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***Abstract*— A recent paper has proved that the classical receiver for coherent detection of differentially encoded M-PSK in AWGN is optimum for the MAP sequence detection criterion. In this letter, we show that asymptotically, as SNR tends to zero, the MAP symbol detection criterion receiver is equivalent to the classical differentially coherent receiver, for M greater than two. An asymptotic relative efficiency figure of merit is defined in order to compare the performances of the classical coherent receiver and the classical differentially coherent receiver.**

***Index Terms*— Differential phase shift keying, maximum $a\ posteriori$ (MAP) symbol detection, differentially coherent receiver, asymptotic relative efficiency.**

I. Introduction

In a paper on coherent detection of differentially encoded M-PSK signals received in AWGN [1], Simon and Divsalar proved that the classical (C) receiver, hitherto assumed to be optimum, is not optimum in the sense of a minimum symbol error probability criterion. In the same paper they derived the maximum $a\ posteriori$ (MAP) receiver under the observation of two consecutive received samples and predicted that both the receivers would exhibit nearly identical symbol error rates (SER) for a large signal-to-noise ratio (SNR). Very recently [2], Colavolpe proved that the classical receiver is optimum under the MAP sequence detection criterion. It was also pointed out that the receiver of [1] is optimum under the MAP symbol detection criterion (MAP-SY) only when the M-ary information symbol phases are independent and uniformly distributed on the set, $\psi = (0, \frac{2\pi}{M}, \frac{2\pi(2)}{M}, .., \frac{2\pi(M-1)}{M})$. Reference [2] concluded using computer simulations that the two detection schemes achieve practically identical SERs. In this letter it is assumed that the information symbols are independent and uniformly distributed on the set ψ. We derive the analytical SER expressions for the C and the MAP-SY receivers under very low SNR conditions. This derivation leads to an asymptotic relative efficiency(ARE) figure of merit, as the SNR tends to zero.

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II. Asymptotic (SNR→ 0) Behavior of the MAP-SY Procedure

We follow closely the notation used in [1] for the detection of a differential M-PSK signal received in AWGN of two-sided power spectral density $\frac{N_o}{2}$. Assuming perfect carrier synchronization and an equivalent baseband representation for the received signal over the k^{th} symbol time, $(k-1)T_s \le t \le kT_s$, the sampled matched filter output takes the following form:

$$R_k = |R_k| e^{j\eta_k} = S_k + N_k \tag{1}$$

where N_k is the complex noise with the real and imaginary parts of N_k being $i.i.d$ as Gaussian having zero mean and variance $\sigma^2 = \frac{N_o}{2}$, $S_k = \sqrt{2P} e^{j\Theta_k}$, $\Theta_k = \Theta_{k-1} + \Delta\theta_k$, Θ_k is the differentially encoded transmitted phase, and $\Delta\theta_k$ is the phase of the M-PSK information symbol. The notation in (1) is similar to that in [1], except that we have dropped for convenience the tilda($\sim$) superscripts in [1]. From [1]-[2] the optimal MAP-SY detection procedure is given by

$$\text{Choose } \hat{\Delta\theta}_k = \arg \max_{\Delta\theta_k \in \psi} \sum_{i=1}^{M} F_i \tag{2}$$

$F_i = \exp\left\{ \frac{\sqrt{2P}}{\sigma_n^2} \left[\text{Re}\left\{ e^{-j\beta_i} (R_{k-1} + R_k e^{-j\Delta\theta_k}) \right\} \right] \right\}$, $\beta_i = \frac{2\pi(i-1)}{M}$, i=1,2,..,M, i.e., it assumes values on the set ψ. The signal power-to-noise power ratio can be defined as SNR $= \frac{P}{\sigma_n^2} = \frac{PT_s}{\sigma_n^2 T_s} = \frac{E_s}{N_o}$. For convenience, let $a = \sqrt{\frac{E_s}{N_o}}$. Asymptotically, as a tends to zero, the exponential term inside the summation in (2) can be expanded in a Taylor series yielding the equivalent test:
Choose $\Delta\theta_k \in \psi$ that maximizes

$$\begin{aligned} \sum_{i=1}^{M} \frac{\sqrt{2P}}{\sigma_n^2} & \left[|R_{k-1}| \cos(\eta_{k-1} - \beta_i) + |R_k| \cos(\eta_k - \beta_i - \Delta\theta_k) \right] \\ & + \frac{P}{\sigma_n^4} \left[|R_{k-1}| \cos(\eta_{k-1} - \beta_i) + |R_k| \cos(\eta_k - \beta_i - \Delta\theta_k) \right]^2 \\ & + 0(a^2) \end{aligned} \tag{3}$$

where $0(a^2)$ contains terms that go to zero faster than a^2 , as a^2 tends to zero. Using the following results,

$$\sum_{\beta_i \in \psi} \cos(\beta_i) = \sum_{\beta_i \in \psi} \sin(\beta_i) = \sum_{\beta_i \in \psi} \sin(2\beta_i) = 0$$
$$\sum_{\beta_i \in \psi} \cos(2\beta_i) = \begin{cases} 0 & M > 2 \\ 2 & M = 2. \end{cases} \quad (4)$$

it can be seen that the first term inside the summation of (3) is zero and that the second term yields the following equivalent test for the asymptotic case.

$$\text{Choose } \hat{\Delta\theta_k} = \arg \max_{\Delta\theta_k \in \psi} \cos(\eta_k - \eta_{k-1} - \Delta\theta_k) \qquad M > 2$$
$$\text{Decide } \Delta\theta_k = \begin{cases} 0 & \text{if } \cos(\eta_k)\cos(\eta_{k-1}) > 0 \\ \pi & \text{otherwise.} \end{cases} \qquad M = 2 \quad (5)$$

For $M > 2$, the asymptotic test procedure is to decide that value of $\Delta\theta_k$ which is closest to (η_k - η_{k-1})mod2π. This is the classical differentially coherent (DC) test for differentially encoded M-PSK [3]. The procedure, for M = 2, is the classical receiver. The equivalence between the MAP-SY and the C receivers for M=2 and any SNR was shown in [1].

III. Asymptotic (SNR$\to 0$) SER for the DC and the C Receivers

A. DC Receiver

The symbol error probability for the DC receiver is given by ((7.7) in [3])

$$P_s = 1 - P_c^{DC} = \frac{\sin\frac{\pi}{M}}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\exp\left(-a^2(1-\cos(\frac{\pi}{M})\cos(t))\right)}{1-\cos(\frac{\pi}{M})\cos(t)} dt \quad (6)$$

The Taylor series expansion of the exponential term in the integrand with respect to a gives the probability of correct decision as

$$\lim_{a\to 0} P_c^{DC} = \frac{1}{M} + \frac{a^2}{2}\sin(\frac{\pi}{M}) \quad (7)$$

B. Classical Receiver

Using the equations (4.197) and (4.199) in [3], an asymptotic expression for the probability of correct decision for the classical receiver can be obtained. The Taylor series expansion of $e^{-2ar\cos(\theta)}$ in equation (4.197) of [3] with respect to a followed by routine manipulations yield an asymptotic expression for the probability of correct decision for the classical receiver:

$$\lim_{a\to 0} P_c^C = \begin{cases} \frac{1}{M} + a^2 \frac{M}{2\pi} \sin^2(\frac{\pi}{M}) & M > 2 \\ \frac{1}{2} + a^2 \frac{2}{\pi} & M = 2. \end{cases} \quad (8)$$

A useful measure of comparison of performances of the DC and the C receivers is the asymptotic relative efficiency (ARE) of the C receiver with respect to the DC receiver defined by

$$ARE = \lim_{a\to 0} \frac{P_c^C - \frac{1}{M}}{P_c^{DC} - \frac{1}{M}} = \begin{cases} \frac{M}{\pi}\sin(\frac{\pi}{M}) & M > 2 \\ \frac{4}{\pi} & M = 2. \end{cases} \quad (9)$$

IV. Discussion

The ARE of the C receiver with respect to the MAP-SY receiver is 1 for $M = 2$ and is the same as (9) for $M > 2$. From (9) the ARE of the C receiver with respect to the MAP-SY receiver is 0.9 for M = 4, 0.9745 for M = 8 and is very close to 1 for $M \geq 16$. Under a weak signal condition, the MAP-SY receiver performs slightly better than the classical one when M = 4 but they exhibit nearly identical performance when $M \geq 8$. Colavolpe [2] differentiates between practically identical performance and negligible performance loss and claims that the C and the MAP-SY receivers exhibit practically identical performance, contrary to the expected behaviors of the MAP sequence detection schemes and the MAP symbol detection schemes under higher SER values. The ARE gives us a correct interpretation of the performances, i.e., the two receivers exhibit negligible performance difference with respect to each other under higher SER values. The analytical results are further substantiated by a computer simulation study, similar to the one quoted in [2]. Results for very low SNR values were not shown in [2]. Though not shown here also, for very low SNR values, we have verified that the AREs (equation (9)) estimated through simulation approximately agree with the theoretical values. For low SNR values and small values of M, except 2, the MAP-SY receiver achieves slightly smaller, albeit negligible, SER than the classical receiver. Practically, the two receivers show nearly identical SER performance, as stated in [2]. Because of this and the ease of its implementation, C will be the receiver of choice for coherent differentially encoded M-PSK demodulation.

V. Conclusion

In this letter we have considered the coherent detection of a differentially encoded M-PSK signal received in AWGN and derived the weak signal SER expressions for the MAP-symbol detection criterion receiver and the classical receiver. An asymptotic relative efficiency measure points out a subtle difference in the performances of these receivers.

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