int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty} e^{-x^2/2} dx\right] \\ &\leq \exp\left\{-\frac{2}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty} e^{-x^2/2} dx\right\}. \end{aligned} \quad (4)$$

There exists  $N_0 > 1$  (independent of  $k \geq 1$ ) such that

$$\int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty} e^{-x^2/2} dx \geq \frac{\nu_k}{2\sqrt{2}N} e^{-\frac{N^2}{\nu_k^2}} \quad (5)$$

for all  $N \geq N_0$  and all  $k \geq 1$ .

Combine (4) and (5) and use hypothesis (1) of the theorem to get

$$P\left(\sup_{k \geq 1} |Z_k| < N\right) = 0$$

for all  $N \geq N_0$ . Hence

$$P\left(\sup_{k \geq 1} |Z_k| < \infty\right) = 0.$$

This contradicts (3)(cf. Dudley [Du], JFA, 1967).

Since  $Y(t_0, \cdot, \omega)$  is locally unbounded, it must be non-linear because of Douady's Theorem:

*Every Borel measurable linear map between two Banach spaces is continuous.*  
(Schwartz [Sc], Radon Measures, Part II, 1973, pp. 155-160).  $\square$

Note that the pathological phenomenon in Theorem III.3 is peculiar to the delay case  $r > 0$ . The proof of the theorem suggests that this pathology is due to the *Gaussian nature* of the Wiener process  $W$  coupled with the *infinite-dimensionality* of the state space  $M_2$ . Because of this, one may expect similar difficulties in certain types of linear spde's driven by *multi-dimensional* white noise (Flandoli and Schaumlöffel [F-S], Stochastics, 1990).

**Problem.**

Classify all finite signed measures  $\nu$  on  $[-r, 0]$  for which (II') is regular.

Note that (I) automatically satisfies the conditions of Theorem III.3, and hence its trajectory field *explodes on every small neighborhood of  $0 \in M_2$* . Because of the singular nature of (I), it is surprising that the maximal exponential growth rate of the trajectory of (I) is *negative* for small  $\sigma$  and is bounded away from zero *independently of the choice of the initial path* in  $M_2$ . This will be shown later in Lecture V (Theorem V.1).

**Regular Linear Systems. White Noise**

SDE's on  $\mathbf{R}^d$  driven by  $m$ -dimensional Brownian motion  $W := (W_1, \dots, W_m)$ , with smooth coefficients.

$$\left. \begin{aligned} dx(t) &= H(x(t - d_1), \dots, x(t - d_N), x(t), x_t)dt \\ &\quad + \sum_{i=1}^m g_i x(t) dW_i(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (VIII)$$

(VIII) is defined on

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) =$  canonical complete filtered Wiener space:

$\Omega :=$  space of all continuous paths  $\omega : \mathbf{R}^+ \rightarrow \mathbf{R}^m$ ,  $\omega(0) = 0$ , in Euclidean space  $\mathbf{R}^m$ , with compact open topology;

$\mathcal{F} :=$  completed Borel  $\sigma$ -field of  $\Omega$ ;

$\mathcal{F}_t :=$  completed sub- $\sigma$ -field of  $\mathcal{F}$  generated by the evaluations  $\omega \rightarrow \omega(u)$ ,  $0 \leq u \leq t$ ,  $t \geq 0$ ;

$P :=$  Wiener measure on  $\Omega$ ;

$dW_i(t) =$  Itô stochastic differentials,  $1 \leq i \leq m$ .

Several finite delays  $0 < d_1 < d_2 < \dots < d_N \leq r$  in drift term; *no delays in diffusion coefficient.*

$H : (\mathbf{R}^d)^{N+1} \times L^2([-r, 0], \mathbf{R}^d) \rightarrow \mathbf{R}^d$  is a fixed continuous linear map;  $g_i$ ,  $i = 1, 2, \dots, m$ , fixed (deterministic)  $d \times d$ -matrices.

**Theorem III.4.** ([Mo], Stochastics, 1990)]

(VIII) is regular with respect to the state space  $M_2 = \mathbf{R}^d \times \mathbf{L}^2([-r, 0], \mathbf{R}^d)$ . There is a measurable version  $X : \mathbf{R}^+ \times M_2 \times \Omega \rightarrow M_2$  of the trajectory field  $\{(x(t), x_t) : t \in \mathbf{R}^+, (x(0), x_0) = (v, \eta) \in M_2\}$  with the following properties:

- (i) For each  $(v, \eta) \in M_2$  and  $t \in \mathbf{R}^+$ ,  $X(t, (v, \eta), \cdot) = (x(t), x_t)$  a.s., is  $\mathcal{F}_t$ -measurable and belongs to  $L^2(\Omega, M_2; P)$ .
- (ii) There exists  $\Omega_0 \in \mathcal{F}$  of full measure such that, for all  $\omega \in \Omega_0$ , the map  $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times M_2 \rightarrow M_2$  is continuous.
- (iii) For each  $t \in \mathbf{R}^+$  and every  $\omega \in \Omega_0$ , the map  $X(t, \cdot, \omega) : M_2 \rightarrow M_2$  is continuous linear; for each  $\omega \in \Omega_0$ , the map  $\mathbf{R}^+ \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$  is measurable and locally bounded in the uniform operator norm on  $L(M_2)$ . The map  $[r, \infty) \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$  is continuous for all  $\omega \in \Omega_0$ .
- (iv) For each  $t \geq r$  and all  $\omega \in \Omega_0$ , the map

$$X(t, \cdot, \omega) : M_2 \rightarrow M_2$$

is compact.

Proof uses variational technique to reduce the problem to the solution of a random family of classical integral equations involving *no stochastic integrals*.

Compactness of semi-flow for  $t \geq r$  will be used later to define hyperbolicity for (VIII) and the associated exponential dichotomies (Lecture IV).

### Regular Linear Systems. Semimartingale Noise

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  a complete filtered probability space satisfying the usual conditions.

Linear systems driven by semimartingale noise, and memory driven by a measure-valued process

$\nu : \mathbf{R} \times \Omega \rightarrow \mathcal{M}([-r, 0], \mathbf{R}^{d \times d})$ , where  $\mathcal{M}([-r, 0], \mathbf{R}^{d \times d})$  is the space of all  $d \times d$ -matrix-valued Borel measures on  $[-r, 0]$  (or  $\mathbf{R}^{d \times d}$ -valued functions of bounded variation on  $[-r, 0]$ ). This space is given the  $\sigma$ -algebra generated by all evaluations. The space  $\mathbf{R}^{d \times d}$  of all  $d \times d$ -matrices is given the Euclidean norm  $\|\cdot\|$ .

$$\left. \begin{aligned} dx(t) &= \left\{ \int_{[-r, 0]} \nu(t)(ds) x(t+s) \right\} dt + dN(t) \int_{-r}^0 K(t)(s) x(t+s) ds \\ &\quad + dL(t) x(t-), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (IX)$$

### Hypotheses (R)

- (i) The process  $\nu : \mathbf{R} \times \Omega \rightarrow \mathcal{M}([-r, 0], \mathbf{R}^{d \times d})$  is measurable and  $(\mathcal{F}_t)_{t \geq 0}$ -adapted. For each  $\omega \in \Omega$  and  $t \geq 0$  define the positive measure  $\bar{\nu}(t, \omega)$  on  $[-r, \infty)$  by

$$\bar{\nu}(t, \omega)(A) := |\nu|(t, \omega)\{(A-t) \cap [-r, 0]\}$$



for all Borel subsets  $A$  of  $[-r, \infty)$ , where  $|\nu|$  is the total variation measure of  $\nu$  wrt the Euclidean norm on  $\mathbf{R}^{d \times d}$ . Therefore the equation

$$\mu(\omega)(\cdot) := \int_0^\infty \bar{\nu}(t, \omega)(\cdot) dt$$

defines a positive measure on  $[-r, \infty)$ . For each  $\omega \in \Omega$  suppose that  $\mu(\omega)$  has a density wrt Lebesgue measure which is locally essentially bounded.

(*Exercise:* This condition is automatically satisfied if  $\nu(t, \omega)$  is independent of  $(t, \omega)$ .)

- (ii)  $K : \mathbf{R} \times \Omega \rightarrow L^\infty([-r, 0], \mathbf{R}^{d \times d})$  is measurable and  $(\mathcal{F}_t)_{t \geq 0}$ -adapted. Define the random field  $\tilde{K}(t, s, \omega)$  by

$\tilde{K}(t, s, \omega) := K(t, \omega)(s - t)$  for  $t \geq 0$ ,  $-r \leq s - t \leq 0$ . Assume that  $\tilde{K}(t, s, \omega)$  is absolutely continuous in  $t$  for Lebesgue a.a.  $s$  and all  $\omega \in \Omega$ . For every  $\omega \in \Omega$ ,  $\frac{\partial \tilde{K}}{\partial t}(t, s, \omega)$  and  $\tilde{K}(t, s, \omega)$  are locally essentially bounded in  $(t, s)$ .  $\frac{\partial \tilde{K}}{\partial t}(t, s, \omega)$  is jointly measurable.

- (iii)  $L = M + V$ ,  $M$  continuous local martingale,  $V$  B.V. process.

**Theorem III.5.** ([M-S], I, AIHP, 1996)

Under hypotheses (R), equation (IX) is regular w.r.t.  $M_2$  with a measurable flow  $X : \mathbf{R}^+ \times M_2 \times \Omega \rightarrow M_2$ . This flow satisfies Theorem III.4.

**Proof.**

This is achieved via a construction in ([M-S], I, AIHP, 1996) which reduces (IX) to a random linear integral equation with *no stochastic integrals* ([M-S], AIHP, 1996, pp. 85-96). Do a complicated pathwise analysis on the integral equation to establish existence and regularity properties of the semiflow.  $\square$

## Regular Non-linear Systems

### (a) SFDE's with Ordinary Diffusion Coefficients

In the sfde,

$$\left. \begin{aligned} dx(t) &= H(x_t)dt + \sum_{i=1}^m g_i(x(t))dW_i(t) \\ x_0 &= \eta \in C \end{aligned} \right\} \quad (XV)$$

let  $H : C \rightarrow \mathbf{R}^d$  be globally Lipschitz and  $g_i : \mathbf{R}^d \rightarrow \mathbf{R}^d$   $C^2$ -bounded maps satisfying the Frobenius condition (vanishing Lie brackets):

$$Dg_i(v)g_j(v) = Dg_j(v)g_i(v), \quad 1 \leq i, j \leq m, \quad v \in \mathbf{R}^d;$$

and  $W := (W_1, W_2, \dots, W_m)$  is  $m$ -dimensional Brownian motion. Note that the diffusion coefficient in (XV) has no memory.

**Theorem III.6** ([Mo], Pitman Books, 1984)

Suppose the above conditions hold. Then the trajectory field  $\{\eta x_t : t \geq 0, \eta \in C\}$  of (XV) has a measurable version  $X : \mathbf{R}^+ \times C \times \Omega \rightarrow C$  satisfying the following properties. For each  $\alpha \in (0, 1/2)$ , there is a set  $\Omega_\alpha \subset \Omega$  of full measure such that for every  $\omega \in \Omega_\alpha$

- (i)  $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times C \rightarrow C$  is continuous;
- (ii)  $X(\cdot, \cdot, \omega) : [r, \infty) \times C \rightarrow C^\alpha$  is continuous;
- (iii) for each  $t \geq r$ ,  $X(t, \cdot, \omega) : C \rightarrow C$  is compact;
- (iv) for each  $t \geq r$ ,  $X(t, \cdot, \omega) : C \rightarrow C^\alpha$  is Lipschitz on every bounded set in  $C$ , with a Lipschitz constant independent of  $t$  in compact sets. Hence each map  $X(t, \cdot, \omega) : C \rightarrow C$  is compact: viz. takes bounded sets into relatively compact sets.

### Proof of Theorem III.6.

([Mo], Pitman Books, 1984, Theorem (2.1), Chapter (V), §2, p. 121). This latter result is proved using a non-linear variational method originally due to Sussman ([Su], Ann. Prob., 1978) and Doss ([Do], AIHP, 1977) in the non-delay case  $r = 0$ . Write  $g := (g_1, g_2, \dots, g_m) : \mathbf{R}^d \rightarrow \mathbf{R}^{d \times m}$ . By the Frobenius condition, there is a  $C^2$  map  $F : \mathbf{R}^m \times \mathbf{R}^d \rightarrow \mathbf{R}^d$  such that  $\{F(\underline{t}, \cdot) : \underline{t} \in \mathbf{R}^m\}$  is a group of  $C^2$  diffeomorphisms  $\mathbf{R}^d \rightarrow \mathbf{R}^d$  satisfying

$$\begin{aligned} D_1 F(\underline{t}, x) &= g(F(\underline{t}, x)), \\ F(\underline{0}, x) &= x \end{aligned}$$

for all  $\underline{t} \in \mathbf{R}^m, x \in \mathbf{R}^d$ .

Define

$$W^0(t) := \begin{cases} W(t) - W(0), & t \geq 0 \\ 0 & -r \leq t < 0 \end{cases}$$

and  $\tilde{H} : \mathbf{R}^+ \times C \times \Omega \rightarrow \mathbf{R}^d$ , by

$$\begin{aligned} \tilde{H}(t, \eta, \cdot) &:= D_2 F(W^0(t), \eta(0))^{-1} \{ H[F \circ (W_t^0, \eta)] \\ &\quad - \frac{1}{2} \text{trace}(Dg[F(W^0(t), \eta(0))] \circ g[F(W^0(t), \eta(0))]) \} \end{aligned}$$

where the expression under the “trace” is viewed as a bilinear form  $\mathbf{R}^m \times \mathbf{R}^m \rightarrow \mathbf{R}^d$ , and the trace has values in  $\mathbf{R}^d$ . Then for each  $\omega$ ,  $\tilde{H}(t, \eta, \omega)$  is jointly continuous, Lipschitz in  $\eta$  in bounded subsets of  $C$  uniformly for  $t$  in compact sets, and satisfies a global linear growth condition in  $\eta$  ([Mo], Pitman Books, 1984, pp. 114-126).

Therefore solve the fde

$$\begin{aligned} {}^\eta \xi'_t &= \tilde{H}(t, {}^\eta \xi_t, \cdot) & t \geq 0 \\ {}^\eta \xi_0 &= \eta. \end{aligned}$$

Define the semiflow

$$X(t, \eta, \omega) = F \circ (W_t^0(\omega), \eta x_t(\omega)).$$

Check that  $X$  satisfies all assertions of theorem ([Mo], 1984, pp.126-133). □

**(b) SFDE's with Smooth Memory**

$$\left. \begin{aligned} dx(t) &= H(dt, x(t), x_t) + G(dt, x(t), g(x_t)), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 \end{aligned} \right\} \quad (XVI)$$

Coefficients  $H$  and  $G$  in (XVI) are semimartingale-valued random fields on  $M_2 = \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$  and  $\mathbf{R}^d \times \mathbf{R}^m$ , respectively. The memory is driven by a functional  $g : L^2([-r, 0], \mathbf{R}^d) \rightarrow \mathbf{R}^m$  with the smoothness property that the process  $t \mapsto g(x_t)$  has absolutely continuous paths for each adapted process  $x$ . Under (technical) but general regularity and boundedness conditions on the characteristics of  $H$  and  $G$ , equation (XVI) is regular:

**Theorem III.7** ([M-S], 1996)

Let

$$\Delta := \{(t_0, t) \in \mathbf{R}^2 : t_0 \leq t\}.$$

Under suitable regularity conditions on  $H, G, g$  in (XVI), there exists a random field  $X : \Delta \times M_2 \times \Omega \rightarrow M_2$  satisfying the following properties:

(i) For each  $(v, \eta) \in M_2, (t_0, t) \in \Delta, X(t_0, t, (v, \eta), \cdot) = (x^{t_0, (v, \eta)}(t), x_t^{t_0, (v, \eta)})$  a.s., where  $x^{t_0, (v, \eta)}$  is the unique solution of (XVI) with  $x_{t_0}^{t_0, (v, \eta)} = (v, \eta)$ .

(ii) For each  $(t_0, t, \omega) \in \Delta \times \Omega$ , the map

$$X(t_0, t, \cdot, \omega) : M_2 \rightarrow M_2$$

is  $C^\infty$ .

(iii) For each  $\omega \in \Omega$  and  $(t_0, t) \in \Delta$  with  $t > t_0 + r$ , the map

$$X(t_0, t, \cdot, \omega) : M_2 \rightarrow M_2$$

carries bounded sets into relatively compact sets.

## **IV. ERGODIC THEORY OF REGULAR LINEAR SFDE's**

**Geilo, Norway**

**Thursday, August 1, 1996**

**14:00-14:50**

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## IV. ERGODIC THEORY OF LINEAR SFDE'S

### 1. Plan

Use state space  $M_2$ . For regular linear sfde's (VIII), (IX), consider the following themes:

- I) Existence of a “perfect” cocycle on  $M_2$  that is a modification of the trajectory field  $(x(t), x_t) \in M_2$ .
- II) Existence of almost sure Lyapunov exponents

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|(x(t), x_t)\|_{M_2}$$

The multiplicative ergodic theorem and *hyperbolicity* of the cocycle.

- III) *The Stable Manifold Theorem*, (viz. “random saddles”) for hyperbolic systems.

## 2. Regular Linear Systems. White Noise

Linear sfde's on  $\mathbf{R}^d$  driven by  $m$ -dimensional Brownian motion  $W := (W_1, \dots, W_m)$ , with smooth coefficients.

$$\left. \begin{aligned} dx(t) &= H(x(t-d_1), \dots, x(t-d_N), x(t), x_t)dt \\ &\quad + \sum_{i=1}^m g_i x(t) dW_i(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (VIII)$$

(VIII) is defined on

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}}, P) =$  canonical complete filtered Wiener space.

$\Omega :=$  space of all continuous paths  $\omega : \mathbf{R} \rightarrow \mathbf{R}^m$ ,  $\omega(0) = 0$ , in Euclidean space  $\mathbf{R}^m$ , with compact open topology;

$\mathcal{F} :=$  completed Borel  $\sigma$ -field of  $\Omega$ ;

$\mathcal{F}_t :=$  completed sub- $\sigma$ -field of  $\mathcal{F}$  generated by the evaluations  $\omega \rightarrow \omega(u)$ ,  $u \leq t$ ,  $t \in \mathbf{R}$ .

$P :=$  Wiener measure on  $\Omega$ .

$dW_i(t) =$  Itô stochastic differentials.

Several finite delays  $0 < d_1 < d_2 < \dots < d_N \leq r$  in drift term; *no delays in diffusion coefficient.*

$H : (\mathbf{R}^d)^{N+1} \times L^2([-r, 0], \mathbf{R}^d) \rightarrow \mathbf{R}^d$  is a fixed continuous linear map,  $g_i$ ,  $i = 1, 2, \dots, m$ , fixed (deterministic)  $d \times d$ -matrices.

Recall regularity theorem:

**Theorem III.4.**([Mo], Stochastics, 1990)]



(VIII) is regular with respect to the state space  $M_2 = \mathbf{R}^d \times \mathbf{L}^2([-r, 0], \mathbf{R}^d)$ .

There is a measurable version  $X : \mathbf{R}^+ \times M_2 \times \Omega \rightarrow M_2$  of the trajectory field  $\{(x(t), x_t) : t \in \mathbf{R}^+, (x(0), x_0) = (v, \eta) \in M_2\}$  of (VIII) with the following properties:

- (i) For each  $(v, \eta) \in M_2$  and  $t \in \mathbf{R}^+$ ,  $X(t, (v, \eta), \cdot) = (x(t), x_t)$  a.s., is  $\mathcal{F}_t$ -measurable and belongs to  $L^2(\Omega, M_2; P)$ .
- (ii) There exists  $\Omega_0 \in \mathcal{F}$  of full measure such that, for all  $\omega \in \Omega_0$ , the map  $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times M_2 \rightarrow M_2$  is continuous.
- (iii) For each  $t \in \mathbf{R}^+$  and every  $\omega \in \Omega_0$ , the map  $X(t, \cdot, \omega) : M_2 \rightarrow M_2$  is continuous linear; for each  $\omega \in \Omega_0$ , the map  $\mathbf{R}^+ \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$  is measurable and locally bounded in the uniform operator norm on  $L(M_2)$ . The map  $[r, \infty) \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$  is continuous for all  $\omega \in \Omega_0$ .
- (iv) For each  $t \geq r$  and all  $\omega \in \Omega_0$ , the map

$$X(t, \cdot, \omega) : M_2 \rightarrow M_2$$

is compact.

Compactness of semi-flow for  $t \geq r$  will be used below to define hyperbolicity for (VIII) and the associated exponential dichotomies.

## **Lyapunov Exponents. Hyperbolicity**

Version  $X$  of the flow constructed in Theorem III.4 is a multiplicative  $L(M_2)$ -valued linear cocycle over the canonical Brownian shift  $\theta : \mathbf{R} \times \Omega \rightarrow \Omega$  on Wiener space:

$$\theta(t, \omega)(u) := \omega(t + u) - \omega(t), \quad u, t \in \mathbf{R}, \quad \omega \in \Omega.$$

Indeed we have

**Theorem IV.1** ([M], 1990)

*There is an  $\mathcal{F}$ -measurable set  $\hat{\Omega}$  of full  $P$ -measure such that  $\theta(t, \cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$  for all  $t \geq 0$  and*

$$X(t_2, \cdot, \theta(t_1, \omega)) \circ X(t_1, \cdot, \omega) = X(t_1 + t_2, \cdot, \omega)$$

*for all  $\omega \in \hat{\Omega}$  and  $t_1, t_2 \geq 0$ .*

## *The Cocycle Property*

**Proof of Theorem IV.1.** (Sketch)

For simplicity consider the case of a single delay  $d_1$ ; i.e.  $N = 1$ .

*First step.*

Approximate the Brownian motion  $W$  in (VIII) by smooth adapted processes  $\{W^k\}_{k=1}^\infty$ :

$$W^k(t) := k \int_{t-(1/k)}^t W(u) du - k \int_{-(1/k)}^0 W(u) du, \quad t \geq 0, k \geq 1. \quad (1)$$

*Exercise:* Check that each  $W^k$  is a *helix* (i.e. has stationary increments):

$$W^k(t_1 + t_2, \omega) - W^k(t_1, \omega) = W^k(t_2, \theta(t_1, \omega)), \quad t_1, t_2 \in \mathbf{R}, \omega \in \Omega. \quad (2)$$

Let  $X^k : \mathbf{R}^+ \times M_2 \times \Omega \rightarrow M_2$  be the stochastic (semi)flow of the random fde's:

$$\left. \begin{aligned} dx^k(t) &= H(x^k(t - d_1), x^k(t), x_t^k) dt \\ &+ \sum_{i=1}^m g_i x(t) (W_i^k)'(t) dt - \frac{1}{2} \sum_{i=1}^m g_i^2 x^k(t) dt \quad t > 0 \\ (x^k(0), x_0^k) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (VIII - k)$$

If  $X : \mathbf{R}^+ \times M_2 \times \Omega \rightarrow M_2$  is the flow of (VIII) constructed in Theorem III.4, then

$$\lim_{k \rightarrow \infty} \sup_{0 \leq t \leq T} \|X^k(t, \cdot, \omega) - X(t, \cdot, \omega)\|_{L(M_2)} = 0 \quad (3)$$

for every  $0 < T < \infty$  and all  $\omega$  in a Borel set  $\hat{\Omega}$  of full Wiener measure which is invariant under  $\theta(t, \cdot)$  for all  $t \geq 0$  ([Mo], Stochastics,

1990). This convergence may be proved using the following stochastic variational method:

Let  $\phi : \mathbf{R}^+ \times \Omega \rightarrow \mathbf{R}^{d \times d}$  be the  $d \times d$ -matrix-valued solution of the linear Itô sode (without delay):

$$\left. \begin{aligned} d\phi(t) &= \sum_{i=1}^m g_i \phi(t) dW_i(t) & t > 0 \\ \phi(0, \omega) &= I \in \mathbf{R}^{d \times d} & \text{a.a. } \omega \end{aligned} \right\} \quad (4)$$

Denote by  $\phi^k : \mathbf{R}^+ \times \Omega \rightarrow \mathbf{R}^{d \times d}$ ,  $k \geq 1$ , the  $d \times d$ -matrix solution of the random family of linear ode's:

$$\left. \begin{aligned} d\phi^k(t) &= \sum_{i=1}^m g_i \phi^k(t) (W_i^k)'(t) - \frac{1}{2} \sum_{i=1}^m g_i^2 \phi^k(t) dt & t > 0 \\ \phi^k(0, \cdot) &= I \in \mathbf{R}^{d \times d}. \end{aligned} \right\} \quad (4')$$

Let  $\hat{\Omega}$  be the sure event of all  $\omega \in \Omega$  such that

$$\phi(t, \omega) := \lim_{k \rightarrow \infty} \phi^k(t, \omega) \quad (5)$$

exists uniformly for  $t$  in compact subsets of  $\mathbf{R}^+$ . Each  $\phi^k$  is an  $\mathbf{R}^{d \times d}$ -valued *cocycle over  $\theta$* , viz.

$$\phi^k(t_1 + t_2, \omega) = \phi^k(t_2, \theta(t_1, \omega)) \phi^k(t_1, \omega) \quad (6)$$

for all  $t_1, t_2 \in \mathbf{R}^+$  and  $\omega \in \Omega$ . From the definition of  $\hat{\Omega}$  and passing to the limit in (6) as  $k \rightarrow \infty$ , conclude that  $\{\phi(t, \omega) : t > 0, \omega \in \Omega\}$ , is an  $\mathbf{R}^{d \times d}$ -valued *perfect cocycle over  $\theta$* , viz.

$$(i) \quad P(\hat{\Omega}) = 1;$$

- (ii)  $\theta(t, \cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$  for all  $t \geq 0$ ;
- (iii)  $\phi(t_1 + t_2, \omega) = \phi(t_2, \theta(t_1, \omega))\phi(t_1, \omega)$  for all  $t_1, t_2 \in \mathbf{R}^+$  and every  $\omega \in \hat{\Omega}$ ;
- (iv)  $\phi(\cdot, \omega)$  is continuous for every  $\omega \in \hat{\Omega}$ .

Alternatively use the perfection theorem in ([M-S], AIHP, 1996, Theorem 3.1, p. 79-82) for crude cocycles with values in a metrizable second countable topological group. Observe that  $\phi(t, \omega) \in GL(\mathbf{R}^d)$ .

Define  $\hat{H} : \mathbf{R}^+ \times \mathbf{R}^d \times M_2 \times \Omega \rightarrow \mathbf{R}^d$  by

$$\begin{aligned} \hat{H}(t, v_1, v, \eta, \omega) \\ := \phi(t, \omega)^{-1} [H(\phi_t(\cdot, \omega)(-d_1, v_1), \phi(t, \omega)(v), \phi_t(\cdot, \omega) \circ (id_J, \eta))] \end{aligned} \quad (7)$$

for  $\omega \in \Omega, t \geq 0, v, v_1 \in \mathbf{R}^d, \eta \in L^2([-r, 0], \mathbf{R}^d)$ , where

$$\phi_t(\cdot, \omega)(s, v) = \begin{cases} \phi(t+s, \omega)(v) & t+s \geq 0 \\ v & -r \leq t+s < 0 \end{cases}$$

and

$$(id_J, \eta)(s) = (s, \eta(s)), \quad s \in J.$$

Define  $\hat{H}^k : \mathbf{R}^+ \times \mathbf{R}^d \times M_2 \times \Omega \rightarrow \mathbf{R}^d$  by a relation similar to (7) with  $\phi$  replaced by  $\phi^k$ . Then the random fde's

$$\left. \begin{aligned} y'(t) &= \hat{H}(t, y(t-d_1), y(t), y_t, \omega) & t > 0 \\ (y(0), y_0) &= (v, \eta) \in M_2 \end{aligned} \right\} \quad (8)$$

$$\left. \begin{aligned} y^{k'}(t) &= \hat{H}^k(t, y^k(t-d_1), y^k(t), y_t^k, \omega) & t > 0 \\ (y^k(0), y_0^k) &= (v, \eta) \in M_2 \end{aligned} \right\} \quad (9)$$

have unique *non-explosive* solutions

$$y, y^k : [-r, \infty) \times \Omega \rightarrow \mathbf{R}^d$$

([Mo], Stochastics, 1990, pp. 93-98). Itô's formula implies that

$$X(t, v, \eta, \omega) = (\phi(t, \omega)(y(t, \omega)), \phi_t(\cdot, \omega) \circ (id_J, y_t)) \quad (10)$$

The chain rule gives a similar relation for  $X^k$  with  $\phi$  replaced by  $\phi^k$  (*Exercise*; [Mo], Stochastics, 1990, pp. 96-97).

Get the convergence

$$\lim_{k \rightarrow \infty} |\hat{H}^k(t, v_1, v, \eta, \omega) - \hat{H}(t, v_1, v, \eta, \omega)| = 0 \quad (11)$$

uniformly for  $(t, v_1, v, \eta)$  in bounded sets of  $\mathbf{R}^+ \times \mathbf{R}^d \times M_2$ . Use Gronwall's lemma and (11) to deduce (3).

*Second step.*

Fix  $\omega \in \hat{\Omega}$  and use uniqueness of solutions to the approximating equation (VIII-k) and the helix property (2) of  $W^k$  to obtain the cocycle property for  $(X^k, \theta)$ :

$$X^k(t_2, \cdot, \theta(t_1, \omega)) \circ X^k(t_1, \cdot, \omega) = X^k(t_1 + t_2, \cdot, \omega)$$

for all  $\omega \in \hat{\Omega}$  and  $t_1, t_2 \geq 0, k \geq 1$ .

*Third step.*

Pass to limit as  $k \rightarrow \infty$  in the above identity and use the convergence (3) *in operator norm* to get the perfect cocycle property for  $X$ .

□



The a.s. Lyapunov exponents

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, (v(\omega), \eta(\omega)), \omega)\|_{M_2},$$

(for a.a.  $\omega \in \Omega$ ,  $(v, \eta) \in L^2(\Omega, M_2)$ ) of the system (VIII) are characterized by the following “spectral theorem”. Each  $\theta(t, \cdot)$  is ergodic and preserves Wiener measure  $P$ . The proof of Theorem IV.2 below uses compactness of  $X(t, \cdot, \omega) : M_2 \rightarrow M_2$ ,  $t \geq r$ , together with an infinite-dimensional version of Oseledec’s multiplicative ergodic theorem due to Ruelle (1982).

**Theorem IV.2.** ([Mo], Stochastics, 1990)

*Let  $X : \mathbf{R}^+ \times M_2 \times \Omega \rightarrow M_2$  be the flow of (VIII) given in Theorem III.4.*

*Then there exist*

- (a) *an  $\mathcal{F}$ -measurable set  $\Omega^* \subseteq \Omega$  such that  $P(\Omega^*) = 1$  and  $\theta(t, \cdot)(\Omega^*) \subseteq \Omega^*$  for all  $t \geq 0$ ,*
- (b) *a fixed (non-random) sequence of real numbers  $\{\lambda_i\}_{i=1}^\infty$ , and*
- (c) *a random family  $\{E_i(\omega) : i \geq 1, \omega \in \Omega^*\}$  of (closed) finite-codimensional subspaces of  $M_2$ , with the following properties:*
  - (i) *If the **Lyapunov spectrum**  $\{\lambda_i\}_{i=1}^\infty$  is infinite, then  $\lambda_{i+1} < \lambda_i$  for all  $i \geq 1$  and  $\lim_{i \rightarrow \infty} \lambda_i = -\infty$ ; otherwise there is a fixed (non-random) integer  $N \geq 1$  such that  $\lambda_N = -\infty < \lambda_{N-1} < \cdots < \lambda_2 < \lambda_1$ ;*
  - (ii) *each map  $\omega \mapsto E_i(\omega)$ ,  $i \geq 1$ , is  $\mathcal{F}$ -measurable into the Grassmannian of  $M_2$ ;*

(iii)  $E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2$ ,  $i \geq 1$ ,  $\omega \in \Omega^*$ ;

(iv) for each  $i \geq 1$ ,  $\text{codim } E_i(\omega)$  is fixed independently of  $\omega \in \Omega^*$ ;

(v) for each  $\omega \in \Omega^*$  and  $(v, \eta) \in E_i(\omega) \setminus E_{i+1}(\omega)$ ,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} = \lambda_i, \quad i \geq 1;$$

(vi) **Top Exponent:**

$$\lambda_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \cdot, \omega)\|_{L(M_2)} \quad \text{for all } \omega \in \Omega^*;$$

(vii) **Invariance:**

$$X(t, \cdot, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

for all  $\omega \in \Omega^*$ ,  $t \geq 0$ ,  $i \geq 1$ .

## *Spectral Theorem*

Proof of Theorem IV.2 is based on Ruelle's discrete version of Oseledec's multiplicative ergodic theorem in Hilbert space ([Ru], Ann. of Math. 1982, Theorem (1.1), p. 248 and Corollary (2.2), p. 253):

**Theorem IV.3** ([Ru], 1982)

Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $\tau : \Omega \rightarrow \Omega$  a  $P$ -preserving transformation. Assume that  $H$  is a separable Hilbert space and  $T : \Omega \rightarrow L(H)$  a measurable map (w.r.t. the Borel field on the space of all bounded linear operators  $L(H)$ ). Suppose that  $T(\omega)$  is compact for almost all  $\omega \in \Omega$ , and  $E \log^+ \|T(\cdot)\| < \infty$ . Define the family of linear operators  $\{T^n(\omega) : \omega \in \Omega, n \geq 1\}$  by

$$T^n(\omega) := T(\tau^{n-1}(\omega)) \circ \cdots \circ T(\tau(\omega)) \circ T(\omega)$$

for  $\omega \in \Omega, n \geq 1$ .

Then there is a set  $\Omega_0 \in \mathcal{F}$  of full  $P$ -measure such that  $\tau(\Omega_0) \subseteq \Omega_0$ , and for each  $\omega \in \Omega_0$ , the limit

$$\lim_{n \rightarrow \infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm and is a positive compact self-adjoint operator on  $H$ . Furthermore each  $\Lambda(\omega)$  has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \dots$$

where the  $\mu_i$ 's are distinct. If  $\{\mu_i\}_{i=1}^\infty$  is infinite, then  $\mu_i \downarrow -\infty$ ; otherwise they terminate at  $\mu_{N(\omega)} = -\infty$ . If  $\mu_i(\omega) > -\infty$ , then  $e^{\mu_i(\omega)}$  has finite multiplicity  $m_i(\omega)$  and finite-dimensional eigen-space  $F_i(\omega)$ , with  $m_i(\omega) := \dim F_i(\omega)$ . Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[ \bigoplus_{j=1}^{i-1} F_j(\omega) \right]^\perp, \quad E_\infty(\omega) := \ker \Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = H$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|T^n(\omega)x\|_H = \begin{cases} \mu_i(\omega), & \text{if } x \in E_i(\omega) \setminus E_{i+1}(\omega) \\ -\infty & \text{if } x \in \ker \Lambda(\omega). \end{cases}$$

**Proof.**

[Ru], Ann. of Math., 1982, pp. 248-254. □

The following “perfect” version of Kingman’s subadditive ergodic theorem is also used to construct the shift invariant set  $\Omega^*$  appearing in Theorem IV.2 above.

**Theorem IV.4**([M, 1990])(“Perfect” Subadditive Ergodic Theorem)

Let  $f : \mathbf{R}^+ \times \Omega \rightarrow \mathbf{R} \cup \{-\infty\}$  be a measurable process on the complete probability space  $(\Omega, \mathcal{F}, P)$  such that

$$(i) \ E \sup_{0 \leq u \leq 1} f^+(u, \cdot) < \infty, \ E \sup_{0 \leq u \leq 1} f^+(1 - u, \theta(u, \cdot)) < \infty;$$

$$(ii) \ f(t_1 + t_2, \omega) \leq f(t_1, \omega) + f(t_2, \theta(t_1, \omega)) \text{ for all } t_1, t_2 \geq 0 \text{ and every } \omega \in \Omega.$$

Then there exist a set  $\hat{\Omega} \in \mathcal{F}$  and a measurable  $\tilde{f} : \Omega \rightarrow \mathbf{R} \cup \{-\infty\}$  with the properties:

$$(a) \ P(\hat{\Omega}) = 1, \ \theta(t, \cdot)(\hat{\Omega}) \subseteq \hat{\Omega} \text{ for all } t \geq 0;$$

$$(b) \ \tilde{f}(\omega) = \tilde{f}(\theta(t, \omega)) \text{ for all } \omega \in \hat{\Omega} \text{ and all } t \geq 0;$$

$$(c) \ \tilde{f}^+ \in \mathbf{L}^1(\Omega, \mathbf{R}; P);$$

$$(d) \ \lim_{t \rightarrow \infty} (1/t)f(t, \omega) = \tilde{f}(\omega) \text{ for every } \omega \in \hat{\Omega}.$$

If  $\theta$  is ergodic, then there exist  $f^* \in \mathbf{R} \cup \{-\infty\}$  and  $\tilde{\tilde{\Omega}} \in \mathcal{F}$  such that

$$(a)' \ P(\tilde{\tilde{\Omega}}) = 1, \ \theta(t, \cdot)(\tilde{\tilde{\Omega}}) \subseteq \tilde{\tilde{\Omega}}, \ t \geq 0;$$

$$(b)' \ \tilde{f}(\omega) = f^* = \lim_{t \rightarrow \infty} (1/t)f(t, \omega) \text{ for every } \omega \in \tilde{\tilde{\Omega}}.$$

**Proof.**

[Mo], Stochastics, 1990, Lemma 7, pp. 115–117. □

Proof of Theorem IV.2 is an application of Theorem IV.3. Requires Theorem IV.4 and the following sequence of lemmas.

**Lemma 1**

For each integer  $k \geq 1$  and any  $0 < a < \infty$ ,

$$E \sup_{0 \leq t \leq a} \|\phi(t, \omega)^{-1}\|^{2k} < \infty;$$

$$E \sup_{0 \leq t_1, t_2 \leq a} \|\phi(t_2, \theta(t_1, \cdot))\|^{2k} < \infty.$$

**Proof.**

Follows by standard sode estimates, the cocycle property for  $\phi$  and Hölder's inequality. ([M], pp. 106-108).  $\square$

The next lemma is a crucial estimate needed to apply Ruelle-Oseledec theorem (Theorem IV.3).

**Lemma 2**

$$E \sup_{0 \leq t_1, t_2 \leq r} \log^+ \|X(t_2, \cdot, \theta(t_1, \cdot))\|_{L(M_2)} < \infty.$$

**Proof.**

If  $y(t, (v, \eta), \omega)$  is the solution of the fde (8), then using Gronwall's inequality, taking  $E \sup_{0 \leq t_1, t_2 \leq r} \log^+ \sup_{\|(v, \eta)\| \leq 1}$  and applying Lemma 1, gives

$$E \sup_{0 \leq t_1, t_2 \leq r} \log^+ \sup_{\|(v, \eta)\| \leq 1} \|(y(t_2, (v, \eta), \theta(t_1, \cdot)), y_{t_2}(\cdot, (v, \eta), \theta(t_1, \cdot)))\|_{M_2} < \infty.$$

Conclusion of lemma now follows by replacing  $\omega'$  with  $\theta(t_1, \omega)$  in the formula

$$\begin{aligned} X(t_2, (v, \eta), \omega') \\ = (\phi(t_2, \omega')(y(t_2, (v, \eta), \omega')), \phi_{t_2}(\cdot, \omega') \circ (id_J, y_{t_2}(\cdot, (v, \eta), \omega'))) \end{aligned}$$

and Lemma 1. □

The existence of the Lyapunov exponents is obtained by interpolating the discrete limit

$$\frac{1}{r} \lim_{k \rightarrow \infty} \frac{1}{k} \log \|X(kr, (v(\omega), \eta(\omega)), \omega)\|_{M_2}, \quad (12)$$

a.a.  $\omega \in \Omega$ ,  $(v, \eta) \in L^2(\Omega, M_2)$ , between delay periods of length  $r$ . This requires the next two lemmas.



### Lemma 3

Let  $h : \Omega \rightarrow \mathbf{R}^+$  be  $\mathcal{F}$ -measurable and suppose  $E \sup_{0 \leq u \leq r} h(\theta(u, \cdot))$  is finite.

Then

$$\Omega_1 := \left( \lim_{t \rightarrow \infty} \frac{1}{t} h(\theta(t, \cdot)) = 0 \right)$$

is a sure event and  $\theta(t, \cdot)(\Omega_1) \subseteq \Omega_1$  for all  $t \geq 0$ .

### Proof.

Use interpolation between delay periods and the discrete ergodic theorem applied to the  $L^1$  function

$$\hat{h} := \sup_{0 \leq u \leq r} h(\theta(u, \cdot)).$$

([Mo], Stochastics, 1990, Lemma 5, pp. 111-113.) □

### Lemma 4

Suppose there is a sure event  $\Omega_2$  such that  $\theta(t, \cdot)(\Omega_2) \subseteq \Omega_2$  for all  $t \geq 0$ , and the limit (12) exists (or equal to  $-\infty$ ) for all  $\omega \in \Omega_2$  and all  $(v, \eta) \in M_2$ . Then there is a sure event  $\Omega_3$  such that  $\theta(t, \cdot)(\Omega_3) \subseteq \Omega_3$  and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} = \frac{1}{r} \lim_{k \rightarrow \infty} \frac{1}{k} \log \|X(kr, (v, \eta), \omega)\|_{M_2}, \quad (13)$$

for all  $\omega \in \Omega_3$  and all  $(v, \eta) \in M_2$ .

**Proof:**

Take  $\Omega_3 := \hat{\Omega} \cap \Omega_1 \cap \Omega_2$ . Use cocycle property for  $X$ , Lemma 2 and Lemma 3 to interpolate. ([Mo], Stochastics 1990, Lemma 6, pp. 113-114.) □

## Proof of Theorem IV.2. (Sketch)

Apply Ruelle-Oseledec Theorem (Theorem IV.3) with

$T(\omega) := X(r, \omega) \in L(M_2)$ , compact linear for  $\omega \in \hat{\Omega}$ ;

$\tau : \Omega \rightarrow \Omega$ ;  $\tau := \theta(r, \cdot)$ .

Then cocycle property for  $X$  implies

$$\begin{aligned} X(kr, \omega, \cdot) &= T(\tau^{k-1}(\omega)) \circ T(\tau^{k-2}(\omega)) \circ \dots \circ T(\tau(\omega)) \circ T(\omega) \\ &:= T^k(\omega) \end{aligned}$$

for all  $\omega \in \hat{\Omega}$ .

Lemma 2 implies

$$E \log^+ \|T(\cdot)\|_{L(M_2)} < \infty.$$

Theorem IV.3 gives a random family of compact self-adjoint positive linear operators  $\{\Lambda(\omega) : \omega \in \Omega_4\}$  such that

$$\lim_{n \rightarrow \infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm and is a positive compact operator on  $M_2$  for  $\omega \in \Omega_4$ , a (continuous) shift-invariant set of full measure.

Furthermore each  $\Lambda(\omega)$  has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \dots$$

where the  $\mu'_i$ s are distinct, with no accumulation points except possibly  $-\infty$ . If  $\{\mu_i\}_{i=1}^\infty$  is infinite, then  $\mu_i \downarrow -\infty$ ; otherwise they terminate at

$\mu_{N(\omega)} = -\infty$ . If  $\mu_i(\omega) > -\infty$ , then  $e^{\mu_i(\omega)}$  has finite multiplicity  $m_i(\omega)$  and finite-dimensional eigen-space  $F_i(\omega)$ , with  $m_i(\omega) := \dim F_i(\omega)$ . Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[ \bigoplus_{j=1}^{i-1} F_j(\omega) \right]^\perp, \quad E_\infty(\omega) := \ker \Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2.$$

Note that  $\text{codim } E_i(\omega) = \sum_{j=1}^{i-1} m_j(\omega) < \infty$ . Also

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \|X(kr, (v, \eta), \omega)\|_{M_2} = \begin{cases} \mu_i(\omega), & \text{if } (v, \eta) \in E_i(\omega) \setminus E_{i+1}(\omega) \\ -\infty & \text{if } (v, \eta) \in \ker \Lambda(\omega). \end{cases}$$

The functions

$$\omega \mapsto \mu_i(\omega), \quad \omega \mapsto m_i(\omega), \quad \omega \mapsto N(\omega)$$

are invariant under the ergodic shift  $\theta(r, \cdot)$ . Hence they take the fixed values  $\mu_i, m_i, N$  almost surely, respectively.

Lemma 4 gives a continuous-shift-invariant sure event  $\Omega^* \subseteq \Omega_4$  such that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} &= \frac{1}{r} \lim_{k \rightarrow \infty} \frac{1}{k} \log \|X(kr, (v, \eta), \omega)\|_{M_2} \\ &= \frac{\mu_i}{r} =: \lambda_i, \end{aligned}$$

for  $(v, \eta) \in E_i(\omega) \setminus E_{i+1}(\omega)$ ,  $\omega \in \Omega^*$ ,  $i \geq 1$ .

$\{\lambda_i := \frac{\mu_i}{r} : i \geq 1\}$  is the *Lyapunov spectrum* of (VIII).

Since Lyapunov spectrum is discrete with no finite accumulation points, then  $\{\lambda_i : \lambda_i > \lambda\}$  is finite for all  $\lambda \in \mathbf{R}$ .

To prove invariance of the Oseledec space  $E_i(\omega)$  under the cocycle  $(X, \theta)$  use the random field

$$\lambda((v, \eta), \omega) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} \quad (v, \eta) \in M_2, \quad \omega \in \Omega^*$$

and the relations

$$E_i(\omega) := \{(v, \eta) \in M_2 : \lambda((v, \eta), \omega) \leq \lambda_i\},$$

$$\lambda(X(t, (v, \eta), \omega), \theta(t, \omega)) = \lambda((v, \eta), \omega), \quad \omega \in \Omega^*, \quad t \geq 0$$

([Mo], Stochastics 1990, p. 122). □

The non-random nature of the Lyapunov exponents  $\{\lambda_i\}_{i=1}^\infty$  of (VIII) is a consequence of the fact the  $\theta$  is ergodic. (VIII) is said to be *hyperbolic* if  $\lambda_i \neq 0$  for all  $i \geq 1$ . When (VIII) is hyperbolic the flow satisfies a *stochastic saddle-point property* (or exponential dichotomy) (cf. the deterministic case with  $E = C([-r, 0], \mathbf{R}^d)$ ,  $g_i \equiv 0$ ,  $i = 1, \dots, m$ , in Hale [H], Theorem 4.1, p. 181).

**Theorem IV.5** (*Random Saddles*)([Mo], Stochastics, 1990)

Suppose the sfde (VIII) is hyperbolic. Then there exist

(a) a set  $\tilde{\Omega}^* \in \mathcal{F}$  such that  $P(\tilde{\Omega}^*) = 1$ , and  $\theta(t, \cdot)(\tilde{\Omega}^*) = \tilde{\Omega}^*$  for all  $t \in \mathbf{R}$ ,

and

(b) a measurable splitting

$$M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \quad \omega \in \tilde{\Omega}^*,$$

with the following properties:

- (i)  $\mathcal{U}(\omega)$ ,  $\mathcal{S}(\omega)$ ,  $\omega \in \tilde{\Omega}^*$ , are closed linear subspaces of  $M_2$ ,  $\dim \mathcal{U}(\omega)$  is finite and fixed independently of  $\omega \in \tilde{\Omega}^*$ .
- (ii) The maps  $\omega \mapsto \mathcal{U}(\omega)$ ,  $\omega \mapsto \mathcal{S}(\omega)$  are  $\mathcal{F}$ -measurable into the Grassmannian of  $M_2$ .
- (iii) For each  $\omega \in \tilde{\Omega}^*$  and  $(v, \eta) \in \mathcal{U}(\omega)$  there exists  $\tau_1 = \tau_1(v, \eta, \omega) > 0$  and a positive  $\delta_1$ , independent of  $(v, \eta, \omega)$  such that

$$\|X(t, (v, \eta), \omega)\|_{M_2} \geq \|(v, \eta)\|_{M_2} e^{\delta_1 t}, \quad t \geq \tau_1.$$

(iv) For each  $\omega \in \tilde{\Omega}^*$  and  $(v, \eta) \in \mathcal{S}(\omega)$  there exists  $\tau_2 = \tau_2(v, \eta, \omega) > 0$  and a positive  $\delta_2$ , independent of  $(v, \eta, \omega)$  such that

$$\|X(t, (v, \eta), \omega)\|_{M_2} \leq \|(v, \eta)\|_{M_2} e^{-\delta_2 t}, \quad t \geq \tau_2.$$

(v) For each  $t \geq 0$  and  $\omega \in \tilde{\Omega}^*$ ,

$$X(t, \omega, \cdot)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)),$$

$$X(t, \omega, \cdot)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)).$$

In particular, the restriction

$$X(t, \omega, \cdot) |_{\mathcal{U}(\omega)} : \mathcal{U}(\omega) \rightarrow \mathcal{U}(\theta(t, \omega))$$

is a linear homeomorphism onto.

**Proof.**

[Mo], Stochastics, 1990, Corollary 2, pp. 127-130. □

## *The Stable Manifold Theorem*



## 5. Regular Linear Systems. Helix Noise

$$\left. \begin{aligned}
 dx(t) = & \left\{ \int_{[-r,0]} \nu(t)(ds) x(t+s) \right\} dt + dN(t) \int_{-r}^0 K(t)(s) x(t+s) ds \\
 & + dL(t) x(t-), \quad t > 0 \\
 (x(0), x_0) = & (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)
 \end{aligned} \right\} \quad (IX)$$

Linear systems driven by helix semimartingale noise, and memory driven by a measure-valued process  $\nu$  on a complete filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}}, P)$ .

### Hypotheses (C)

- (i) The processes  $\nu, K$  are stationary ergodic in the sense that there is a measurable ergodic  $P$ -preserving flow  $\theta : \mathbf{R} \times \Omega \rightarrow \Omega$  such that for each  $t \in \mathbf{R}$ ,  $\mathcal{F}_t = \theta(t, \cdot)^{-1}(\mathcal{F}_0)$  and

$$\nu(t, \omega) = \nu(0, \theta(t, \omega)), \quad t \in \mathbf{R}, \omega \in \Omega$$

$$K(t, \omega) = K(0, \theta(t, \omega)), \quad t \in \mathbf{R}, \omega \in \Omega.$$

- (ii)  $L = M + V$ ,  $M$  continuous local martingale,  $V$  B.V. process. The processes  $N, L, M$  have jointly stationary ergodic increments:

$$N(t+h, \omega) - N(t, \omega) = N(h, \theta(t, \omega)),$$

$$L(t+h, \omega) - L(t, \omega) = L(h, \theta(t, \omega)),$$

$$M(t+h, \omega) - M(t, \omega) = M(h, \theta(t, \omega)),$$

for  $t \in \mathbf{R}$ ,  $\omega \in \Omega$ .

Semimartingales satisfying Hypothesis (C)(ii) were studied by de Sam Lazaro and P.A. Meyer ([S-M], 1971, 1976), Çinlar, Jacod, Protter and Sharpe [CJPS], Protter [P], 1986.

Equation (IX) is regular w.r.t.  $M_2$  with a measurable flow  $X : \mathbf{R}^+ \times M_2 \times \Omega \rightarrow M_2$ . This flow satisfies Theorems III.4 and the cocycle property. This is achieved via a construction in ([M-S], AIHP, 1996) based on the following consequence of Hypothesis (C)(ii):

**Theorem IV.6** ([Mo], Survey paper, 1992, [M-S], AIHP, 1996)

Suppose  $M$  satisfies Hypothesis (C)(ii). Then there is an  $(\mathcal{F}_t)_{t \geq 0}$ -adapted version  $\phi : \mathbf{R}^+ \times \Omega \rightarrow \mathbf{R}^{d \times d}$  of the solution to the matrix equation

$$\left. \begin{aligned} d\phi(t) &= dM(t)\phi(t) \quad t > 0 \\ \phi(0) &= I \in \mathbf{R}^{d \times d} \end{aligned} \right\} \quad (X)$$

and a set  $\Omega_1 \in \mathcal{F}$  such that

- (i)  $P(\Omega_1) = 1$ ;
- (ii)  $\theta(t, \cdot)(\Omega_1) \subseteq \Omega_1$  for all  $t \geq 0$ ;
- (iii)  $\phi(t_1 + t_2, \omega) = \phi(t_2, \theta(t_1, \omega))\phi(t_1, \omega)$  for all  $t_1, t_2 \in \mathbf{R}^+$  and every  $\omega \in \Omega_1$ ;
- (iv)  $\phi(\cdot, \omega)$  is continuous for every  $\omega \in \Omega_1$ .

A proof of Theorem IV.6 is given in ([Mo], Survey, 1992; [M-S], AIHP, 1996): either by a double-approximation argument or via perfection techniques.

The existence of a discrete non-random Lyapunov spectrum  $\{\lambda_i\}_{i=1}^\infty$  for the sfde (IX) is proved via Ruelle-Oseledec multiplicative ergodic theorem which requires the integrability property (Lemma 2):

$$E \sup_{0 \leq t_1, t_2 \leq r} \log^+ \|X(t_1, \theta(t_2, \cdot), \cdot)\|_{L(M_2)} < \infty.$$

The above integrability property is established under the following set of hypotheses on  $\nu$ ,  $K$ ,  $N$ ,  $L$ :

## Hypotheses (I)

(i)

$$\begin{aligned} & \sup_{-r \leq s \leq 2r} \left| \frac{d\bar{\nu}(\cdot)(s)}{ds} \right|^2, \quad \sup_{0 \leq t \leq 2r, -r \leq s \leq 0} \|K(t, \cdot)(s)\|^3, \\ & \sup_{0 \leq t \leq 2r, -r \leq s \leq 0} \left\| \frac{\partial}{\partial t} K(t, \cdot)(s) \right\|^3, \quad \sup_{0 \leq t \leq 2r, -r \leq s \leq 0} \left\| \frac{\partial}{\partial s} K(t, \cdot)(s) \right\|^3, \\ & \{|V|(2r, \cdot)\}^4, \end{aligned}$$

are all integrable, where

$$\bar{\nu}(\omega)(A) := \int_0^\infty |\nu(t, \omega)| \{(A-t) \cap [-r, 0]\} dt, \quad A \in \text{Borel}[-r, \infty)$$

has a locally (essentially) bounded density  $\frac{d\bar{\nu}(\cdot)(s)}{ds}$ ; and  $|V| =$  total variation of  $V$  w.r.t. the Euclidean norm  $\|\cdot\|$  on  $\mathbf{R}^{d \times d}$ .

(ii) Let  $N = N^0 + V^0$  where the local  $(\mathcal{F}_t)_{t \geq 0}$ -martingale  $N^0 = (N_{ij}^0)_{i,j=1}^d$  and the bounded variation process

$V^0 = (V_{ij}^0)_{i,j=1}^d$  are such that

$$\{[N_{ij}^0](2r, \cdot)\}^2, \quad \{|V_{ij}^0|(2r, \cdot)\}^4, \quad i, j = 1, 2, \dots, d$$

are integrable.

$|V_{ij}^0|(2r, \cdot) =$  total variation of  $V_{ij}^0$  over  $[0, 2r]$ .

(iii)  $[M_{ij}](1) \in L^\infty(\Omega, \mathbf{R})$ ,  $i, j = 1, 2, \dots, d$ .

The integrability property of the cocycle  $(X, \theta)$  is a consequence of

$$E \log^+ \sup_{0 \leq t_1, t_2 \leq r, \|(v, \eta)\| \leq 1} |x(t_1, (v, \eta), \theta(t_2, \cdot))| < \infty.$$

Proof of latter property uses lengthy argument based on establishing the existence of suitable higher order moments for the coefficients of an associated random integral equation. (See Lemmas (5.1)-(5.5) in [M-S],I, AIHP, 1996.)

Since  $\theta$  is ergodic, the multiplicative ergodic theorem ( Theorem IV.3, Ruelle) now gives a fixed discrete set of Lyapunov exponents

**Theorem IV.7** ([Mo], Survey, 1992; [M-S], AIHP, 1996)

*Under Hypotheses (C) & (I), the statements of Theorems IV.2 and IV.5 hold true for the linear sfde (IX).*

Note that the Lyapunov spectrum of (IX) does not change if one uses the state space  $D([-r, 0], \mathbf{R}^d)$  with the supremum norm  $\|\cdot\|_\infty$  ([M-S], AIHP 1996).

**V. STABILITY**  
**EXAMPLES AND CASE STUDIES**

**Geilo, Norway**

**Friday, August 2, 1996**

**14:00-14:50**

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## V. STABILITY. EXAMPLES AND CASE STUDIES

### 1. Plan.

- I) Estimates on the “maximal exponential growth rate” for the singular noisy feedback loop. Use of Lyapunov functionals.
- II) Examples and case studies of linear sfde’s: Existence of the stochastic semiflow and its Lyapunov spectrum.
- III) Study almost sure asymptotic stability via upper bounds on the top Lyapunov exponent  $\lambda_1$ .
- IV) Lyapunov spectrum for sdde’s with Poisson noise.

*Lyapunov exponents for linear sode's (without memory)*: studied by many authors: e.g. Arnold, Kliemann and Oeljeklaus, 1989, Arnold, Oeljeklaus and Pardoux, 1986, Baxendale, 1985, Pardoux and Wihstutz [PW1], 1988, Pinsky and Wihstutz [PW2], 1988, and the references therein.

*Asymptotic stability of sfde's*: treated in Kushner [K], JDE, 1968, Mizel and Trutzer [MT], 1984, Mohammed [M1]-[M4], 1984, 1986, 1990, 1992, Mohammed and Scheutzow [MS], 1996, Scheutzow [S], 1988, Kolmanovskii and Nosov [KN], 1986. Mao ([Ma], 1994, Chapter 5) gives several results concerning top exponential growth rate for sdde's driven by  $C$ -valued semimartingales. Assumes that second-order characteristics of the driving semimartingales are *time-dependent and decay to zero exponentially fast in time, uniformly in the space variable*.



## 2. Noisy Feedback Loop Revisited Once More!

Noisy feedback loop is modelled by the one-dimensional linear sdde

$$\left. \begin{aligned} dx(t) &= \sigma x(t-r) dW(t), & t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R} \times L^2([-r, 0], \mathbf{R}), \end{aligned} \right\} \quad (I)$$

driven by a Wiener process  $W$  with a *positive delay*  $r$ .

(I) is singular with respect to  $M_2$  (Theorem III.3).

Consider the more general one-dimensional linear sfde:

$$\left. \begin{aligned} dx(t) &= \int_{-r}^0 x(t+s) d\nu(s) dW(t), & t > 0 \\ (x(0), x_0) &\in M_2 := \mathbf{R} \times L^2([-r, 0], \mathbf{R}) \end{aligned} \right\} \quad (II')$$

where  $W$  is a Wiener process and  $\nu$  is a fixed finite real-valued Borel measure on  $[-r, 0]$ .

(II') is regular if  $\nu$  has a  $C^1$  (or even  $L^2_1$ ) density with respect to Lebesgue measure on  $[-r, 0]$  ([M-S], I, 1996). If  $\nu$  satisfies Theorem III.3, then (II') is singular.

In the singular case, there is no stochastic flow (Theorem III.3) and we do not know whether a (discrete) set of Lyapunov exponents

$$\lambda((v, \eta), \cdot) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \|(x(t, (v, \eta)), x_t(\cdot, (v, \eta)))\|_{M_2}, \quad (v, \eta) \in M_2$$

exists. Existence of Lyapunov exponents for singular equations is hard. But can still define the *maximal exponential growth rate*

$$\bar{\lambda}_1 := \sup_{(v, \eta) \in M_2} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|(x(t, (v, \eta)), x_t(\cdot, (v, \eta)))\|_{M_2}$$

for the trajectory random field  $\{(x(t, (v, \eta)), x_t(\cdot, (v, \eta))) : t \geq 0, (v, \eta) \in M_2\}$ .  $\bar{\lambda}_1$  may depend on  $\omega \in \Omega$ . But  $\bar{\lambda}_1 = \lambda_1$  in the regular case.

Inspite of the *extremely erratic dependence on the initial paths* of solutions of (I), it is shown in Theorem V.1 that for small noise variance, *uniform almost sure global asymptotic stability* still persists. For small  $\sigma$ ,  $\bar{\lambda}_1 \leq -\sigma^2/2 + o(\sigma^2)$  uniformly in the initial path (Theorem V.1, and Remark (iii)). For large  $|\sigma|$  and  $\nu = \delta_{-r}$ ,

$$\frac{1}{2r} \log |\sigma| + o(\log |\sigma|) \leq \bar{\lambda}_1 \leq \frac{1}{r} \log |\sigma|$$

([M-S], II, 1996, Remark (ii) after proof of Theorem 2.3 ). This result is in sharp contrast with the non-delay case ( $r = 0$ ), where  $\lambda_1 = -\sigma^2/2$  for all values of  $\sigma$ . Proofs of Theorems V.1, V.2 involve very delicate constructions of new types of Lyapunov functionals on the underlying state space.

**Theorem V.1.** ([M-S], II, 1996).

Let  $\nu$  be a probability measure on  $[-r, 0]$ ,  $r > 0$ , and consider the sfde

$$\left. \begin{aligned} dx(t) &= \sigma \left( \int_{[-r, 0]} x(t+s) d\nu(s) \right) dW(t), \quad t \geq 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 \end{aligned} \right\} \quad (II')$$

with  $\sigma \in \mathbf{R}$ ,  $(v, \eta) \in M_2$ ,  $W$  standard Brownian motion, and  $x(\cdot, (v, \eta))$  the solution of (II') through  $(v, \eta) \in M_2$ . Then there exists  $\sigma_0 > 0$  and a continuous strictly negative nonrandom function  $\phi : (-\sigma_0, \sigma_0) \rightarrow \mathbf{R}^-$  (independent of  $(v, \eta) \in M_2$  and  $\nu$ ) such that

$$P \left( \limsup_{t \rightarrow \infty} \frac{1}{t} \log \| (x(t, (v, \eta)), x_t(\cdot, (v, \eta))) \|_{M_2} \leq \phi(\sigma) \right) = 1.$$

for all  $(v, \eta) \in M_2$  and all  $-\sigma_0 < \sigma < \sigma_0$ .

**Remark:**

Theorem also holds for state space  $C$  with  $\|\cdot\|_\infty$ .

**Proof of Theorem V.1.** (Sketch)

Sufficient to consider (II') on  $C \equiv C([-r, 0], \mathbf{R})$ , because  $C$  is continuously embedded in  $M_2$ . W.l.o.g., assume that  $\sigma > 0$ .

- Use Lyapunov functional  $V : C \rightarrow \mathbf{R}^+$

$$V(\eta) := (R(\eta) \vee |\eta(0)|)^\alpha + \beta R(\eta)^\alpha, \quad \eta \in C.$$

where  $R(\eta) := \bar{\eta} - \underline{\eta}$ , the diameter of the range of  $\eta$ ,  $\bar{\eta} := \sup_{-r \leq s \leq 0} \eta(s)$  and  $\underline{\eta} := \inf_{-r \leq s \leq 0} \eta(s)$ .

- Fix  $0 < \alpha < 1$  and *arrange* for  $\beta = \beta(\sigma)$  for sufficiently small  $\sigma$  such that

$$E(V({}^n x_r)) \leq \delta(\sigma)V(\eta), \quad \eta \in C, \quad (1)$$

and  $\delta(\sigma) \in (0, 1)$  is a continuous function of  $\sigma$  defined near 0. There is a positive  $K = K(\alpha)$  (independent of  $\eta, \nu$ ) such that  $\delta(\sigma) \sim (1 - K\sigma^2)$ . Set

$$\phi(\sigma) := \frac{1}{\alpha} \log \delta(\sigma).$$

Estimate (1) is hard ([M-S], II, 1996, pp. 12-18).

- $\{{}^n x_{nr}\}_{n=1}^\infty$  is a Markov process in  $C$ . So (1) implies that  $\delta(\sigma)^{-n}V({}^n x_{nr})$ ,  $n \geq 1$ , is a non-negative  $(\mathcal{F}_{nr})$  supermartingale.
- There exists  $Z : \Omega \rightarrow [0, \infty)$  such that

$$\lim_{n \rightarrow \infty} \frac{V({}^n x_{nr})}{\delta(\sigma)^n} = Z \quad \text{a.s.} \quad (2)$$

- Form of  $V$  and (2) imply

$$\begin{aligned} \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \log |x(t)| &\leq \overline{\lim}_{n \rightarrow \infty} \frac{1}{nr} \log[|x(nr)| + R(x_{nr})] \\ &= \frac{1}{\alpha} \overline{\lim}_{n \rightarrow \infty} \frac{1}{nr} \log V(x_{nr}) \leq \frac{1}{\alpha} \log \delta(\sigma) = \phi(\sigma) < 0. \end{aligned}$$

- $\delta(\sigma)$ ,  $\phi(\sigma)$  independent of  $\eta$ ,  $\nu$ . “Domain” of  $\phi$  also independent of  $\eta$ ,  $\nu$ . □

**Remarks.**

- (i) Choice of  $\sigma_0$  in Theorem V.1 depends on  $r$ . In (I) the scaling  $t \mapsto t/r$  has the effect of replacing  $r$  by 1 and  $\sigma$  by  $\sigma\sqrt{r}$ . If  $\bar{\lambda}_1(r, \sigma)$  is the maximal exponential growth rate of (I), then  $\bar{\lambda}_1(r, \sigma) = \frac{1}{r}\bar{\lambda}_1(1, \sigma\sqrt{r})$  (*Exercise*). Hence  $\sigma_0$  decreases (like  $\frac{1}{\sqrt{r}}$ ) as  $r$  increases. Thus (for a fixed  $\sigma$ ), a *small delay*  $r$  tends to *stabilize* equation (I). A *large delay* in (I) has a *destabilizing* effect (Theorem V.2 below).
- (ii) Using a Lyapunov function(al) argument, Theorem V.2 below shows that for sufficiently large  $\sigma$ , the singular delay equation (I) is unstable. Result is in sharp contrast with the non-delay case  $r = 0$ , where

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log |x(t)| = -\sigma^2/2 < 0$$

for *all*  $\sigma \in \mathbf{R}$  (even when  $\sigma$  is large).

- (iii) The growth rate function  $\phi$  in Theorem V.1 satisfies

$$\phi(\sigma) = -\sigma^2/2 + o(\sigma^2)$$

as  $\sigma \rightarrow 0^+$ . Agrees with non-delay case  $r = 0$ . Above relation follows by modifying proof of Theorem V.1.

**Theorem V.2.**

Consider the equation

$$\left. \begin{aligned} dx(t) &= \sigma x(t-r) dW(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R} \times L^2([-r, 0], \mathbf{R}), \end{aligned} \right\} \quad (I)$$

driven by a standard Wiener process  $W$  with a *positive delay*  $r$  and  $\sigma \in \mathbf{R}$ . Then there exists a continuous function  $\psi : (0, \infty) \rightarrow \mathbf{R}$  which is increasing to infinity such that

$$P\left(\liminf_{t \rightarrow \infty} \frac{1}{t} \log \|(x(t, (v, \eta)), x_t(\cdot, (v, \eta)))\|_{M_2} \geq \psi(|\sigma|)\right) = 1,$$

for all  $(v, \eta) \in M_2 \setminus \{0\}$  and all  $\sigma \neq 0$ . The function  $\psi$  is independent of the choice of  $(v, \eta) \in M_2 \setminus \{0\}$ .

**Remarks.**

- (i)  $\|\cdot\|_{M_2}$  can be replaced by the sup-norm on  $C$ .
- (ii) Proof shows  $\psi(\sigma) \sim \frac{1}{2} \log \sigma$  for large  $\sigma$ .

## Proof of Theorem V.2.

Use the continuous Lyapunov functional

$$V : M_2 \setminus \{0\} \rightarrow [0, \infty)$$

$$V((v, \eta)) := \left( v^2 + |\sigma| \int_{-1}^0 \eta^2(s) ds \right)^{-1/4}$$

[M-S], Part II, 1996, pp. 20-24. □

### 3. Regular one-dimensional linear sfde's

To outline a general scheme for obtaining estimates on the top Lyapunov exponent for a class of one-dimensional regular linear sfde's. Then apply scheme to specific examples within the above class.

Scheme applies to multidimensional linear equations with multiple delays.

*Note:* Approach in ([Ku], JDE, 1968) uses Lyapunov functionals and yields strictly weaker estimates in all cases.

Consider the class of one-dimensional linear sfde's

$$\left. \begin{aligned} dx(t) = & \left\{ \nu_1 x(t) + \mu_1 x(t-r) + \int_{-r}^0 x(t+s) \sigma_1(s) ds \right\} dt \\ & + \left\{ \nu_2 x(t) + \int_{-r}^0 x(t+s) \sigma_2(s) ds \right\} dM(t), \end{aligned} \right\} \quad (XVII)$$

where  $r > 0, \sigma_1, \sigma_2 \in C^1([-r, 0], \mathbf{R})$ , and  $M$  is a continuous helix local martingale on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  with (stationary) ergodic increments. Ergodic theorem gives the a.s. deterministic limit  $\beta := \lim_{t \rightarrow \infty} \frac{\langle M \rangle(t)}{t}$ . Assume that  $\beta < \infty$  and  $\langle M \rangle(1) \in L^\infty(\Omega, \mathbf{R})$ .

Hence (XVII) is regular with respect to  $M_2$  and has a sample-continuous stochastic semiflow  $X : \mathbf{R}^+ \times M_2 \times \Omega \rightarrow M_2$  (Theorem III.5). The stochastic semiflow  $X$  has a fixed (non-random) Lyapunov spectrum (Theorem IV.7). Let  $\lambda_1$  be its top exponent. We wish to develop

an upper bound for  $\lambda_1$ . By the spectral theorem (Theorem IV.7, cf. Theorem IV.2), there is a shift-invariant set  $\Omega^* \in \mathcal{F}$  of full  $P$ -measure and a measurable random field  $\lambda : M_2 \times \Omega \rightarrow \mathbf{R} \cup \{-\infty\}$ ,

$$\lambda((v, \eta), \omega) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2}, \quad (v, \eta) \in M_2, \omega \in \Omega^*, \quad (1)$$

giving the Lyapunov spectrum of (XVII).

Introduce family of equivalent norms

$$\|(v, \eta)\|_\alpha := \left\{ \alpha v^2 + \int_{-r}^0 \eta(s)^2 ds \right\}^{1/2}, \quad (v, \eta) \in M_2, \quad \alpha > 0, \quad (2)$$

on  $M_2$ . Then

$$\lambda((v, \eta), \omega) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_\alpha, \quad (v, \eta) \in M_2, \omega \in \Omega^* \quad (3)$$

for all  $\alpha > 0$ ; i.e. the Lyapunov spectrum of (XVII) with respect to  $\|\cdot\|_\alpha$  is independent of  $\alpha > 0$ .

Let  $x$  be the solution of (XVII) starting at  $(v, \eta) \in M_2$ . Define

$$\rho_\alpha(t)^2 := \|X(t)\|_\alpha^2 = \alpha x(t)^2 + \int_{t-r}^t x(u)^2 du, \quad t > 0, \quad \alpha > 0. \quad (4)$$

For each fixed  $(v, \eta) \in M_2$ , define the set  $\Omega_0 \in \mathcal{F}$  by  $\Omega_0 := \{\omega \in \Omega : \rho_\alpha(t, \omega) \neq 0 \text{ for all } t > 0\}$ . If  $P(\Omega_0) = 0$ , then by uniqueness there is a random time  $\tau_0$  such that a.s.  $X(t, (v, \eta), \cdot) = 0$  for all  $t \geq \tau_0$ . Hence  $\lambda_1 = -\infty$ . So suppose that  $P(\Omega_0) > 0$ . Itô's formula implies

$$\begin{aligned} \log \rho_\alpha(t) &= \log \rho_\alpha(0) + \int_0^t Q_\alpha(a(u), b(u), I_1(u)) du \\ &\quad + \int_0^t \tilde{Q}_\alpha(a(u), I_2(u)) d\langle M \rangle(u) + \int_0^t R_\alpha(a(u), I_2(u)) dM(u), \end{aligned} \quad (5)$$

for  $t > 0$ , a.s. on  $\Omega_0$ , where

$$\left. \begin{aligned} Q_\alpha(z_1, z_2, z_3) &:= \nu_1 z_1^2 + \sqrt{\alpha} \mu_1 z_1 z_2 + \sqrt{\alpha} z_1 z_3 + \frac{1}{2} \frac{z_1^2}{\alpha} - \frac{1}{2} z_2^2 \\ \tilde{Q}_\alpha(z_1, z'_3) &:= \alpha \left( \frac{1}{2} - z_1^2 \right) \left( \frac{\nu_2}{\sqrt{\alpha}} z_1 + z'_3 \right)^2 \\ R_\alpha(z_1, z'_3) &:= \nu_2 z_1^2 + \sqrt{\alpha} z_1 z'_3, \quad \|\sigma_i\|_2 := \left\{ \int_{-r}^0 \sigma_i(s)^2 ds \right\}^{1/2}, \end{aligned} \right\} \quad (6)$$

$i = 1, 2$ , and

$$a(t) := \frac{\sqrt{\alpha} x(t)}{\rho_\alpha(t)}, \quad b(t) := \frac{x(t-r)}{\rho_\alpha(t)}, \quad I_i(t) := \frac{\int_{-r}^0 x(t+s) \sigma_i(s) ds}{\rho_\alpha(t)} \quad (7)$$

for  $i = 1, 2$ ,  $t > 0$ , a.s. on  $\Omega_0$ .

Since

$$|I_i(t)| \leq \frac{1}{\rho_\alpha(t)} \left( \int_{-r}^0 x(t+s)^2 ds \right)^{1/2} \|\sigma_i\|_2 = \sqrt{1 - a^2(t)} \|\sigma_i\|_2,$$

$i = 1, 2$ , a.s. on  $\Omega_0$  the variables  $z_1, z_2, z_3, z'_3$  in (6) must satisfy

$$|z_1| \leq 1, \quad z_2 \in \mathbf{R}, \quad |z_3|^2 \leq (1 - z_1^2) \|\sigma_1\|_2^2, \quad |z'_3|^2 \leq (1 - z_1^2) \|\sigma_2\|_2^2.$$

Let  $\tau_1 := \inf\{t > 0 : \rho_\alpha(t) = 0\}$ . Then the local martingale

$$\int_0^{t \wedge \tau_1} R_\alpha(a(u), I_2(u)) dM(u), \quad t > 0,$$

is a time-changed (possibly stopped) Brownian motion. Since  $|R_\alpha(a(u), I_2(u))| \leq |\nu_2| + \sqrt{\alpha} \|\sigma_2\|_2$  for all  $u \in [0, \tau_1)$ , a.s., then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^{t \wedge \tau_1} R_\alpha(a(u), I_2(u)) dM(u) = 0 \quad \text{a.s.} \quad (8)$$



Divide (5) by  $t$ , let  $t \rightarrow \infty$ , to get

$$\begin{aligned} \lambda((v, \eta), \omega) \leq & \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t Q_\alpha(a(u), b(u), I_1(u)) du \\ & + \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \tilde{Q}_\alpha(a(u), I_2(u)) d\langle M \rangle(u). \end{aligned} \quad (9)$$

a.s. on  $\Omega_0$ , for all  $\alpha > 0$ .

Wish to develop upper bounds on  $\lambda_1$  in the following cases.

One-dimensional linear sfde (smooth memory in white-noise term):

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\} dt + \left\{ \int_{-r}^0 x(t+s) \sigma_2(s) ds \right\} dW(t), \quad t > 0 \quad (VII)$$

with real constants  $\nu_1, \mu_1$  and  $\sigma_2 \in C^1([-r, 0], \mathbf{R})$ . It is a special case of (XVII). Hence (VII) is regular with respect to  $M_2$ . The process  $\int_{-r}^0 x(t+s) \sigma_2(s) ds$  has  $C^1$  paths in  $t$ . Hence the stochastic differential  $dW$  in (VII) may be interpreted in the Itô or Stratonovich sense *without changing the solution  $x$* .

### Theorem V.3.

Suppose  $\lambda_1$  is the top a.s. Lyapunov exponent of (VII). Define the function

$$\theta(\delta, \alpha) := -\delta + \left( \nu_1 + \delta + \frac{1}{2} \alpha \mu_1^2 e^{2\delta r} + \frac{1}{2\alpha} \right) \vee \left( \frac{\alpha}{2} \|\sigma_2\|_2^2 e^{2\delta+r} \right)$$

for all  $\alpha \in \mathbf{R}^+, \delta \in \mathbf{R}$ , where  $\delta^+ := \max\{\delta, 0\}$ .

Then

$$\lambda_1 \leq \inf\{\theta(\delta, \alpha) : \delta \in \mathbf{R}, \alpha \in \mathbf{R}^+\}. \quad (10)$$

### Proof.

Maximize the integrand on the right-hand-side of (9) (with  $M = W$ ); then use exponential shift by  $\delta$  to refine the resulting estimate. Then minimize over  $\alpha, \delta$  ([M-S], II, 1996, pp. 34-35).  $\square$

Corollary below shows that the estimate in Theorem V.3 reduces to well-known estimate in deterministic case  $\sigma_2 \equiv 0$  (Hale [Ha], pp.17-18).

**Corollary V.3.1.**

In (VII), suppose  $\mu_1 \neq 0$  and let  $\delta_0$  be the unique real solution of the transcendental equation

$$\nu_1 + \delta + |\mu_1|e^{\delta r} = 0. \tag{11}$$

Then

$$\lambda_1 \leq -\delta_0 + \frac{1}{2} \frac{\|\sigma_2\|_2^2}{|\mu_1|} e^{|\delta_0|r}. \tag{12}$$

If  $\mu_1 = 0$  and  $\nu_1 \geq 0$ , then  $\lambda_1 \leq \frac{1}{2}(\nu_1 + \sqrt{\nu_1^2 + \|\sigma_2\|_2^2})$ . If  $\mu_1 = 0$  and  $\nu_1 < 0$ , then  $\lambda_1 \leq \nu_1 + \frac{1}{2}\|\sigma_2\|_2 e^{-\nu_1 r}$ .

**Proof.**

Suppose  $\mu_1 \neq 0$ . Denote by  $f(\delta)$ ,  $\delta \in \mathbf{R}$ , the left-hand-side of (11). Then  $f(\delta)$  is an increasing function of  $\delta$ .  $f$  has a unique real zero  $\delta_0$ . Using (10), we may put  $\delta = \delta_0$  and  $\alpha = |\mu_1|^{-1}e^{-\delta_0 r}$  in the expression for  $\theta(\delta, \alpha)$ . This gives (12).

Suppose  $\mu_1 = 0$ . Put  $\delta = (-\nu_1)^+$  in  $\theta(\delta, \alpha)$  and minimize the resulting expression over all  $\alpha > 0$ . This proves the last two assertions of the corollary ([M-S], II, 1996, pp. 35-36). □

**Remarks.**

- (i) Upper bounds for  $\lambda_1$  in Theorem (V.3) and Corollary V.3.1 agree with corresponding bounds in the deterministic case (for  $\mu_1 \geq 0$ ), but are not optimal when  $\mu_1 = 0$  and  $\sigma_2$  is strictly positive and sufficiently small; cf. Theorem V.1 for small  $\|\sigma_2\|_2$ .
- (ii) *Problem:* What are the asymptotics of  $\lambda_1$  for small delays  $r \downarrow 0$ ?

Our second example is the stochastic delay equation

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t - r)\} dt + x(t)dM(t), \quad t > 0, \tag{XVIII}$$

where  $M$  is the helix local martingale appearing in (XVII) and satisfying the conditions therein. Hence (XVIII) is regular with respect to  $M_2$ . Theorem below gives estimate on its top exponent.

**Theorem V.4.**

*In (XVIII) define  $\delta_0$  as in Corollary V.3.1. Then the top a.s. Lyapunov exponent  $\lambda_1$  of (XVIII) satisfies*

$$\lambda_1 \leq -\delta_0 + \frac{\beta}{16}. \tag{13}$$

**Proof.**

Maximize the following functions separately over their appropriate ranges:

$$Q_\alpha(z_1, z_2) := \nu_1 z_1^2 + \sqrt{\alpha} \mu_1 z_1 z_2 + \frac{1}{2} \frac{z_1^2}{\alpha} - \frac{1}{2} z_2^2,$$

$$\tilde{Q}_\alpha(z_1) := \left(\frac{1}{2} - z_1^2\right) z_1^2, \quad |z_1| \leq 1, z_2 \in \mathbf{R}.$$

Then use an exponential shift of the Lyapunov spectrum by an amount  $\delta$ . Minimize the resulting bound over all  $\alpha$  (for fixed  $\delta$ ) and then over all  $\delta \in \mathbf{R}$ . This minimum is attained if  $\delta$  solves the transcendental equation (11). Hence the conclusion of the theorem ([M-S], II, 1996, pp. 36-37). □

**Remark.**

The above estimate for  $\lambda_1$  is sharp in the deterministic case  $\beta = 0$  and  $\mu_1 \geq 0$ , but is not sharp when  $\beta \neq 0$ ; e.g.  $M = W$ , one-dimensional standard Brownian motion in the non-delay case ( $\mu_1 = 0$ ). When  $M = \nu_2 W$  for a fixed real  $\nu_2$ , the above bound may be considerably sharpened as in Theorem V.5 below. The sdde in this theorem is a model of dye circulation in

the blood stream (cf. Bailey and Williams [B-W], 1996; Lenhart and Travis, 1986).

**Theorem V.5.** ([M-S], II, 1996).

For the equation

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}dt + \nu_2 x(t) dW(t) \quad (VI)$$

set

$$\phi(\delta) := -\delta + \frac{1}{4\nu_2^2} \left[ \left( |\mu_1| e^{\delta r} + \nu_1 + \delta + \frac{1}{2}\nu_2^2 \right)^+ \right]^2, \quad (14)$$

for  $\nu_2 \neq 0$ . Then

$$\lambda_1 \leq \inf_{\delta \in \mathbf{R}} \phi(\delta). \quad (15)$$

In particular, if  $\delta_0$  is the unique solution of the equation

$$\nu_1 + \delta + |\mu_1| e^{\delta r} + \frac{1}{2}\nu_2^2 = 0, \quad (16)$$

then  $\lambda_1 \leq -\delta_0$ .

**Proof.**

Maximize

$$Q_\alpha(z_1, z_2, 0) + \tilde{Q}_\alpha(z_1, 0) = \left( \nu_1 + \frac{1}{2\alpha} + \frac{\nu_2^2}{2} \right) z_1^2 + \sqrt{\alpha} \mu_1 z_1 z_2 - \frac{1}{2} z_2^2 - \nu_2^2 z_1^4 \quad (17)$$

for  $|z_1| \leq 1, z_2 \in \mathbf{R}$  and then minimize the resulting bound for  $\lambda_1$  over  $\alpha > 0$ . Get

$$\lambda_1 \leq \frac{1}{16\nu_2^2} \left[ (2\nu_1 + 2|\mu_1| + \nu_2^2)^+ \right]^2.$$

The first assertion of the theorem follows from above estimate by applying an exponential shift to (VI). Last assertion of the theorem is obvious ([M-S], II, 1996, pp. 38-39.)  $\square$

*Problem:* Is  $\lambda_1 = \inf_{\delta \in \mathbf{R}} \phi(\delta)$  ?

**Remark.**

Estimate in Theorem V.5 agrees with the non-delay case  $\mu_1 = 0$  whereby  $\lambda_1 = \nu_1 - \frac{1}{2}\nu_2^2 = \inf_{\delta \in \mathbf{R}} \phi(\delta)$ . Cf. also [AOP], 1986, [B], 1985, and [AKO], 1989.

**4. SDDE with Poisson Noise.**

Consider the one-dimensional linear delay equation

$$\left. \begin{aligned} dx(t) &= x((t-1)-) dN(t) & t > 0 \\ x_0 &= \eta \in D := D([-1, 0], \mathbf{R}). \end{aligned} \right\} \quad (V)$$

The process  $N(t) \in \mathbf{R}$  is a Poisson process with i.i.d. inter-arrival times  $\{T_i\}_{i=1}^{\infty}$  which are exponentially distributed with the same parameter  $\mu$ . The jumps  $\{Y_i\}_{i=1}^{\infty}$  of  $N$  are i.i.d. and independent of all the  $T_i$ 's. Let

$$j(t) := \sup \left\{ j \geq 0 : \sum_{i=1}^j T_i \leq t \right\}.$$

Then

$$N(t) = \sum_{i=1}^{j(t)} Y_i.$$

Equation (V) can be solved a.s. in forward steps of lengths 1, using the relation

$$x^\eta(t) = \eta(0) + \sum_{i=1}^{j(t)} Y_i x \left( \left( \sum_{j=1}^i T_j - 1 \right) - \right) \quad \text{a.s.}$$

Trajectory  $\{x_t : t \geq 0\}$  is a Markov process in the state space  $D$  (with the supremum norm  $\|\cdot\|_\infty$ ). Furthermore, the above relation implies that (V) is regular in  $D$ ; i.e., it admits a measurable flow  $X : \mathbf{R}^+ \times D \times \Omega \rightarrow D$  with  $X(t, \cdot, \omega) = {}^\eta x_t(\cdot, \omega)$ , continuous linear in  $\eta$  for all  $t \geq 0$  and a.a.  $\omega \in \Omega$  (cf. the singular equation (I) ).

The a.s. Lyapunov spectrum of (V) may be characterized directly (without appealing to the Oseledec Theorem) by interpolating between the sequence of random times:

$$\begin{aligned}\tau_0(\omega) &:= 0, \\ \tau_1(\omega) &:= \inf \left\{ n \geq 1 : \sum_{j=1}^k T_j \notin [n-1, n] \quad \text{for all } k \geq 1 \right\}, \\ \tau_{i+1}(\omega) &:= \inf \left\{ n > \tau_i(\omega) : \sum_{j=1}^k T_j \notin [n-1, n] \quad \text{for all } k \geq 1 \right\}, \quad i \geq 1.\end{aligned}$$

It is easy to see that  $\{\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \dots\}$  are i.i.d. and  $E\tau_1 = e^\mu$ .

**Theorem V.6.** ([M-S], II, 1996)

Let  $\xi \in D$  be the constant path  $\xi(s) = 1$  for all  $s \in [-1, 0]$ . Suppose  $E \log \|X(\tau_1(\cdot), \xi, \cdot)\|_\infty$  exists (possibly  $= +\infty$  or  $-\infty$ ). Then the a.s. Lyapunov spectrum

$$\lambda(\eta) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \eta, \omega)\|_\infty, \quad \eta \in D, \omega \in \Omega$$

of (V) is  $\{-\infty, \lambda_1\}$  where

$$\lambda_1 = e^{-\mu} E \log \|X(\tau_1(\cdot), \xi, \cdot)\|_\infty.$$

In fact,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \eta, \omega)\|_\infty = \begin{cases} \lambda_1 & \eta \notin \text{Ker } X(\tau_1(\omega), \cdot, \omega) \\ -\infty & \eta \in \text{Ker } X(\tau_1(\omega), \cdot, \omega). \end{cases}$$

**Proof.**

The i.i.d. sequence

$$S_i := \frac{\|(X(\tau_i, \xi, \cdot))\|}{\|(X(\tau_{i-1}, \xi, \cdot))\|} \quad i = 1, 2, \dots$$

and the LLN give

$$\lim_{n \rightarrow \infty} \frac{1}{\tau_n} \log \|(X(\tau_n, \xi, \omega))\| = e^{-\mu}(E \log S_1)$$

for a.a.  $\omega \in \Omega$ .

Interpolate between the times  $\tau_1, \tau_2, \tau_3, \dots$  to get the continuous limit ([M-S], II, 1996, pp. 27-28).  $\square$

## **VI. MISCELLANY**

**Geilo, Norway**

**Saturday, August 3, 1996**

**14:00-14:50**

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## VI. MISCELLANY

### 1. Malliavin Calculus of SFDE's

#### Objectives.

- I) To establish the existence of smooth densities for solutions of  $\mathbf{R}^d$ -valued sfde's of the form

$$dx(t) = H(t, x_t) dt + g(t, x(t-r)) dW(t). \quad (XIX)$$

In the above equation,  $W$  is an  $m$ -dimensional Wiener process,  $r$  is a positive time delay,  $H$  is a functional  $C([-r, 0], \mathbf{R}^d) \rightarrow \mathbf{R}^d$  and  $g : [0, \infty) \times \mathbf{R}^d \rightarrow \mathbf{R}^{d \times m}$  is a  $d \times m$ -matrix-valued function, such that for fixed  $t$  the

$d \times d$ -matrix  $g(t, x)g(t, x)^*$  has degeneracies of polynomial order as  $x$  runs on a hypersurface in  $\mathbf{R}^d$ .

- II) Method of proof gives a very general criterion for the hypoellipticity of a class of degenerate parabolic second-order time-dependent differential operators with space-independent principal part.
- III) More generally the analysis works when  $H$  is replaced by a non-anticipating functional which may depend on the *whole history* of the path ([B-M], Ann. Prob., 1995).

Case  $H \equiv 0$  studied by (Bell and Mohammed [B-M], (J.F.A., 1991). Solution  $x(t)$  has smooth density wrt Lebesgue measure on  $\mathbf{R}^d$  if  $g(t, x)g(t, x)^*$  degenerates like  $|x|^2$  near 0 (e.g.  $g(t, \cdot)$  linear.) Proof uses Malliavin calculus.

#### Difficulties.

- (i) The infinitesimal generator of the trajectory Feller process  $\{x_t : t \geq 0\}$  is a highly degenerate second-order differential operator

on the state space: its principal part degenerates on a surface of *finite* codimension (Lecture II, Theorem II.3). Hence cannot use existing techniques from pde's.

- (ii) Analysis by Malliavin calculus requires derivation of probabilistic lower bounds on the *Malliavin covariance matrix* of the solution  $x$ . These bounds are difficult because there is no stochastic flow in the singular case (Lecture III, Theorem III.3). cf. sode case, where stochastic flow is invertible. See work by Kusuoka and Stroock in the uniformly elliptic case ([K-S], I, Taniguchi Sympos. 1982).
- (iii) The form of the Malliavin covariance allows *polynomial (finite-type)* rate of degeneracy near a hypersurface, coupled with limited contact of the initial path with the hypersurface. cf. sode case where degeneracies of *infinite type* are compatible with hypoellipticity (Bell and Mohammed [B-M], Duke Math. Journal, 1995).

## Hypotheses (H).

- (i)  $W : [0, \infty) \times \Omega \rightarrow \mathbf{R}^m$  is standard  $m$ -dimensional Wiener process, defined on a complete filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ .
- (ii)  $g : [0, \infty) \times \mathbf{R}^d \rightarrow \mathbf{R}^{d \times m}$  is a continuous map into the space of  $d \times m$  matrices, with bounded Fréchet derivatives of all orders in the space variables .
- (iii)  $r$  is a positive real number, and  $\eta : [-r, 0] \rightarrow \mathbf{R}^d$  is a continuous initial path.
- (iv)  $H : [0, \infty) \times C \rightarrow \mathbf{R}^d$  is a globally bounded continuous map with all Fréchet derivatives of  $H(t, \eta)$  wrt  $\eta$  globally bounded in  $(t, \eta) \in \mathbf{R}^+ \times C$ . Think of  $H(t, \xi_t)$  as a smooth  $\mathbf{R}^d$ -valued functional in  $\xi \in C([-r, t], \mathbf{R}^d)$ . Denote its Fréchet derivative wrt  $\xi \in C([-r, t], \mathbf{R}^d)$  by  $H_\xi(t, \xi)$ . Set

$$\alpha_t := \sup\{\|H_\xi(u, \xi)\| : u \in [0, t], \xi \in C([-r, u], \mathbf{R}^d)\}, \quad t > 0,$$

and

$$\alpha_\infty := \sup\{\|H_\xi(u, \xi)\| : u \in [0, \infty), \xi \in C([-r, u], \mathbf{R}^d)\},$$

where  $\|H_\xi(u, \xi)\|$  is the operator norm of the partial Fréchet derivative  $H_\xi(u, \xi) : C([-r, u], \mathbf{R}^d) \rightarrow \mathbf{R}^d$ .

## Theorem VI.1.

Assume Hypotheses (H) for the sfde (XIX). Suppose there exist positive constants  $\rho, \delta$ , an integer  $p \geq 2$  and a function  $\phi : [0, \infty) \times \mathbf{R}^d \rightarrow \mathbf{R}$  satisfying the following conditions

$$(i) \quad g(t, x)g(t, x)^* \geq \begin{cases} |\phi(t, x)|^p I, & |\phi(t, x)| < \rho \\ \delta I, & |\phi(t, x)| \geq \rho \end{cases} \quad (1)$$

for  $(t, x) \in [0, \infty) \times \mathbf{R}^d$ .

- (ii)  $\phi(t, x)$  is  $C^1$  in  $t$  and  $C^2$  in  $x$ , with bounded first derivatives in  $(t, x)$  and bounded second derivatives in  $x \in \mathbf{R}^d$ .
- (iii) There is a positive constant  $c$  such that

$$\|\nabla\phi(t, x)\| \geq c > 0 \tag{2}$$

for all  $(t, x) \in [0, \infty) \times \mathbf{R}^d$ , with  $|\phi(t, x)| \leq \rho$ . In (2),  $\nabla$  denotes the gradient operator with respect to the space variable  $x \in \mathbf{R}^d$ .

- (iv) There is a positive number  $\delta_0$  such that  $\delta_0 < (3\alpha_\infty)^{-1} \wedge r$  and for every Borel set  $J \subseteq [-r, 0]$  of Lebesgue measure  $\delta_0$  the following holds

$$\int_J \phi(t+r, \eta(t))^2 dt > 0. \tag{3}$$

Define  $s_0 \in [-r, 0]$  by

$$s_0 := \sup\{s \in [-r, 0] : \int_{-r}^s \phi(u+r, \eta(u))^2 du = 0\}.$$

Then for all  $t > s_0 + r$  the solution  $x(t)$  of (XIX) is absolutely continuous with respect to  $d$ -dimensional Lebesgue measure, and has a  $C^\infty$  density.

**Remark.**

Condition (iv) of Theorem VI.1 may be replaced by the following equivalent condition:

- (iv)' The set  $\{s : s \in [-r, 0], \phi(s, \eta(s)) = 0\}$  has Lebesgue measure less than  $(3\alpha_\infty)^{-1} \wedge r$ .

**Theorem VI.2**

In the sfde

$$\left. \begin{aligned} dy(t) &= H(t, y_t) dt + F(t)dW(t), & t > a \\ y(t) &= x(t), & a-r \leq t \leq a, \quad a \geq r \end{aligned} \right\} \tag{XX}$$

suppose that  $F : [a, \infty) \rightarrow \mathbf{R}^{d \times n}$  and  $x : [0, a] \rightarrow \mathbf{R}^d$  are continuous. Assume that  $H$  satisfies regularity hypotheses analogous to (H). For  $t > a$  let

$$\alpha'_t := \sup\{\|H_\xi(u, \xi)\| : u \in [0, t], \xi \in C\}.$$

Suppose there exists  $\delta^* < 1/(3\alpha'_t)$  such that

$$\int_{t-\delta^*}^t \mu_1(s) ds > 0, \tag{4}$$

where  $\mu_1(s)$ ,  $s \geq a$ , is the smallest eigenvalue of the non-negative definite matrix  $F(s)F(s)^*$ . Then the solution  $y(t)$  of (XX) has an absolutely continuous distribution with respect to  $d$ -dimensional Lebesgue measure and has a  $C^\infty$  density.

In the special case when  $H(t, y) = h(t, y(t))$  in equation (XX) for some Lipschitz function  $h : \mathbf{R}^+ \times \mathbf{R}^d \rightarrow \mathbf{R}^d$ , then  $y$  is a (time-inhomogeneous) diffusion process. In this case the proof of Theorem VI.2 gives the following pde result.

### Theorem VI.3

For each  $t > 0$ , let  $A(t) = [a_{ij}(t)]_{i,j=1}^d$  denote a symmetric non-negative definite  $d \times d$ -matrix. Let  $\mu_2(t)$  be the smallest eigenvalue of  $A(t)$ . Assume the following:

- (i) The map  $t \mapsto A(t)$  is continuous.
- (ii) There exists  $T > 0$  such that

$$\int_0^T \mu_2(s) ds > 0. \quad (5)$$

- (iii) The functions  $b_i$ ,  $i = 1, \dots, d$ ,  $c : \mathbf{R}^+ \times \mathbf{R}^d \rightarrow \mathbf{R}$  are bounded, jointly continuous in  $(t, x)$  and have partial derivatives of all orders in  $x$ , all of which are bounded in  $(t, x)$ . Let  $T_0 := \sup\{T > 0 : \int_0^T \mu_2(s) ds = 0\}$ , and let  $L_{t,x}$  denote the differential operator

$$L_{t,x} := \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t, x) \frac{\partial}{\partial x_i} + c(t, x). \quad (6)$$

Then the parabolic equation  $\frac{\partial u}{\partial t} = L_{t,x}u$  has a fundamental solution  $\Gamma(t, x, y)$  defined on  $(T_0, \infty) \times \mathbf{R}^{2d}$ , which is  $C^1$  in  $t$  and  $C^\infty$  in  $(x, y)$ . Furthermore, if the coefficients  $a_{ij}(t)$ ,  $b_i(t, x)$ ,  $c(t, x)$ ,  $i, j = 1, \dots, d$ , are  $C^\infty$  in  $(t, x)$ , and

$$\lim_{t \rightarrow T_0^+} (t - T_0) \log \left\{ \int_{T_0}^t \mu_2(s) ds \right\} = 0, \quad (7)$$

then  $\frac{\partial}{\partial t} - L_{t,x}$  is a hypoelliptic operator on  $(T_0, \infty) \times \mathbf{R}^d$ ; (viz. if  $\phi$  is a distribution on  $(T_0, \infty) \times \mathbf{R}^d$  such that  $\left(\frac{\partial}{\partial t} - L_{t,x}\right)\phi$  is  $C^\infty$ , then  $\phi$  is also  $C^\infty$ .)

The *mean ellipticity* hypothesis (5) is much weaker than classical *pointwise ellipticity*.

#### Problem.

Can Theorem VI.3 be proved using existing pde's techniques?

## Proof of Theorem VI.1. (Outline)

Objective is to get good probabilistic lower bounds on the Malliavin covariance matrix of the solution  $x(t)$  of (XIX). Do this using the following steps:

(cf.[B-M 2], Ann. Prob. 1995, “conditioning argument” using (XX)).

*Step 1.*

We use piecewise linear approximations of  $W$  in (XIX) to compute the Malliavin covariance matrix  $C(T)$  of  $x(T)$  as

$$C(T) = \int_0^T Z(u)g(u, x(u-r))g(u, x(u-r))^* Z(u)^* du,$$

where the  $(d \times d)$ -matrix-valued process  $Z : [0, T] \times \Omega \rightarrow \mathbf{R}^{d \times d}$  satisfies the advanced *anticipating* Stratonovich integral equation

$$\begin{aligned} Z(t) = I + \int_{T \wedge (t+r)}^T Z(u)Dg(u, x(u-r))(\cdot) \circ dW(u) \\ + \int_t^T Z(u)[\{H_x(u, x)^*(\cdot)\}'(t)]^* du, \quad 0 \leq t \leq T. \end{aligned}$$

In the above integral equation,  $H_x(u, x)$  is the Fréchet partial derivative of the map

$$(u, x) \rightarrow H(u, x_u)$$

with respect to  $x \in C([-r, u], \mathbf{R}^d)$ . If  $W^{1,2}$  is the Cameron Martin subspace of  $C([-r, u], \mathbf{R}^d)$ , let  $H_x(u, x)^*$  denote the Hilbert-space adjoint of the restriction

$$H_x(u, x)|_{W^{1,2}} : W^{1,2} \rightarrow \mathbf{R}^d.$$

We solve the above integral equation as follows.

Start with the terminal condition  $Z(T) = I$ . On the last delay period  $[(T-r) \vee 0, T]$  define  $Z$  to be the unique solution of the linear integral equation

$$Z(t) = I + \int_t^T Z(u)[\{H_x(u, x)^*(\cdot)\}'(t)]^* du$$

for a.e.  $t \in ((T-r) \vee 0, T)$ . When  $T > r$ , use successive approximations to solve the anticipating integral equation, treating the stochastic integral as a predefined random *forcing term*. This gives a unique solution of the integral equation by successive backward steps of length  $r$ . The matrix  $Z(t)$  need not be invertible for small  $t$ . Compare  $Z(t)$  with the analogous process for the diffusion case (sode). In this case  $Z(t)$  is invertible for all  $t$  and anticipating integrals are not needed.

*Step 2.*

Since  $H_x(u, x)$  is globally bounded in  $(u, x)$ , then so is  $[H_x(u, x)^*(\cdot)]'(t)$  in  $(u, x, t)$  ([B-M], Ann. Prob. 1995, Lemma 3.3). Hence can choose a *deterministic* time  $t_0 < T$  sufficiently close to  $T$  such that almost surely  $Z(t)$  is invertible and  $\|Z(t)^{-1}\| \leq 2$  for a.e.  $t \in (t_0, T)$ . ([B-M], Ann. Prob. 1995, Lemma 3.4).

*Step 3.*

The above lower bound on  $\|Z(t)\|$  and the representation of  $C(T)$  imply that

$$\det C(T) \geq \left[ \frac{1}{4} \int_{t_0}^T \hat{g}(u, x(u-r))^2 du \right]^d \quad \text{a.s.}$$

where

$$\hat{g}(u, v) := \inf \{ |g(u, v)^*(e)| : e \in \mathbf{R}^d, |e| = 1 \},$$

for all  $u \geq 0, v \in \mathbf{R}^d$ .

*Step 4.*

Prove the

*Propagation Lemma:*

Let  $-r < a < b < a + r$ . Then the statement

$$P\left(\int_a^b |\phi(u+r, x(u))|^2 du < \epsilon\right) = o(\epsilon^k)$$



as  $\epsilon \rightarrow 0+$  for every  $k \geq 1$ ,

implies that

$$P\left(\int_{a+r}^{b+r} |\phi(u+r, x(u))|^2 du < \epsilon\right) = o(\epsilon^k)$$

as  $\epsilon \rightarrow 0+$  for every  $k \geq 1$ . (Proof uses Itô's formula, the lower bound on  $\|\nabla\phi\|$ , the polynomial degeneracy condition and the Kusuoka-Stroock  $\epsilon^{1/(18)}$ -lemma!).

*Step 5.*

By successively applying Step 4, we propagate the “limited contact” hypothesis on the initial path  $\eta$  in order to get the estimate:

$$P\left(\int_{t_0}^T |\phi(u, x(u-r))|^2 du < \epsilon\right) = o(\epsilon^k)$$

as  $\epsilon \rightarrow 0+$  for every  $k \geq 1$ .

*Step 6.*

Using the polynomial degeneracy hypothesis, Step 5, Jensen's inequality, and Lemma 4.3 of ([B-M], Ann. Prob. 1995), we obtain

$$P\left(\int_{t_0}^T \hat{g}(u, x(u-r))^2 du < \epsilon\right) = o(\epsilon^k)$$

as  $\epsilon \rightarrow 0+$  for every  $k \geq 1$ .

*Step 7.*

Combining steps 3 and 6 gives

$$P(\det C(T) < \epsilon) = o(\epsilon^k)$$

as  $\epsilon \rightarrow 0+$  for every  $k \geq 1$ . This implies that  $C(T)^{-1}$  exists a.s. and  $\det C(T)^{-1} \in \bigcap_{q=1}^{\infty} L^q(\Omega, \mathbf{R})$ . □

## 2. Back to Square One: Diffusions via SDDE's

Objective is to prove the following existence theorem for classical diffusions using approximations by small delays: (Caratheodory)

**Theorem VI.4.**( Itô, Gihman-Skorohod,..)

Let  $h : \mathbf{R}^d \rightarrow \mathbf{R}^d$ ,  $g : \mathbf{R}^d \rightarrow \mathbf{R}^{d \times m}$  be globally Lipschitz, and  $W$   $m$ -dimensional Brownian motion. Suppose  $x_0 \in \mathbf{R}^d$ . Then the sode

$$\begin{aligned} dx(t) &= h(x(t)) dt + g(x(t)) dW(t), \quad t > 0 \\ x(0) &= x_0 \end{aligned}$$

has an adapted solution with continuous sample paths.

**Proof.**([B-M], Stochastics, 1989)

For simplicity assume that  $h \equiv 0$  and  $d = m = 1$ .

Fix  $0 < T < \infty$ . For each integer  $k \geq 1$ , define

$$\left. \begin{aligned} x^k(t) &= x_0 + \int_0^t g\left(x^k\left(u - \frac{1}{k}\right)\right) dW(u), \quad t \geq 0 \\ x^k(t) &= x_0, \quad -\frac{1}{k} \leq t \leq 0 \end{aligned} \right\} \quad (*)$$

Note that  $x^k$  exists, is adapted and continuous.

*Step 1.*

$x^k : [0, \infty) \rightarrow L^2(\Omega, \mathbf{R})$  is  $(\frac{1}{2})$ -Hölder, with Hölder constant independent of  $k$ . (*Exercise:* To prove this observe first that by (\*) and the linear growth property of  $g$ , there is a positive constant  $K$  independent of  $k$  and  $t \in [0, T]$  such that

$$E \sup_{0 \leq u \leq t} (|x^k(t)|^2 + |g(x^k(t))|^2) \leq K$$

for all  $k \geq 1$  and all  $t \in [0, T]$ . Then  $E[x^k(t) - x^k(s)]^2 \leq K(t - s)$  for all  $t, s \geq 0$ .)

*Step 2.*

For each  $t \geq 0$ ,  $x^k(t)$  converges to a limit  $x(t)$  in  $L^2(\Omega, \mathbf{R}^d)$ : Let  $L$  be the Lipschitz constant for  $g$ .

For  $l > k$ , we have

$$\begin{aligned}
E[x^l(t) - x^k(t)]^2 &= E\left\{\int_0^t \left[g\left(x^l\left(u - \frac{1}{l}\right)\right) - g\left(x^k\left(u - \frac{1}{k}\right)\right)\right] dW(u)\right\}^2 \\
&\leq L^2 \int_0^t E\left[x^l\left(u - \frac{1}{l}\right) - x^k\left(u - \frac{1}{l}\right)\right]^2 du \\
&\leq 2L^2 \int_0^t E\left[x^l\left(u - \frac{1}{l}\right) - x^k\left(u - \frac{1}{l}\right)\right]^2 du \\
&\quad + 2L^2 \int_0^t E\left[x^k\left(u - \frac{1}{l}\right) - x^k\left(u - \frac{1}{k}\right)\right]^2 du \\
&\leq 2L^2 \int_{-\frac{1}{l}}^{t-\frac{1}{l}} E[x^l(u) - x^k(u)]^2 du + 2KL^2t\left(\frac{1}{k} - \frac{1}{l}\right) \\
&\leq 2L^2 \int_0^t E[x^l(u) - x^k(u)]^2 du + 2t\|a\|_\infty\left(\frac{1}{k} - \frac{1}{l}\right)
\end{aligned}$$

by Step 1. Thus, by Gronwall's lemma

$$E[x^l(t) - x^k(t)]^2 \leq 2TK\left(\frac{1}{k} - \frac{1}{l}\right)e^{2L^2t}$$

Thus convergence holds. Also

$$E[x(t) - x^k(t)]^2 \leq \frac{2TK}{k}e^{2L^2t}$$

*Step 3.*

The process  $x$  satisfies original sode:

Simply take limits as  $k \rightarrow \infty$  in both sides of (\*). Then  $LHS \rightarrow x(t)$  in  $L^2$ .  $x$  is adapted, since each  $x^k$  is. Also

$$\begin{aligned}
& E \left\{ \int_0^t \left[ g \left( x^k \left( u - \frac{1}{k} \right) \right) - g(x(u)) \right] dW(u) \right\}^2 \\
& \leq L^2 \int_0^t E \left[ x^k \left( u - \frac{1}{k} \right) - x(u) \right]^2 du \\
& \leq 2L^2 \int_0^t E \left[ x^k \left( u - \frac{1}{k} \right) - x^k(u) \right]^2 du + 2L^2 \int_0^t E [x^k(u) - x(u)]^2 du \\
& \leq \frac{2L^2 K t}{k} + \frac{2L^2 K T}{k} \int_0^t 2e^{2L^2 u} du \\
& \leq \frac{2KL^2}{k} \left[ t + \frac{1}{2L^2} \left( e^{2L^2 t} - 1 \right) \right] \\
& \rightarrow 0 \text{ as } k \rightarrow \infty
\end{aligned}$$

Thus

$$RHS \rightarrow x_0 + \int_0^t g(x(u)) dW(u)$$

i.e.  $x$  satisfies the sode.

Since the Itô integral has an a.s. continuous modification, it follows from Doob's inequality that  $x$  has such a modification.  $\square$

### 3. Affine SFDE's. A Simple Model of Population Growth

Recall simple population growth model (Lecture I):

$$dx(t) = \{-\alpha x(t) + \beta x(t-r)\} dt + \sigma dW(t), \quad t > 0 \quad (II)$$

for a large population  $x(t)$  with  
constant birth rate =  $\beta > 0$  (per capita);  
constant death rate =  $\alpha > 0$  (per capita);  
development period  $r = (9) > 0$ ;

migration, white noise, variance  $\sigma$ .

To determine stability, growth rates of population, consider the general affine system :

$$\left. \begin{aligned} dx(t) &= \left\{ \int_{[-r,0]} \mu(ds)x(t+s) \right\} dt + dQ(t), \quad t > 0 \\ x_0 &= \eta \in D := D([-r,0], \mathbf{R}^d). \end{aligned} \right\} \quad (X)$$

$D :=$  space of all cadlag paths  $[-r,0] \rightarrow \mathbf{R}^d$  with  $\|\cdot\|_\infty$ -norm.

**Theorem VI.5.**

The Lyapunov spectrum of  $(X)$  coincides with the set of all real parts  $\{\beta_i : i \geq 1\}$  of the spectrum of the generator  $A$  of the homogeneous equation corresponding to  $Q \equiv 0$ , together with possibly  $-\infty$ .

We now consider the *hyperbolic case* when  $\beta_i \neq 0$  for all  $i \geq 1$ . In this case, the following result ([M-S1], Theorem 20) establishes the existence of a hyperbolic splitting along a unique stationary solution of  $(X)$ .

**Theorem VI.6.** ([M-S], 1990)

Suppose that  $Q$  is cadlag and has stationary increments. Assume that the characteristic equation

$$\det\left(\lambda I - \int_{[-r,0]} e^{\lambda s} \mu(ds)\right) = 0$$

has no roots on the imaginary axis; i.e., the associated homogeneous equation ( $Q \equiv 0$ ) has no zero Lyapunov exponents. Suppose also that

$$\overline{\lim}_{t \rightarrow \pm\infty} \frac{1}{|t|} \log |Q(t)| < |\operatorname{Re} \lambda| \quad \text{a.s.}$$

for all characteristic roots  $\lambda$ . Then there is a unique  $D$ -valued random variable  $\eta_\infty$  such that the trajectory  $\{x_t^{\eta_\infty} : t \geq 0\}$  of  $(X)$  is a  $D$ -valued stationary process. The random variable  $\eta_\infty$  is measurable with respect to the  $\sigma$ -algebra generated by  $\{Q(t) : t \in \mathbf{R}\}$ . Furthermore, let  $\beta_1 > \beta_2 > \dots$ , be an ordering of the Lyapunov spectrum of  $(X)$  and suppose  $\beta_m > 0$ ,  $\beta_{m+1} < 0$ . Then there exists a decreasing sequence of finite-codimensional subspaces  $\{E_i : i \geq 1\}$  of  $D$  such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|x_t(\omega)\|_\infty = \beta_i, \quad i \geq 1$$

if  $x_0(\omega) \in \eta_\infty + E_{i-1} \setminus E_i$ ,  $1 \leq i \leq m$ , and

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \log \|x_t(\omega)\|_\infty \leq \beta_{m+1}$$

if  $x_0(\omega) \in \eta_\infty + E_m$ .

Results on the existence of  $p$ -th moment Lyapunov exponents appear in ([M-S], Stochastics, 1990). Under mild non-degeneracy condition on the stationary solution, one gets the existence of *only* one  $p$ -th moment exponent ( $= p\beta_1$ ) which is independent of all *random* (possibly *anticipating*) initial conditions in  $D$ . This result is in agreement with the affine linear finite-dimensional non-delay case ( $r = 0$ ) ([AOP],[B],[AKO]).

**Problem.**

Under what conditions on the parameters  $\alpha, \beta$  does (II) have a stationary solution?

**Interesting Fact:**

The affine hereditary system (X) may be viewed as a *finite-dimensional* stochastic perturbation of the associated *infinitely degenerate* deterministic homogeneous system ( $Q \equiv 0$ ) with *countably many* Lyapunov exponents. However, these finite-dimensional perturbations provide noise that is generically rich enough to account for a *single*  $p$ -th moment Lyapunov exponent in the affine stochastic system (X).

#### **4. Random Delays**

See [Mo], Pitman Books, 1984, pp. 167-186. Delays are allowed to be random (independently) of the noise and essentially bounded. Markov property fails, but get a measure-valued process with random *Markov* transition measures on state space.

#### **5. Infinite Delays. Stationary Solutions**

Pioneering work of Itô and Nisio, [I-N], J. Math. Kyoto University, 1964, pp. 1-75. Results summarized in [Mo], 1984, pp. 230-233.



## 6. Summary

Main themes are:

SMOOTH HISTORY  $\iff$  REGULARITY OF SFDE

FINITE HISTORY  $\implies$  LOCAL COMPACTNESS OF SEMI-FLOW  $\implies$  DISCRETE LYAPUNOV SPECTRUM  $\implies$  HYPERBOLICITY AND STABLE MANIFOLDS

HELIX NOISE  $\implies$  NON-IVERTIBLE MULTIPLICATIVE COCYCLE ON HILBERT SPACE

DISCRETE FINITE HISTORY  $\implies$  SINGULAR SYSTEM  $\implies$  NONEXISTENCE OF FLOW

**STOCHASTIC DIFFERENTIAL SYSTEMS  
WITH MEMORY**

**THEORY, EXAMPLES AND APPLICATIONS**

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## BIBLIOGRAPHIES (TENTATIVE)

### Ergodic Theory and Lyapunov Exponents

- [AKO] Arnold, L., Kliemann, W. and Oeljeklaus, E. Lyapunov exponents of linear stochastic systems, in *Lyapunov Exponents*, Springer Lecture Notes in Mathematics **1186** (1989), 85–125.
- [AOP] Arnold, L. Oeljeklaus, E. and Pardoux, E., Almost sure and moment stability for linear Itô equations, in *Lyapunov Exponents*, Springer Lecture Notes in Mathematics **1186** (ed. L. Arnold and V. Wihstutz) (1986), 129–159.
- [B] Baxendale, P.H., Moment stability and large deviations for linear stochastic differential equations, in Ikeda, N. (ed.) *Proceedings of the Taniguchi Symposium on Probabilistic Methods in Mathematical Physics*, Katata and Kyoto (1985), 31–54, Tokyo: Kinokuniya (1987).
- [CJPS] Cinlar, E., Jacod, J., Protter, P. and Sharpe, M. Semimartingales and Markov processes, *Z. Wahrsch. Verw. Gebiete* **54** (1980), 161–219.
- [FS] Flandoli, F. and Schaumlöffel, K.-U. Stochastic parabolic equations in bounded domains: Random evolution operator and Lyapunov exponents, *Stochastics and Stochastic Reports* **29**, 4 (1990), 461–485.
- [M1] Mohammed, S.-E.A. *Stochastic Functional Differential Equations*, Research Notes in Mathematics **99**, Pitman Advanced Publishing Program, Boston-London-Melbourne (1984).

- [M2] Mohammed, S.-E.A. Non-linear flows for linear stochastic delay equations, *Stochastics* **17**, 3 (1986), 207–212.
- [M3] Mohammed, S.-E.A. The Lyapunov spectrum and stable manifolds for stochastic linear delay equations, *Stochastics and Stochastic Reports* **29** (1990), 89–131.
- [M4] Mohammed, S.-E.A. Lyapunov exponents and stochastic flows of linear and affine hereditary systems, (1992) (Survey article), Birkhäuser (1992), 141–169.
- [MS] Mohammed, S.-E.A. and Scheutzow, M.K.R. Lyapunov exponents of linear stochastic functional differential equation driven by semimartingales. Part I: The multiplicative ergodic theory, (preprint, 1990), AIHP, vol. 32, no. 1, (1996), 69-105.
- [P] Protter, Ph.E. Semimartingales and measure-preserving flows, *Ann. Inst. Henri Poincaré, Probabilités et Statistiques*, vol. 22, (1986), 127-147.
- [R] Ruelle, D. Characteristic exponents and invariant manifolds in Hilbert space, *Annals of Mathematics* **115** (1982), 243–290.
- [S] Scheutzow, M.K.R. *Stationary and Periodic Stochastic Differential Systems: A study of Qualitative Changes with Respect to the Noise Level and Asymptotics*, Habilitationsschrift, University of Kaiserslautern, W. Germany (1988).
- [Sc] Schwartz, L., *Radon Measures on Arbitrary Topological Spaces and Cylindrical measures*, Tata Institute of Fundamental Research, Oxford University Press, (1973).
- [Sk] Skorohod, A. V., *Random Linear Operators*, D. Reidel Publishing Company (1984).

- [SM] de Sam Lazaro, J. and Meyer, P.A. Questions de théorie des flots, *Seminaire de Probab. IX*, Springer Lecture Notes in Mathematics **465**, (1975), 1–96.
- [E] Elworthy, K. D., Stochastic differential equations on manifolds, Cambridge (Cambridgeshire) ; New York : Cambridge University Press, (18), 326 p. ; 23 cm. 1982, London Mathematical Society lecture note series ; 70.
- [D] Dudley, R.M., The sizes of compact subsets of Hilbert space and continuity of Gaussian processes, *J. Functional Analysis*, 1, (1967), 290-330.
- [FS] Flandoli, F. and Schaumlöffel, K.-U. Stochastic parabolic equations in bounded domains: Random evolution operator and Lyapunov exponents, *Stochastics and Stochastic Reports* **29**, 4 (1990), 461–485.
- [H] Hale, J.K., *Theory of Functional Differential Equations*, Springer-Verlag, New York, Heidelberg, Berlin, (1977).
- [Ha] Has'minskiĭ, R. Z., *Stochastic Stability of Differential Equations*, Sijthoff & Noordhoff (1980).
- [KN] Kolmanovskii, V.B. and Nosov, V.R., *Stability of Functional Differential Equations*, Academic Press, London, Orlando (1986) .
- [K] Kushner, H.J., On the stability of processes defined by stochastic differential-difference equations, *J. Differential equations*, 4, (1968), 424-443.
- [Ma] Mao, X.R., *Exponential Stability of Stochastic Differential Equations*, Pure and Applied Mathematics, Marcel Dekker, New York-Basel-Hong Kong (1994).

- [MT] Mizel, V.J. and Trutzer, V., Stochastic hereditary equations: existence and asymptotic stability, *J. Integral Equations*, (1984), 1-72
- [M1] Mohammed, S.-E.A., *Stochastic Functional Differential Equations*, Research Notes in Mathematics **99**, Pitman Advanced Publishing Program, Boston-London-Melbourne (1984).
- [M2] Mohammed, S.-E.A., Non-linear flows for linear stochastic delay equations, *Stochastics* **17**, 3 (1986), 207–212.
- [M3] Mohammed, S.-E.A., The Lyapunov spectrum and stable manifolds for stochastic linear delay equations, *Stochastics and Stochastic Reports* **29** (1990), 89–131.
- [M4] Mohammed, S.-E.A., Lyapunov exponents and stochastic flows of linear and affine hereditary systems, (1992) (Survey article), in *Diffusion Processes and Related Problems in Analysis, Volume II*, edited by Pinsky, M., and Wihstutz, V., Birkhäuser (1992), 141–169.
- [MS] Mohammed, S.-E.A. and Scheutzow, M.K.R., Lyapunov exponents of linear stochastic functional differential equation driven by semimartingales. Part I: The multiplicative ergodic theory, *Ann. Inst. Henri Poincaré, Probabilités et Statistiques*, Vol. 32, no. 1 (1996), 69-105.
- [PW1] Pardoux, E. and Wihstutz, V., Lyapunov exponent and rotation number of two-dimensional stochastic systems with small diffusion, *SIAM J. Applied Math.*, 48, (1988), 442-457.
- [PW2] Pinsky, M. and Wihstutz, V., Lyapunov exponents of nilpotent Itô systems, *Stochastics*, 25, (1988), 43-57.
- [R] Ruelle, D. Characteristic exponents and invariant manifolds in Hilbert space, *Annals of Mathematics* **115** (1982), 243–290.

[S] Scheutzow, M.K.R. *Stationary and Periodic Stochastic Differential Systems: A study of Qualitative Changes with Respect to the Noise Level and Asymptotics*, Habilitationsschrift, University of Kaiserslautern, W. Germany (1988).

[Sc] Schwartz, L., *Radon Measures on Arbitrary Topological Vector Spaces and Cylindrical Measures*, Tata Institute of Fundamental Research, Oxford University, Academic Press, London, Orlando (1986) .

## Malliavin Calculus and SDDE's

[B] D.R. Bell, *The Malliavin Calculus*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 34, Longman, 1987.

[B-M.1] D.R. Bell and S.-E. A. Mohammed, The Malliavin calculus and stochastic delay equations, *J. Funct. Anal.* **99**, No. 1 (1991), 75–99.

[B-M.2] D.R. Bell and S.-E. A. Mohammed, An extension of Hörmander's theorem for infinitely degenerate parabolic operators, preprint.

[B-M.3] D.R. Bell and S.-E. A. Mohammed, Opérateurs paraboliques hypoelliptiques avec dégénérescences exponentielles, *C.R. Acad. Sci. Paris*, t. 317, Série I, (1993), 1059-1064.

[I-N] K. Itô and M. Nisio, On stationary solutions of a stochastic differential equation, *J. Math. Kyoto University*, 4-1 (1964), 1–75.

[G-S] I.I. Gihman and A.V. Skorohod, *Stochastic Differential Equations*, Springer-Verlag, Berlin/Heidelberg/New York, 1981.

[K-S.1] S. Kusuoka, and D.W. Stroock, Applications of the Malliavin calculus, I, *Taniguchi Sympos. SA Katata* (1982), 271–306.

[K-S.2] S. Kusuoka, and D.W. Stroock, Applications of the Malliavin calculus, II, *J. Fac. Sci. Univ. Tokyo, Sect. 1A Math.* **32**, No. 1 (1985), 1–76.

[M.1] S.-E.A. Mohammed, *Stochastic Functional Differential Equations*, Research Notes in Mathematics, Vol. 99, Pitman Advanced Publishing Program, Boston/London/Melbourne, 1984.

- [M.2] S.-E.A. Mohammed, Non-linear flows for linear stochastic delay equations, *Stochastics*, Vol.17, No. 3 (1986), 207-212.
- [S] D.W. Stroock, The Malliavin calculus, a functional analytic approach, *J. Funct. Anal.* 44 (1981), 212–257.
- [B] D.R. Bell, *The Malliavin Calculus*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 34, Longman, 1987.
- [B-M.1] D.R. Bell and S.-E. A. Mohammed, The Malliavin calculus and stochastic delay equations, *J. Funct. Anal.* 99, No. 1 (1991), 75–99.
- [B-M.2] D.R. Bell and S.-E. A. Mohammed, An extension of Hörmander’s theorem for infinitely degenerate parabolic operators, preprint.
- [B-M.3] D.R. Bell and S.-E. A. Mohammed, Opérateurs paraboliques hypoelliptiques avec dégénérescences exponentielles, *C.R. Acad. Sci. Paris*, t. 317, Série I, (1993), 1059-1064.
- [I-N] K. Itô and M. Nisio, On stationary solutions of a stochastic differential equation, *J. Math. Kyoto University*, 4-1 (1964), 1–75.
- [G-S] I.I. Gihman and A.V. Skorohod, *Stochastic Differential Equations*, Springer-Verlag, Berlin/Heidelberg/New York, 1981.
- [K-S.1] S. Kusuoka, and D.W. Stroock, Applications of the Malliavin calculus, I, *Taniguchi Sympos. SA Katata* (1982), 271–306.
- [K-S.2] S. Kusuoka, and D.W. Stroock, Applications of the Malliavin calculus, II, *J. Fac. Sci. Univ. Tokyo, Sect. 1A Math.* 32, No. 1 (1985), 1–76.
- [M.1] S.-E.A. Mohammed, *Stochastic Functional Differential Equations*, Research Notes in Mathematics, Vol. 99, Pitman Advanced Publishing Program, Boston/London/Melbourne, 1984.
- [M.2] S.-E.A. Mohammed, Non-linear flows for linear stochastic delay equations, *Stochastics*, Vol.17, No. 3 (1986), 207-212.
- [S] D.W. Stroock, The Malliavin calculus, a functional analytic approach, *J. Funct. Anal.* 44 (1981), 212–257.



## General

Antonelli, P.L. & Elliott, R.J., Nonlinear filtering theory for coral/starfish and plant/herbivore interactions, *Stochastic Analysis & Applications*4(1) (1986), 213–255.

Arnold, L. *Stochastic Differential Equations: Theory and Applications*, John Wiley and Sons, New York (1974).

Arnold, L., Papanicolaou, G., & Wihstutz, V., Asymptotic analysis of the Lyapunov exponent and rotation number of the random oscillator and application, *SIAM J. Applied Math.*46 (1986), 427–450.

Arnold, L. & Kliemann, W., Qualitative Theory of Stochastic Systems, *Probabilistic Analysis & Related Topics*, Vol. 3, Academic Press (1983), 1–79.

Arnold, L. & Kliemann, W., Large Deviations of Linear stochastic differential equations, H.J. Engelbert and W. Schmidt (eds.), *Stochastic differential Systems*, Lecture notes in Control & Information Sciences, Vol. 96, Springer-Verlag (1987), 117–151.

Arnold, L. & Lefever, R. (eds.), *Stochastic Nonlinear Systems in Physics, Chemistry & Biology*, Springer Series in Synergetics, Springer-Verlag, Berlin & New York (1981).

Arnold, L. & Wihstutz, V., Lyapunov exponents: A survey, in *Lyapunov Exponents: Bremen 1984*, Springer Lecture Notes in Mathematics 1186, (1986), 1–26.

Arnold, L. & Wihstutz, V. (eds.), *Lyapunov Exponents*, Proceedings, Bremen 1984, Lecture Notes in Mathematics 1186, Springer-Verlag (1986).

Arnold, L. Oeljeklaus, E. & Pardoux, E., Almost sure and moment stability of linear Itô equations, in *Lyapunov Exponents: Bremen 1984*, (1986), 129–159.

Arnold, L., Horsthemke, W. & Stucki, J., The influence of external real and white noise on the Lotka-Volterra model, *Biometrical Journal*, 21, (1979), 451–471.

Ashworth, M.J., *Feedback Design of Systems with Significant Uncertainty*, Research Studies Press (John Wiley & Sons) (1982).

Bailey, H.R. & Reeve, E.B., Mathematical models describing the distribution of  $I^{131}$  – albumin in man, *J. Lab. Clin. Med.* 60, (1962), 923–943.

Bailey, H.R. & Williams, M.Z., Some results on the differential difference equation  $\dot{x}(t) = \sum_{i=0}^N A_i x(t - T_i)$ , *J. Math Ana. Appl.* 15, (1966), 569–587.

Banks, H.T., The representation of solutions of linear functional diff'l equations, *J. Differential Equation*, 5 (1969), 399–410.

Baxendale, P.H., The Lyapunov spectrum of a stochastic flow of diffeomorphisms, *Springer Lecture Notes in Mathematics 1186*, Berlin-Heidelberg-New York-Tokyo (1986), 322–337.

Baxendale, P.H., Asymptotic behaviour of stochastic flows of diffeomorphisms: Two case studies, *Probab. Th. Rel. Fields* 73, (1986), 51–85.

Baxendale, P.H., Moment stability and large deviations for linear stochastic differential equations, in: Ikeda, N. (ed.), *Proceedings of the Taniguchi Symposium on Probabilistic Methods in Mathematical Physics*, Katata and Kyoto 1985, pp. 31–54, Tokyo: Kinokuniya (1987).

Bell, D. & Mohammed, S.E.A., The Malliavin calculus & stochastic delay Equations, *Journal of Functional Analysis* Vol. 99, No. 1, 1991, 75–99.

Bell, D. & Mohammed, S.-E.A., Regularity of a Non-Markov Ito process under  $p$ -th order degeneracy, (1991) (preprint :accepted for publication in the *Journal of Functional Analysis*).

Bellman, R. and Cooke, K.L., *Differential-Difference Equations*, Academic Press, New York-London (1963).

Bismut, J.-M, A generalized formula of Itô and some other properties of stochastic flows, *Z. Wahr. verw. Geb.*, 55, (1981), 331–350.

Bismut, J.-M, *Mécanique Aléatoire*, Springer Lecture Notes in Mathematics (1981).

Carverhill A.P., Chappell, M.J. & Elworthy, K.D., Characteristic exponents for stochastic flows, *Proceedings of BIBOS 1: Stochastic Processes*, Bielefeld, September 1984, Springer Lecture Notes in Mathematics.

Carverhill, A.P., A Markovian approach to the multiplicative ergodic (Oseledec) theorem for nonlinear stochastic dynamical systems, Mathematics Institute, University of Warwick, England (Preprint) (1984).

Carverhill, A.P., A formula for the Lyapunov numbers of a stochastic flow. Application to a perturbation theorem, *Stochastics*, 14, (1985), 209–226.

Carverhill, A.P., Survey: Lyapunov exponents for stochastic flows on manifolds in *Lyapunov Exponents: Bremen 1984*, Springer Lecture Notes in Mathematics 1186, 292–307.

Carverhill, A.P., Flows of stochastic dynamical systems: ergodic theory, *Stochastics*, 14, (1985), 273–318.

Carverhill, A.P. & Elworthy, K.D., Flows of stochastic dynamical systems: The functional analytic approach, *Z. Wahr. Verw. Gebiete*, 65, (1983), 245–267.

Carverhill, A.P. & Elworthy, K.D., Lyapunov exponents for a stochastic analogue of the geodesic flow, University of Warwick (1985) (preprint).

Coleman, B.D. & Mizel, V.J., Norms and semi-groups in the theory of fading memory, *Arch. Rat. Mech. Ana.*, 2, (1966), 87–123.

Coleman, B.D. & Mizel, V.J., On the general theory of fading memory, *Arch. Rat. Mech. Ana.*, 29, (1968), 18–31.

Cushing, J.M., *Integrodifferential Equations and Delay Models in Population Dynamics*, Lecture Notes in Biomathematics (#) 20, Springer-Verlag, Berlin (1977).

Daletskii, Yu.L., Infinite-dimensional elliptic operators and parabolic equations connected with them, *Russ. Math. Surveys*, 22, (1967), 1–53.

Dalke, S., *Invariante Mannigfaltigkeiten für Produkte Zufälliger Diffeomorphismen*, Dissertation, Institut Dynamische Systeme, Bremen (1989).

de Sam Lazzaro, & Meyer, P.A., Questions de théorie des flots, *Seminaire de Probab. 1X*, Springer Lecture Notes in Mathematics 465 (1975), 1–96.

Delfour, M.C. & Mitter, S.K., Hereditary differential systems with constant delays I. General Case, *J. Differential Equations*, 12, (1972), 213–235.

Delfour, M.C. & Mitter, S.K., Hereditary differential systems with constant delays II. A class of affine systems and the adjoint problem, *J. Differential Equations*, 18, (1975), 18–28.

Dorf, R.C., *Modern Control Systems*, Addison-Wesley (1967).

El'sgol'tz, L.E., *Introduction to the Theory of Differential Equations with Deviating Arguments*, Holden-Day, Inc. (1966).

Elworthy, K.D., *Stochastic Differential Equations on Manifolds*, LMS Lecture Notes Series 70, Cambridge University Press, Cambridge (1982).

Flandoli & Schaumlöffel, K. -U., Stochastic Parabolic Equations in bounded domains: Random evolution operator and Lyapunov exponents, *Stochastics and Stochastic Reports*, Vol. 29, No. 4 (1990), 461–485.

Furstenberg H. & Kesten, H., Products of random matrices, *Annals Math. Statist.*, 31, (1960), 457–469.

- Furstenberg, H., Noncommuting random products, *Trans. Amer. Math. Soc.*, 108, (1963), 377–428.
- Gihman, I.I. and Skorohod, A.V., *Stochastic Differential Equations*, Springer-Verlag, New York (1973).
- Halanay, A., *Differential Equations, Stability, Oscillations, Time-Lags*, Academic Press, (1966).
- Hale, J.K., *Theory of Functional Differential Equations*, Springer-Verlag, New York, Heidelberg-Berlin (1977).
- Hale, J.K., Linear functional differential equations with constant coefficients, *Cont. Diff. Eqns.*, 2, (1963), 291–319.
- Hale, J.K., Magalhães, L.T., & Oliva, W.M., *An Introduction to Infinite Dimensional Dynamical Systems-Geometric Theory*, Springer - Verlag, New York Inc., (1984).
- Has'minskii, R.Z., Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems, *Theory Prob. Appl.*, 12, (1967), 144–147.
- Has'minskii, R.Z., *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff (1980).
- Ikeda, N. & Watanabe, S., *Stochastic Differential Equations and Diffusion Processes*, North Holland-Kodansha, Amsterdam-Tokyo (1981).
- Itô, K., *Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces*, CBMS 47, SIAM (1984).
- Itô, K., Stochastic integral, *Proc. Imp. Acad. Tokyo*, 20, (1944), 519–524.
- Itô, K., On a formula concerning stochastic differentials, *Nagoya Math. J.*, 3, (1951), 55–65.
- Itô, K., On stochastic differential equations on a differentiable manifold, *Nagoya Math. J.*, 1, (1950), 35–47.
- Itô, K., *On Stochastic Differential Equations*, Mem. Amer. Math. Soc., 4, (1951).

Itô, K. & McKean, H.P., *Diffusion Processes and Their Sample Paths*, Springer-Verlag, Berlin (1965).

Itô, K. & Nisio, M., On stationary solutions of a stochastic differential equation, *J. Math. Kyoto University*, 4-1, (1964), 1–75.

Jones, G.S., Asymptotic fixed point theorems and periodic solutions of functionals differential equations, *Cont. Diff. Eqns.*, 2, (1963), 385–405.

Kallianpur, G., *Stochastic Differential Equations in Duals of Nuclear Spaces With Some Applications*, I.M.A., University of Minnesota, Preprint #244 (1986).

Kallianpur, B. and Wolpert R., Infinite dimensional stochastic differential equations models for spatially distributed neurons, *Appl. Math. and Opt.*, 12, (1984), 125–172.

Kifer, Y., *Ergodic Theory of Random Transformations*, Birkhäuser, Boston-Basel-Stuttgart (1985).

Kifer, Y., Characteristic exponents for random homeomorphisms of metric spaces, *Lyapunov Exponents: Bremen 1984*, Springer Lecture Notes in Mathematics, 1186, (1986), 74–84.

Kifer, Y., The exit problem for small random perturbations of dynamical systems with a hyperbolic fixed point, *Israel J. of Math.*, 40, #1, (1981), 74–96.

Kolmanovski, V.B., and Nosov, V. R., *Stability of Functional Differential Equations*, Academic Press (1986).

Krasovskii, N., *Stability of Motion*, Moscow (1959) (Translated by Stanford University Press, 1963).

Kubo, R., The fluctuation-dissipation theorem and Brownian motion, in *Many-Body Theory*, Edited by R. Kubo, Syokabo & Benjamin (1966), 1–16.

Kunita, H., On the Decomposition of Solutions of Stochastic Differential Equations, *Lecture Notes in Mathematics 851*, Springer-Verlag, Berlin-Heidelberg-New York (1981), 213–255.

Kunita, H., On backward stochastic differential equations, *Stochastics* (1981)

Kunita, H., Stochastic Differential Equations and Stochastic Flows of Diffeomorphisms, *Lecture Notes in Mathematics 1097*, Springer, Berlin-Heidelberg-New York-Tokyo (1984), 143–303.

Kushner, H., On the stability of processes defined by stochastic differential-difference equations, *J. Differential Equations*, 4, (1968), 424–443.

Kusuoka, S. & Stroock, D., Applications of the Malliavin calculus I, *Taniguchi Sympos. SA Katata* (1982), 271–306.

Kusuoka, S. & Stroock, D., Applications of the Malliavin calculus II, *J. Fac. Sci. Univ. Tokyo Sect. 1A Math.* 32, No. 1 (1985), 1–76.

Le Jan, Y., Equilibre et exposant de Lyapunov de certains flots browniens, *C.R. Acad. Sciences Paris*, t. 298 Série 1 (1984), 361–364.

Leitman, M.J. & Mizel, V.J., On fading memory spaces and hereditary integral equations, *Arch. Rat. Mech. Ana.* 55, (1974), 18–51.

Lenhart, S.M. & Travis, C., Global stability of a biological model with time delay, *Proceedings of the American Mathematical Society*, 96-1, (1986), 75–78.

London, W.P. & Yorke, J.A., Recurrent epidemics of measles, chickenpox and mumps I: Seasonal variation in contact rates., *Amer. J. Epid.*, 98, (1973).

Mallet-Paret, J., Generic and Qualitative Properties of Retarded Functional Differential Equations, *Meeting Func. Diff. Eqns. Braz. Math. Soc.*, São Carlos, July 1975.

Mallet-Paret, J., Generic properties of R.F.D.E.'s, *Bull. Amer. Math. Soc.*, 81, (1975), 750–752.

Malliavin, P., Stochastic Calculus of Variation and Hypoelliptic Operators, *Proc. Intern. Symp. Stoch. Diff. Eqns.*, Kyoto 1976, Edited by K. Itô, Wiley, Tokyo-New York (1978), 195–263.

Mañé, R., Lyapunov exponents and stable manifolds for compact transformations, *Springer Lecture Notes in Mathematics 1007*, (1983), 522–577.

Mao, X., Exponential Stability for delay Itô equations, *Proceedings of IEEE International Conference on Control and Application*, April 1989.

Marcus M. & Mizel, V.J., Stochastic functional differential equations modelling materials with selective recall, *Stochastics*, 25, No.4 (1988), 195–232.

Markus, L., Controllability in topological dynamics, *Proceedings of the International Congress of Mathematicians* (Vancouver B.C. 1974), Vol. 2, 355–360.

Markus, L. and Meyer K.R., Periodic orbits and solenoids in generic Hamiltonian dynamical systems, *Amer. J. Math.*, 102, (1980), 25–92.

Markus, L. and Weerasinghe, A., Stochastic oscillators, University of Minnesota Mathematics Report 85-123 (1985) (preprint).

McDonald, N., *Time Lags in Biological Models*, Lecture Notes in Biomathematics, 27, Springer-Verlag, Berlin (1978).

McKean, H.P., *Stochastic Integrals*, Academic Press, New York (1969).

McShane, E.J., *Stochastic Calculus and Stochastic Models*, Academic Press, London-New York (1974).

Mees, A.I., *Dynamics of Feedback Systems*, John Wiley and Sons (1981).

Metivier, M. and Pellaumail, J., *Stochastic Integration*, Academic Press, London-New York (1980).



Millionscikov, V.M., On the spectral theory of nonautonomous linear systems of differential equations, *Trans. Moscow Math. Soc.*, 18, (1968), 161–206.

Mishkis, A.D., *General Theory of Diff'l Equations with a Retarded Argument*, A.M.S. Translations No. 55 (1951).

Mizel, V.J. & Trutzer, V., Stochastic hereditary equations: existence and asymptotic stability, *Journal of Integral Equations*, 7, (1984), 1–72.

Mohammed, S.E.A., *Retarded Functional Differential Equations: A Global Point of View*, Research Notes in Mathematics, 21, Pitman Books Ltd., London-San Francisco-Melbourne (1978), pp. 147.

Mohammed, S.E.A., Stability of linear delay equations under a small noise, *Proceedings of Edinburgh Mathematical Society*, 29, (1986), 233–254.

Mohammed, S.E.A., Markov solutions of stochastic functional differential equations, *Conference on Functional Differential Systems and Related Topics*, Polish Academy of Sciences (1981).

Mohammed, S.E.A., *Stochastic Functional Differential Equations*, Research Notes in Math, # 99, Pitman Advanced Publishing Program, Boston-London-Melbourne (1984), pp. 245.

Mohammed, S.E.A., Non-linear flows for linear stochastic delay equations, *Stochastics*, Vol. 17, # 3, (1986), 207–212.

Mohammed, S.E.A., The infinitesimal generator of a stochastic functional differential equation, *Springer Lecture Notes in Mathematics 964*, Springer-Verlag (1982), 529–538.

Mohammed, S.E.A., The infinitesimal generator of a stochastic functional differential equation, *Springer Lecture Notes in Mathematics 964*, Springer-Verlag (1982), 529–538.

Mohammed, S.E.A., Stochastic F.D.E.'s and Markov Processes, I, II, (Preprints) School of Math. Sciences, U. of Khartoum (1978).

Mohammed, S.E.A., Unstable invariant distributions for a class of stochastic delay equations, *Proceedings of Edinburgh Mathematical Society*, 31, pp. 1–23 (1988).

Mohammed, S.E.A., The Lyapunov spectrum and stable manifolds for stochastic linear delay equations, *Stochastics & Stochastic Reports*, Vol. 19 (1990), 89–131.

Mohammed, S.E.A., & Scheutzow, M., Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales, Part I: The multiplicative ergodic theory, (preprint) (1990).

Mohammed, S.E.A., Lyapunov exponents & stochastic flows of linear & affine hereditary systems: A survey, *Proceedings of International Conference on Diffusion Processes and Related Problems in Analysis, Vol. II*, (1991), edited by M. Pinsky & V. Wihstutz, Birkhäuser (1992), 141–169.

Mohammed, S.E.A., Scheutzow, M. & Weizsäcker, H.V., Hyperbolic state space decomposition for a linear stochastic delay equation, *SIAM Journal on Control and Optimization*, 24-3, (1986), 543–551.

Mohammed, S.E.A., Scheutzow, M., Lyapunov exponents and stationary solutions for affine stochastic delay equations, *Stochastics and Stochastic Reports*, Vol. 29, No. 2 (1990), 259–283.

Nussbaum, R., Periodic solutions of some non-linear autonomous functional differential equations, *Ann. Mat. Pura Appl.* 10 (1974), 263–306.

Oliva, W.M., Functional differential equations on compact manifolds and an approximation theorem, *J. Differential Equations* 5 (1969), 483–496.

Oliva, W.M., Functional Differential Equations – Generic Theory, in *Dynamical Systems – An International symposium*, vol. 1, Academic Press (1976), 195–209.

Orey, S., Stationary solutions for linear systems with additive noise, *Stochastics* 5, (1981), 241–251.

Oseledec, V.I., A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems, *Trans. Moscow Math Soc.* 19, (1968), 197–231.

Pesin, Y.B., Lyapunov characteristic exponents and smooth ergodic theory, *Russian Math. Survey*, 32, (1977), 55–114.

Pinsky, M.A., Stochastic stability and the Dirichlet problem, *Comm. Pure Appl. Math.* 27, (1974), 311–350.

Pinsky, M.A., Instability of the harmonic oscillator with small noise, *SIAM J. Appl. Math.*, 46 (1986), 451–463.

Pinsky, M. & Wihstutz, V., Lyapunov exponents of nilpotent Itô systems, *Stochastics*, 25 (1988), 43–57.

Prajneshu., Time-dependent solution of the logistic model for population growth in random environment, *J. Appl. Prob.*, 17, (1980), 1083–1086.

Protter, Semimartingales and measure-preserving flows, *Ann. Inst. Henri Poincaré'*, Vol. 22, no. 2, 1986 (127–147).

Pugh, C. & Shub, M., Ergodic attractors, *Trans. Amer. Math. Soc.*, 312 (1989), 1–54.

Rozanov, J.A., Infinite-dimensional Gaussian distribution, American Mathematical Society Trans., *Proceedings of the Steklov Institute of Mathematics* 108 (1971).

Ruelle, D., *Elements of Differentiable Dynamics and Bifurcation Theory*, Academic Press, Inc. (1989).

Ruelle, D., Characteristic exponents and invariant manifolds in Hilbert space, *Annals of Mathematics* 115, (1982), 243–290.

Ruelle, D., Ergodic theory of differentiable dynamical systems, *I.H.E.S. Publications* 50, (1979), 275–305.

- Ruelle, D. & Eckmann, I., Ergodic theory of chaos and strange attractors, *Reviews of Modern Physics* 57, (1985), 617–656.
- Scheutzow, M., Qualitative behaviour of stochastic delay equations with a bounded memory, *Stochastics* 12, (1984), 41–80.
- Skorohod, A.V., *Random Linear Operators*, D. Reidel Publishing Company (1984).
- Stroock, D.W. & Varadhan, S.R.S., *Multidimensional Diffusion processes*, Springer-Verlag, Berlin-Heidelberg-New York (1979).
- Thieullen, P. Fibres dynamiques asymptotiquement compacts exposants de Lyapunov. Entropie. Dimension, *Ann. Inst. Henri Poincaré*, *Anal. Non Linéaire*, 4(1), (1987), 49–97.
- Van Kampen, N.G., *Stochastic Processes in Physics and Chemistry*, North-Holland Publishing Company, Amsterdam-New York-Oxford (1981).
- Varadhan, S.R.S., *Large Deviations and Applications*, Philadelphia, SIAM (1984).
- Waltman, P., *Deterministic Threshold Models in the Theory of Epidemics*, Lecture Notes in Biomathematics, Vol. 1, Springer-Verlag (1974).
- Wihstutz, V., Parameter dependence of the Lyapunov exponent for linear stochastic systems. A survey, *Springer Lecture Notes in Mathematics* 1186, (1986), 200–215.

## Stochastic delay equations

[1] 1 379 172 Bell, Denis R.; Mohammed, Salah-Eldin A. Smooth densities for degenerate stochastic delay equations with hereditary drift. *Ann. Probab.* 23 (1995), no. 4, 1875–1894.60H07 (60H10 60H20 60H30)

[2] 92k:60124 Bell, Denis R.; Mohammed, Salah Eldin A. The Malliavin calculus and stochastic delay equations. *J. Funct. Anal.* 99 (1991), no. 1, 75–99. (Reviewer: David Nualart) 60H10 (60H07)

[3] 92a:60148 Mohammed, Salah Eldin A.; Scheutzow, Michael K. R. Lyapunov exponents and stationary solutions for affine stochastic delay equations. *Stochastics Stochastics Rep.* 29 (1990), no. 2, 259–283. (Reviewer: Wolfgang Kliemann) 60H20 (34D08 93D05)

[4] 91c:49045 Flandoli, Franco Solution and control of a bilinear stochastic delay equation. *SIAM J. Control Optim.* 28 (1990), no. 4, 936–949. (Reviewer: Negash G. Medhin) 49K45 (60H20 93C20)

[5] 90b:60068 Mohammed, S. E. A. Unstable invariant distributions for a class of stochastic delay equations. *Proc. Edinburgh Math. Soc.* (2) 31 (1988), no. 1, 1–23. (Reviewer: Stanislaw Wpolhk edrychowicz) 60H10 (34F05 34K05)

[6] 89b:60154 Tudor, C.; Tudor, M. On approximation of solutions for stochastic delay equations. *Stud. Cerc. Mat.* 39 (1987), no. 3, 265–274. (Reviewer: Seppo Heikkilä) 60H20 (34F05 34K05)

[7] 88j:60082 Chang, Mou Hsiung Discrete approximation of nonlinear filtering for stochastic delay equations. *Stochastic Anal. Appl.* 5 (1987), no. 3, 267–298. (Reviewer: Maurizio Pratelli) 60G35 (60H10 93E11 93E25)

[8] 87j:60097 Mohammed, S.; Scheutzow, M.; von Weizsäcker, H. Hyperbolic state space decomposition for a linear stochastic delay equation. *SIAM J. Control Optim.* 24 (1986), no. 3, 543–551. (Reviewer: Ludwig Arnold) 60H99 (34K20)

[9] 85h:60087 Scheutzow, M. Qualitative behaviour of stochastic delay equations with a bounded memory. *Stochastics* 12 (1984), no. 1, 41–80. (Reviewer: Toshio Yamada) 60H10

[10] 80a:60083 Chojnowska-Michalik, Anna Representation theorem for general stochastic delay equations. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* 26 (1978), no. 7, 635–642. (Reviewer: I. M. Stoyanov) 60H20 (34G99 34K99)

## Stochastic functional differential equations

[1] 1 373 727 Mohammed, Salah-Eldin A.; Scheutzow, Michael K. R. Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales. I. The multiplicative ergodic theory. *Ann. Inst. H. Poincaré Probab. Statist.* 32 (1996), no. 1, 69–105.60H99

[2] 1 355 171 Yasinskii, V. K.; Yurchenko, I. V. Asymptotic stability in the mean square of the trivial solution of systems of stochastic functional-differential equations with discontinuous trajectories. (Ukrainian) *Dopov./Dokl. Akad. Nauk Ukraïni* 1994, no. 11, 24–28.34K50 (34K20 60H99)

[3] 1 318 857 Turo, Jan Successive approximations of solutions to stochastic functional-differential equations. *Period. Math. Hungar.* 30 (1995), no. 1, 87–96.60H10 (34Kxx)

[4] 1 308 293 Kolmanovskii, V.; Shaikhet, L. New results in stability theory for stochastic functional-differential equations (SFDEs) and their applications. *Proceedings of Dynamic Systems and Applications*, Vol. 1 (Atlanta, GA, 1993), 167–171, Dynamic, Atlanta, GA, 1994. 60Hxx (34Kxx 93E15)

[5] 96f:60110 Wang, Xiang Dong; Zhang, Dao Fa Existence of solutions for stochastic functional-differential equations with continuous coefficients. (Chinese) *J. Math. Res. Exposition* 15 (1995), no. 2, 267–270. (Reviewer: Binggen Zhang) 60H99

[6] 95i:34153 Kadiev, R. I. The admissibility of pairs of spaces with respect to some of the variables for linear stochastic functional-differential equations. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.* 1994, no. 5, 13–22. (Reviewer: Binggen Zhang) 34K50 (34K20 60H99)

[7] 94i:60069 Xu, Jia Gu Weak and strong solutions of generalized Itt's stochastic functional-differential equations. *Acta Math.*

Sinica (N.S.) 9 (1993), no. 3, 265–277. (Reviewer: Vivek S. Borkar) 60H10 (34K50)

[8] 93j:60091 Kadiev, R. I. Stability with respect to some of the variables of linear stochastic functional-differential equations. (Russian) Functional-differential equations (Russian), 140–148, Perm. Politekh. Inst., Perm, 1991. 60H99 (34K20 34K50)

[9] 93i:60119 Bainov, D. D.; Kolmanovskii, V. B. Periodic solutions of stochastic functional-differential equations. Math. J. Toyama Univ. 14 (1991), 1–39. 60H99 (34K15 34K20 34K50)

[10] 93i:34147 Kadiev, R. I.; Ponosov, A. V. Stability of linear stochastic functional-differential equations with constantly acting perturbations. (Russian) Differentsialnye Uravneniya 28 (1992), no. 2, 198–207, 364; translation in Differential Equations 28 (1992), no. 2, 173–179 34K50 (34K20)

[11] 93h:60102 Nechaeva, I. G.; Khusainov, D. Ya. Obtaining estimates for the stability of solutions of stochastic functional-differential equations. (Russian) Differentsialnye Uravneniya 28 (1992), no. 3, 405–414, 547; translation in Differential Equations 28 (1992), no. 3, 338–346 60H99 (34K20 34K50)

[12] 93g:49007 Govindan, T. E.; Joshi, M. C. Stability and optimal control of stochastic functional-differential equations with memory. Numer. Funct. Anal. Optim. 13 (1992), no. 3-4, 249–265. 49J25 (34K35 49K45 93E15 93E20)

[13] 92e:60129 Nechaeva, I. G.; Khusainov, D. Ya. Exponential estimates for solutions of linear stochastic functional-differential equations. (Russian) Ukrain. Mat. Zh. 42 (1990), no. 10, 1338–1343; translation in Ukrainian Math. J. 42 (1990), no. 10, 1189–1193 (1991) (Reviewer: Binggen Zhang) 60H99

[14] 91j:60112 Zhang, Dao Fa; Wang, Xiang Dong Some notes on stochastic functional-differential equations. (Chinese) Qufu Shifan Daxue Xuebao Ziran Kexue Ban 16 (1990), no. 3, 24–27. 60H99

[15] 91i:60169 Tsarkov, E. F.; Yasinskii, V. K. Nonlinear stochastic functional-differential equations. (Russian) Akad. Nauk Ukrain. SSR Inst. Mat. Preprint 1989, no. 32, 27–46. (Reviewer: Binggen Zhang) 60H99 (47H99)

- [16] 91g:60075 Marcus, Moshe; Mizel, Victor J. Stochastic functional differential equations modelling materials with selective recall. *Stochastics* 25 (1988), no. 4, 195–232. (Reviewer: S. E. A. Mohammed) 60H99 (34F05 34K99 35R60 93E03)
- [17] 91d:60150 Rodkina, A. E. Stochastic functional-differential equations with respect to a semimartingale. (Russian) *Differentsialnye Uravneniya* 25 (1989), no. 10, 1716–1721, 1835; translation in *Differential Equations* 25 (1989), no. 10, 1195–1120 (1990) (Reviewer: V. G. Kolomiets) 60H99 (34F05 34K99 60G44)
- [18] 90k:60113 Rodkina, A. E. A proof of the solvability of nonlinear stochastic functional-differential equations. (Russian) *Global analysis and nonlinear equations (Russian)*, 127–133, 174, *Novoe Global. Anal., Voronezh. Gos. Univ., Voronezh*, 1988. 60H10 (34F05 34K99)
- [19] 90f:60116 Kadiev, R. I. Solvability of the Cauchy problem for stochastic functional-differential equations. (Russian) *Functional-differential equations (Russian)*, 75–83, vi, *Perm. Politekh. Inst., Perm*, 1987. 60H99
- [20] 89m:60142 Rodkina, A. E. The averaging principle for stochastic functional-differential equations. (Russian) *Differentsialnye Uravneniya* 24 (1988), no. 9, 1543–1551, 1653; translation in *Differential Equations* 24 (1988), no. 9, 1014–1021 (Reviewer: Constantin Tudor) 60H20 (34C29)
- [21] 89e:93215 Sverdan, M. L.; Yasinskaya, L. I. Exponential stability of stochastic functional-differential equations under constantly acting random perturbations. (Russian) *Dokl. Akad. Nauk Ukrain. SSR Ser. A* 1987, no. 4, 18–21.93E15 (34F05 60H99)
- [22] 89b:60153 Tudor, C. On stochastic functional-differential equations. *Differential equations and applications, I, II (Russian)* (Ruse, 1985), 971–974, ‘Angel Kancev’ *Tech. Univ., Ruse*, 1987. (Reviewer: Seppo Heikkilä) 60H20 (34K05)
- [23] 88m:60173 Tudor, Constantin On stochastic functional-differential equations with unbounded delay. *SIAM J. Math. Anal.* 18 (1987), no. 6, 1716–1725. (Reviewer: B. G. Pachpatte) 60H99



[24] 87a:34066 Sverdan, M. L.; Yasinskaya, L. I. Exponential stability in the mean square of solutions of stochastic functional-differential equations in the presence of Poisson perturbations. (Russian) Akad. Nauk Ukrain. SSR Inst. Mat. Preprint 1985, no. 41, 20–26.34F05 (34K20)

[25] 86j:60151 Mohammed, S. E. A. Stochastic functional differential equations. Research Notes in Mathematics, 99. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1984. vi+245 pp. ISBN: 0-273-08593-X (Reviewer: M. Mitivier) 60H99 (34F05)

[26] 86f:60079 Chang, Mou Hsiung On Razumikhin-type stability conditions for stochastic functional differential equations. Math. Modelling 5 (1984), no. 5, 299–307. (Reviewer: S. M. Khrisanov) 60H25 (34F05)

[27] 84g:34119 Mohammed, S. E. A. The infinitesimal generator of a stochastic functional-differential equation. Ordinary and partial differential equations (Dundee, 1982), pp. 529–538, Lecture Notes in Math., 964, Springer, Berlin-New York, 1982. (Reviewer: G. S. Ladde) 34K05 (34F05 60H25)

[28] 83a:60096 Sasagawa, I. Existence of stationary and periodic solutions of stochastic functional-differential equations. Proceedings of the VIIIth International Conference on Nonlinear Oscillations, Vol. II (Prague, 1978), pp. 603–608, Academia, Prague, 1979. 60H10 (34F05)

[29] 82k:60128 Jasinskaja, L. I. Mean square stability of the trivial solution of linear stochastic functional-differential equations with variable coefficients. (Russian) Ukrain. Mat. Zh. 33 (1981), no. 4, 482–488. (Reviewer: D. Bobrowski) 60H10 (93E15)

[30] 82g:93049 Jasinskii, V. K. Stability of solutions of linear stochastic functional-differential equations with variable coefficients. (Russian) Analytic methods for investigating nonlinear oscillations (Russian), pp. 169–177, 185, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1980. 93E15

[31] 82b:60076 Jasinskii , V. K. Stability of the almost surely trivial solution of stochastic functional-differential equations. (Russian) Asymptotic methods of nonlinear mechanics (Russian), pp. 176–184, 200, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1979. (Reviewer: G. N. Milshtein) 60H25 (34F05)

[32] 82a:60087 Jasinskaja, L. I.; Jasinskii , V. K. Asymptotic mean-square stability of the trivial solution of stochastic functional-differential equations. (Russian) Ukrain. Mat. Zh. 32 (1980), no. 1, 89–98, 143. (Reviewer: D. Bobrowski) 60H10 (34K20)

[33] 80m:49023 Chang, Mou Hsiung Optimal control for stochastic functional differential equations. Bull. Inst. Math. Acad. Sinica 7 (1979), no. 1, 21–37. (Reviewer: Virginia M. Warfield) 49B60 (34F05)

[34] 58 #13349 Carkov, E. F. Stability in the first approximation of the trivial solution of stochastic functional-differential equations. (Russian) Mat. Zametki 23 (1978), no. 5, 733–738. (Reviewer: Mou-Hsiung Chang) 60H10 (34K20)

[35] 58 #9692 Carkov, E. F. Exponential  $p$ -stability of the trivial solution of stochastic functional-differential equations. (Russian) Teor. Veroyatnost. i Primenen. 23 (1978), no. 2, 445–448. (Reviewer: I. M. Stojanov) 93E15 (60H20)

[36] 58 #6572 Carkov, E. F. Asymptotic stability of the solutions of stochastic functional-differential equations. (Russian) Approximate methods for the study of nonlinear systems (Russian), pp. 221–232. Izдание Inst. Mat. Akad. Nauk SSR, Kiev, 1976. 34F05 (60H10)

[37] 58 #3046 Carkov, E. F. The structure of the spectrum of the generating operator of a semigroup of covariance operators of the solution of stochastic functional-differential equations. (Russian) Latvinsk. Mat. Ezegodnik Vyp. 21 (1977), 99–107, 236. (Reviewer: N. N. Vahaniya (Vakhania)) 60H99

[38] 57 #10827 Chang, Mou Hsiung On existence and stability results of a class of stochastic functional differential equations. Bull. Inst. Math. Acad. Sinica 4 (1976), no. 2, 209–228. (Reviewer: Kunio Nishioka) 60H99 (34K05 34K20)

[39] 55 #4381 Carkov, E. F. Mean-square asymptotic exponential stability of the trivial solution of stochastic functional-differential equations. (Russian) *Teor. Verojatnost. i Primenen.* 21 (1976), no. 4, 871–875. (Reviewer: I. M. Stojanov) 60H99 (93E15)

[40] 50 #11443 Chang, M. H.; Ladde, G.; Liu, P. T. Stability of stochastic functional differential equations. *J. Mathematical Phys.* 15 (1974), 1474–1478. (Reviewer: E. F. Carkov) 60H10 (34KXX)

[41] 49 #8105 Ladde, G. S. Differential inequalities and stochastic functional differential equations. *J. Mathematical Phys.* 15 (1974), 738–743. (Reviewer: A. T. Bharucha-Reid) 60H10 (34F05)

[42] 49 #6352 Bikis, M. A.; Sadovjak, A. M.; Carkov, E. F. Continuity with respect to a parameter of the solutions of stochastic functional-differential equations. (Russian) *Latvian mathematical yearbook*, 12 (Russian), pp. 39–49. Izdat. "Zinatne", Riga, 1973. (Reviewer: G. S. Ladde) 60H10 (34F05)

[43] 1 222 665 Ponosov, A. V. On the theory of reducible stochastic functional-differential equations with respect to a semimartingale. (Russian) *Functional-differential equations (Russian)*, 111–120, Perm. Politekh. Inst., Perm, 1990. 60H99 (34K50)

[44] 1 221 339 Sergeev, V. A. The property of exponential  $p$ -stability of the trivial solution of a periodic stochastic functional-differential equation. (Russian) *Boundary value problems (Russian)*, 102–106, Perm. Politekh. Inst., Perm, 1991. 34K50 (34K20 60H99)

[45] 1 219 110 Ponosov, A. V. Canonical reducibility of stochastic functional-differential equations. (Russian) *Boundary value problems (Russian)*, 75–79, Perm. Politekh. Inst., Perm, 1989. 34K50

[46] 1 188 476 Yasinsky, V. K. On strong solutions of stochastic functional-differential equations with infinite aftereffect. *Proceedings of the Latvian Probability Seminar*, Vol. 1 (Riga, 1991), 189–215, Riga Tech. Univ., Riga, 1992. 60H99

[47] 1 188 474 Tsarkov, Je.; Yanson, V. A. Investigation of stability of stochastic functional-differential equations with periodic coefficients. *Proceedings of the Latvian Probability Seminar*, Vol. 1 (Riga, 1991), 154–174, Riga Tech. Univ., Riga, 1992. 60H99