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The Stable Manifold Theorem for SDE's (Stochastic Analysis Seminar, MSRI)

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THE STABLE MANIFOLD THEOREM
FOR SDE'S

MSRI, Berkeley : December 3 & 5, 1997

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Outline

- Formulate a *Local Stable Manifold Theorem* for stochastic differential equations (SDE's).
- Theorem holds for Stratonovich and Itô SDE's driven by spatial Kunita-type semimartingales with stationary ergodic increments.
- Start with the existence of a stochastic flow for SDE.
- Concept of a hyperbolic stationary trajectory. The stationary trajectory is a solution of the forward /backward anticipating SDE for all time (Stratonovich case).
- Existence of a stationary random family of asymptotically invariant stable and unstable manifolds within a stationary neighborhood of the hyperbolic stationary solution.
- The stable and unstable manifolds are dynamically characterized using forward and backward solutions of anticipating versions of the (Stratonovich) SDE.
- Proof based on Ruelle-Oseledec (non-linear) multiplicative ergodic theory and anticipating stochastic calculus.

Formulation of The Theorem

Stratonovich SDE

$$dx(t) = h(x(t)) dt + \sum_{i=1}^m g_i(x(t)) \circ dW_i(t), \quad (I)$$

on $\mathbf{R}^d$ driven by m -dimensional Brownian motion $W := (W_1, \dots, W_m)$.

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}}, P) :=$ canonical filtered Wiener space.

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R} \rightarrow \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F} :=$ Borel σ -field of Ω ;

$\mathcal{F}_t :=$ sub- σ -field of $\mathcal{F}$ generated by the evaluations $\omega \rightarrow \omega(u)$, $u \leq t$, $t \in \mathbf{R}$.

$P :=$ Wiener measure on Ω .

$h, g_i : \mathbf{R}^d \rightarrow \mathbf{R}^d$, $1 \leq i \leq m$, vector fields on $\mathbf{R}^d$. For some $k \geq 1, \delta \in (0, 1)$, h is $C_b^{k, \delta}$, viz. h has all derivatives $D^j h$, $1 \leq j \leq k$, continuous and globally bounded, $D^k h$ Hölder continuous with exponent δ .

g_i , $1 \leq i \leq m$, globally bounded and in $C_b^{k+1, \delta}$.

$\theta : \mathbf{R} \times \Omega \rightarrow \Omega$ is the (ergodic) Brownian shift

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in \Omega.$$

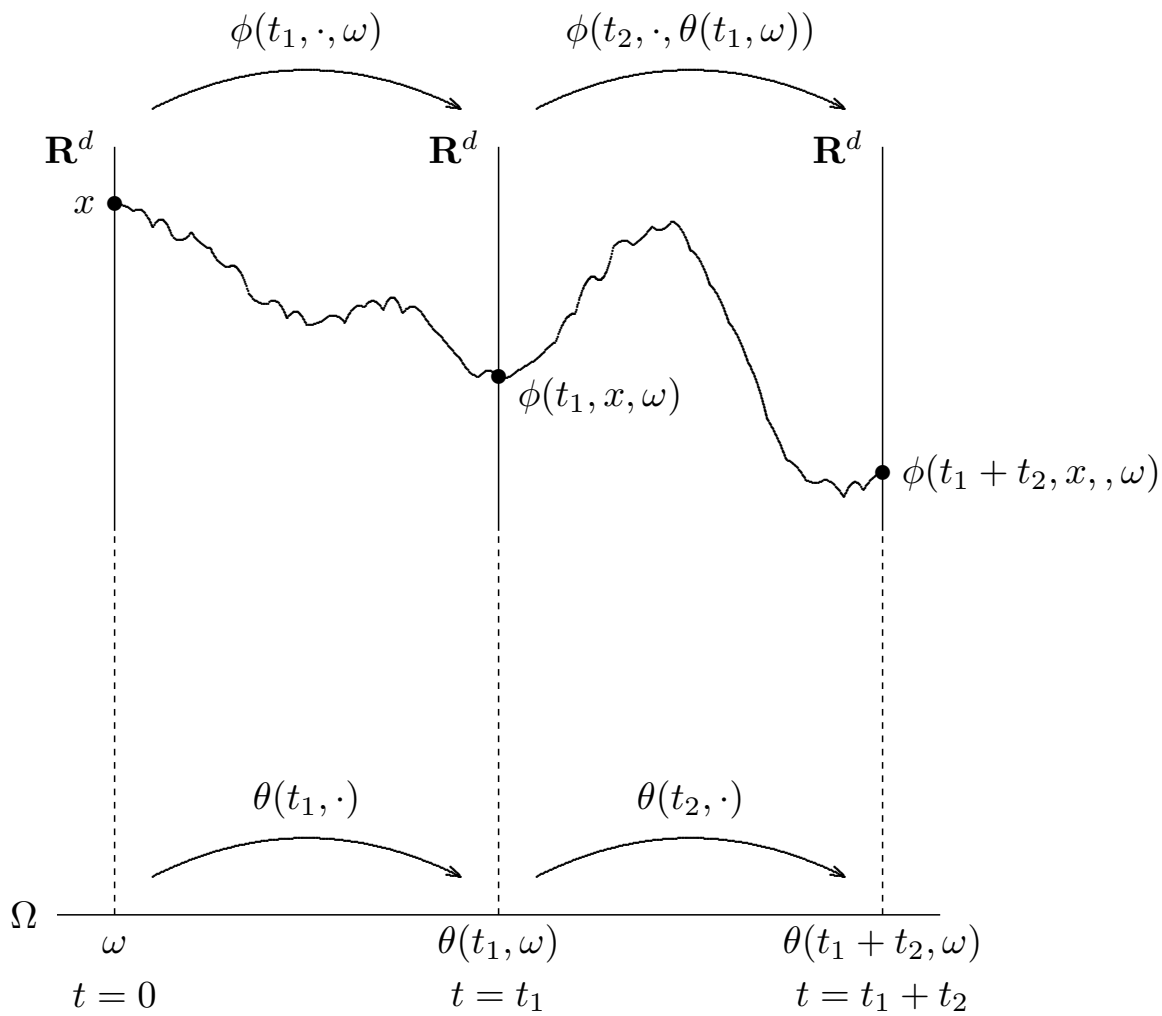
Let $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \rightarrow \mathbf{R}^d$ be the stochastic flow generated by (I) ($\phi(t, \cdot, \omega) = [\phi(-t, \cdot, \theta(t, \omega))]^{-1}, t < 0$). Then ϕ is a perfect cocycle:

$$\phi(t + s, \cdot, \omega) = \phi(t, \cdot, \theta(s, \omega)) \circ \phi(s, \cdot, \omega),$$

for all $s, t \in \mathbf{R}$ and all $\omega \in \Omega$ ([I-W], [A-S], [A]).

Figure illustrates the cocycle property. Vertical solid lines represent random fibers consisting of copies of $\mathbf{R}^d$. (ϕ, θ) is a “random vector-bundle morphism” over the “base” probability space Ω .

The Cocycle



Definition

The SDE (I) has a *stationary trajectory* if there exists an $\mathcal{F}$ -measurable random variable $Y : \Omega \rightarrow \mathbf{R}^d$ such that

$$\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \quad (1)$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $\phi(t, Y) = Y(\theta(t))$.

If (1) holds on a sure event Ω_t that may depend on t , then there are “perfect” versions of the stationary random variable Y and of the flow ϕ such that (1) and the cocycle property hold *for all* $\omega \in \Omega$ ([Sc]).

Let $\phi(t, Y)$ be a stationary solution of (I). Cocycle property of ϕ implies that the linearization

$$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$$

along the stationary solution is also a $d \times d$ -matrix-valued cocycle. Using Kolmogorov’s theorem, the random variables

$$\sup_{x \in \mathbf{R}^d} \frac{|D_2\phi(t, x)|}{(1 + |x|^\gamma)}, \quad \gamma > 0,$$

have moments of all orders. If $E \log^+ |Y| < \infty$, then $E \log^+ |D_2 \phi(1, Y)| < \infty$. Apply Oseledec's Theorem to get a *non-random* finite Lyapunov spectrum:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |D_2 \phi(n, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbf{R}^d).$$

Spectrum takes finitely many values $\{\lambda_i\}_{i=1}^p$ with non-random multiplicities q_i , $1 \leq i \leq p$, and $\sum_{i=1}^p q_i = d$ ([Ru.1], Theorem I.6).

Definition

Stationary trajectory $\phi(t, Y)$ of (I) is *hyperbolic* if $E \log^+ |Y(\cdot)| < \infty$, and if the linearized cocycle $(D_2 \phi(n, Y(\omega), \omega), \theta(n, \omega))$ has a non-vanishing Lyapunov spectrum

$$\{\lambda_p < \cdots < \lambda_{i_0+1} < \lambda_{i_0} < 0 < \lambda_{i_0-1} < \cdots < \lambda_2 < \lambda_1\}$$

i.e. $\lambda_i \neq 0$ for all $1 \leq i \leq p$.

Define $\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all $\lambda_i > 0$, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized flow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Let $\rho \in \mathbf{R}^+$, $x \in \mathbf{R}^d$.

$B(x, \rho) :=$ open ball in $\mathbf{R}^d$, center x and radius ρ ;

$\bar{B}(x, \rho) :=$ corresponding closed ball;

$\mathcal{C}(\mathbf{R}^d) :=$ the class of all non-empty compact subsets of $\mathbf{R}^d$ with Hausdorff metric d^* :

$d^*(A_1, A_2) := \sup\{d(x, A_1) : x \in A_2\} \vee \sup\{d(y, A_2) : y \in A_1\}$ where $A_1, A_2 \in \mathcal{C}(\mathbf{R}^d)$;

$d(x, A_i) := \inf\{|x - y| : y \in A_i\}$, $x \in \mathbf{R}^d$, $i = 1, 2$;

$\mathcal{B}(\mathcal{C}(\mathbf{R}^d)) :=$ Borel σ -algebra on $\mathcal{C}(\mathbf{R}^d)$ with respect to the metric d^* .

Theorem 1 (The Stable Manifold Theorem) (M.+ Scheut-
zow, 1997)

Assume that the coefficients of SDE (I) satisfy the given hypotheses. Suppose $\phi(t, Y)$ is a hyperbolic stationary trajectory of (I) with $E \log^+ |Y| < \infty$.

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

- (i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,
- (ii) $\mathcal{F}$ -measurable random variables $\rho_i, \beta_i : \Omega^* \rightarrow [0, \infty)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

- (a) $\tilde{\mathcal{S}}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$|\phi(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0} \quad (2)$$

for all $x \in \tilde{\mathcal{S}}(\omega)$. Each stable subspace $\mathcal{S}(\omega)$ of the linearized flow $D_2\phi$ is tangent at $Y(\omega)$ to the submanifold $\tilde{\mathcal{S}}(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. In particular, $\dim \tilde{\mathcal{S}}(\omega) = \dim \mathcal{S}(\omega)$ and is non-random.

$$(b) \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{\mathcal{S}}(\omega)}} \left\{ \frac{|\phi(t, x_1, \omega) - \phi(t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq \lambda_{i_0}.$$

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ such that

$$\phi(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega)), \quad t \geq \tau_1(\omega). \quad (3)$$

Also

$$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t, \omega)), \quad t \geq 0. \quad (4)$$

(d) $\tilde{\mathcal{U}}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_2(\omega))$ with the property that

$$|\phi(-n, x, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega) e^{(-\lambda_{i_0-1} + \epsilon_2)n} \quad (5)$$

for all integers $n \geq 0$. Also

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |\phi(-t, x, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1}. \quad (6)$$

for all $x \in \tilde{\mathcal{U}}(\omega)$. Furthermore, the unstable subspace $\mathcal{U}(\omega)$ of $D_2\phi$ is the tangent space to $\tilde{\mathcal{U}}(\omega)$ at $Y(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. In particular, $\dim \tilde{\mathcal{U}}(\omega) = \dim \mathcal{U}(\omega)$ and is non-random.

$$(e) \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{\mathcal{U}}(\omega)}} \left\{ \frac{|\phi(-t, x_1, \omega) - \phi(-t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq -\lambda_{i_0-1}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ such that

$$\phi(-t, \cdot, \omega)(\tilde{\mathcal{U}}(\omega)) \subseteq \tilde{\mathcal{U}}(\theta(-t, \omega)), \quad t \geq \tau_2(\omega). \quad (7)$$

Also

$$D_2\phi(-t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(-t, \omega)), \quad t \geq 0. \quad (8)$$

(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ and $\tilde{\mathcal{S}}(\omega)$ are transversal, viz.

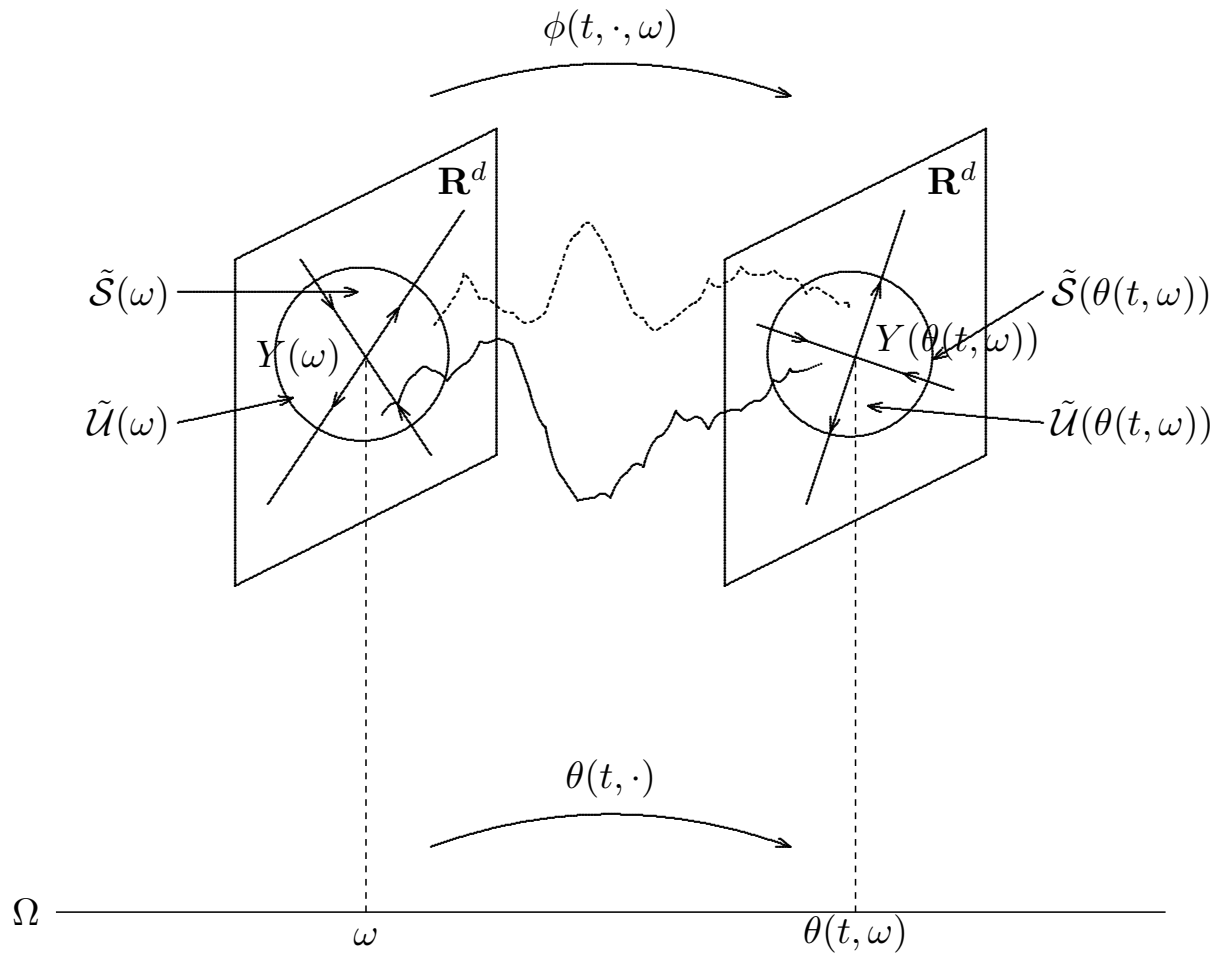
$$\mathbf{R}^d = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega). \quad (9)$$

(h) The mappings

$$\begin{aligned} \Omega &\rightarrow \mathcal{C}(\mathbf{R}^d), & \Omega &\rightarrow \mathcal{C}(\mathbf{R}^d), \\ \omega &\mapsto \tilde{\mathcal{S}}(\omega) & \omega &\mapsto \tilde{\mathcal{U}}(\omega) \end{aligned}$$

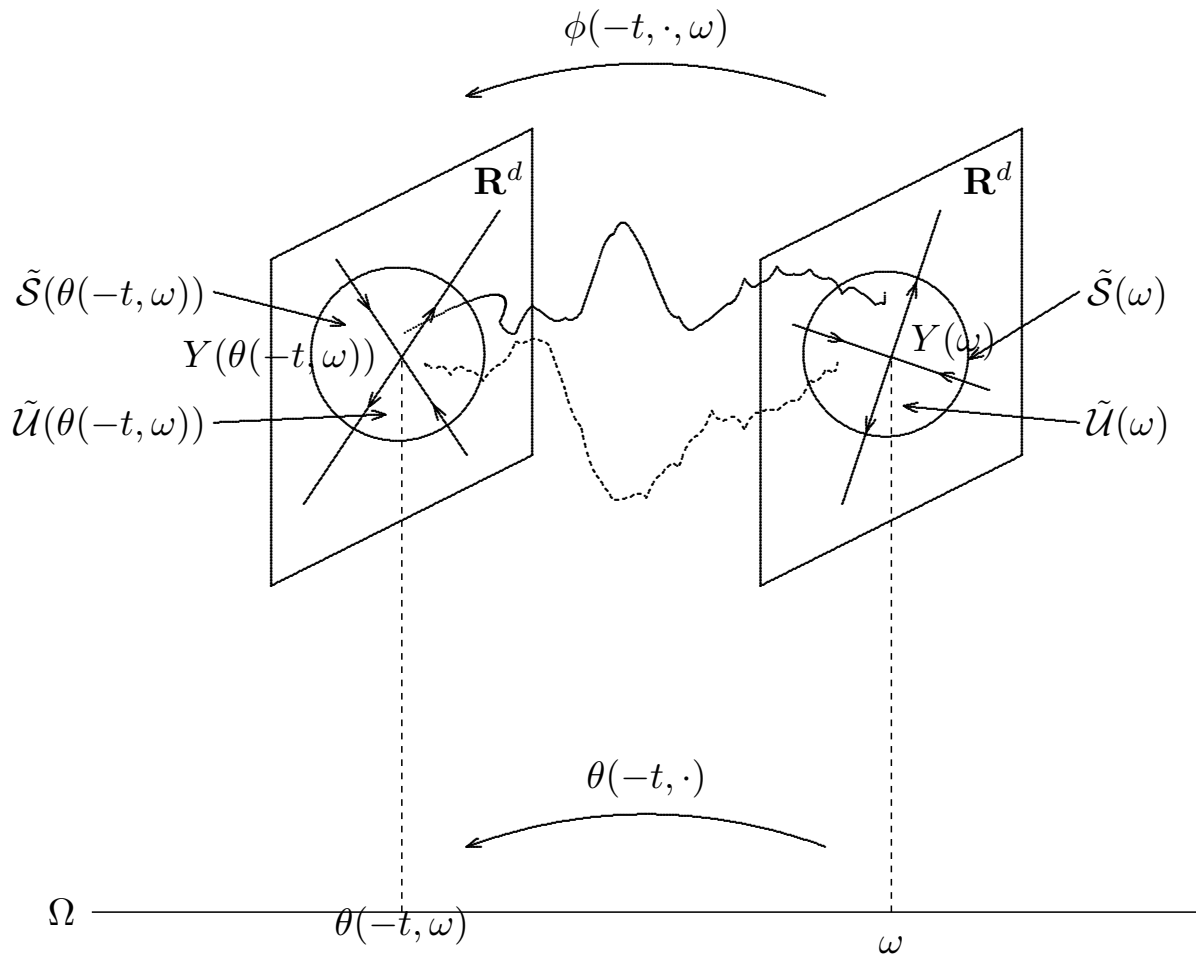
are $(\mathcal{F}, \mathcal{B}(\mathcal{C}(\mathbf{R}^d)))$ -measurable.

Assume, further, that $h, g_i, 1 \leq i \leq m$, are C_b^∞ . Then the local stable and unstable manifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞ .



$$t > \tau_1(\omega)$$

A picture is worth a 1000 words!



$$t > \tau_2(\omega)$$

Remarks

- (i) In Stratonovich SDE (I), replace global boundedness on g'_i s by requiring

$$\mathbf{R}^d \ni x \mapsto \sum_{l=1}^m \frac{\partial g_l^i(x)}{\partial x_j} g_l^j(x) \in \mathbf{R}, \quad 1 \leq i, j \leq d$$

to be in $C_b^{k,\delta}$.

- (ii) Conjecture: The the global boundedness condition is not needed. This conjecture is not hard to check if the vector fields $g_i, 1 \leq i \leq m$ are C_b^∞ and generate a finite-dimensional solvable Lie algebra. See [Ku], Theorem 4.9.10, p. 212.
- (iii) Theorem holds for the Itô SDE

$$dx(t) = h(x(t)) dt + \sum_{i=1}^m g_i(x(t)) dW_i(t), \quad (II)$$

with $h, g_i : \mathbf{R}^d \rightarrow \mathbf{R}^d, 1 \leq i \leq m$, in $C_b^{k,\delta}$.

- (iv) Replace the stationary random variable Y by its invariant distribution μ , then formulate result with respect to the product measure $\mu \otimes P$ and the underlying skew-product flow. This would give stable and unstable manifolds that are defined a.e. $(\mu \otimes P)$; cf. [C] for

the globally asymptotically stable case on a compact manifold.

(v) Replace the SDE (I) with Kunita-type SDE

$$\left. \begin{aligned} d\phi(t) &= \overset{\circ}{F}(\circ dt, \phi(t)), \quad t > s \\ \phi(s) &= x \end{aligned} \right\}$$

where $\overset{\circ}{F}$ is a spatial semimartingale helix (i.e. with stationary ergodic increments) and with local characteristics of class $(B_{ub}^{k+1,\delta}, B_{ub}^{k,\delta})$ and the function

$$[0, \infty) \times \mathbf{R}^d \ni (t, x) \mapsto \sum_{j=1}^d \frac{\partial a^{\cdot,j}(t, x, y)}{\partial x_j} \Big|_{y=x} \in \mathbf{R}^d$$

belongs to $B_{ub}^{k,\delta}$. In the Itô case, last condition is not needed.

$$\begin{aligned} \overset{\circ}{F}(t, x) &= V(t, x) + M(t, x) \\ a^{i,j}(t, x, y) &:= \frac{d}{dt} \langle M^i(\cdot, x), M^j(\cdot, y) \rangle(t) \\ b^i(t, x) &:= \frac{d}{dt} V^i(t, x), \quad x, y \in \mathbf{R}^d, 1 \leq i, j \leq d \end{aligned}$$

Sketch of Proof

Linearization and Substitution

Assume regularity conditions on the coefficients h, g_i . By the *Substitution Rule*, $\phi(t, Y(\omega), \omega)$ is a stationary *solution* of the *anticipating* Stratonovich SDE

$$\left. \begin{aligned} d\phi(t, Y) &= h(\phi(t, Y)) dt + \sum_{i=1}^m g_i(\phi(t, Y)) \circ dW_i(t), \quad t > 0 \\ \phi(0, Y) &= Y. \end{aligned} \right\} \quad (II)$$

([N-P]).

Linearize the SDE (I) along the stationary trajectory. By substitution, match the solution of the linearized equation with the linearized cocycle $D_2\phi(t, Y(\omega), \omega)$. Hence $D_2\phi(t, Y(\omega), \omega)$, $t \geq 0$, solves the SDE:

$$\left. \begin{aligned} dD_2\phi(t, Y) &= Dh(\phi(t, Y))D_2\phi(t, Y) dt \\ &\quad + \sum_{i=1}^m Dg_i(\phi(t, Y))D_2\phi(t, Y) \circ dW_i(t), \quad t > 0 \\ D_2\phi(0, Y) &= I. \end{aligned} \right\} \quad (III)$$

D_2, D denotes spatial (Fréchet) derivatives.

Similarly, the backward trajectories

$$\phi(t, Y), D_2\phi(t, Y), \quad t < 0,$$

solve the corresponding backward Stratonovich SDE's:

$$\left. \begin{aligned} d\phi(t, Y) &= -h(\phi(t, Y)) dt - \sum_{i=1}^m g_i(\phi(t, Y)) \circ \hat{d}W_i(t), \quad t < 0 \\ \phi(0, Y) &= Y. \end{aligned} \right\} \quad (II^-)$$

$$\left. \begin{aligned} dD_2\phi(t, Y) &= -Dh(\phi(t, Y))D_2\phi(t, Y) dt \\ &\quad - \sum_{i=1}^m Dg_i(\phi(t, Y))D_2\phi(t, Y) \circ \hat{d}W_i(t), \quad t < 0 \\ D_2\phi(0, Y) &= I. \end{aligned} \right\} \quad (III^-)$$

Above SDE's (II)-(III)⁻ give dynamic characterizations of the stable and unstable manifolds.

The following lemma is used to construct the shift-invariant sure event appearing in the statement of the local stable manifold theorem. Gives “perfect versions” of the ergodic theorem and Kingman’s subadditive ergodic theorem.

Lemma 1

(i) Let $h : \Omega \rightarrow \mathbf{R}^+$ be $\mathcal{F}$ -measurable and such that

$$\int_{\Omega} \sup_{0 \leq u \leq 1} h(\theta(u, \omega)) dP(\omega) < \infty.$$

Then there is a sure event $\Omega_1 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_1) = \Omega_1$ for all $t \in \mathbf{R}$, and

$$\lim_{t \rightarrow \infty} \frac{1}{t} h(\theta(t, \omega)) = 0$$

for all $\omega \in \Omega_1$.

(ii) Suppose $f : \mathbf{R}^+ \times \Omega \rightarrow \mathbf{R} \cup \{-\infty\}$ is a measurable process on $(\Omega, \mathcal{F}, P)$ satisfying the following conditions

$$(a) \quad E \sup_{0 \leq u \leq 1} f^+(u) < \infty, \quad E \sup_{0 \leq u \leq 1} f^+(1 - u, \theta(u)) < \infty$$

(b) $f(t_1 + t_2, \omega) \leq f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ for all $t_1, t_2 \geq 0$ and all $\omega \in \Omega$.

Then there is sure event $\Omega_2 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_2) = \Omega_2$ for all $t \in \mathbf{R}$, and a fixed number $f^* \in \mathbf{R} \cup \{-\infty\}$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} f(t, \omega) = f^*$$

for all $\omega \in \Omega_2$.

Proof

[Mo.1], Lemma 7. $\square$

Theorem 2 ([O], 1968)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\theta : \mathbf{R}^+ \times \Omega \rightarrow \Omega$ a measurable family of ergodic P -preserving transformations. Let $T : \mathbf{R}^+ \times \Omega \rightarrow L(\mathbf{R}^d)$ be measurable, such that (T, θ) is an $L(\mathbf{R}^d)$ -valued cocycle. Suppose that

$$E \sup_{0 \leq t \leq 1} \log^+ \|T(t, \cdot)\| < \infty, \quad E \sup_{0 \leq t \leq 1} \log^+ \|T(1-t, \theta(t, \cdot))\| < \infty.$$

Then there is a set $\Omega_0 \in \mathcal{F}$ of full P -measure such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \in \mathbf{R}^+$, and for each $\omega \in \Omega_0$, the limit

$$\lim_{n \rightarrow \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. Each $\Lambda(\omega)$ has a discrete non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \dots > e^{\lambda_p}$$

where the λ_i 's are distinct. Each e^{λ_i} has a fixed non-random multiplicity m_i and eigen-space $F_i(\omega)$, with $m_i := \dim F_i(\omega)$. Define

$$E_1(\omega) := \mathbf{R}^d, \quad E_i(\omega) := [\oplus_{j=1}^{i-1} F_j(\omega)]^\perp, \quad 1 < i \leq p.$$

Then

$$E_p(\omega) \subset \dots \subset E_{i+1}(\omega) \subset E_i(\omega) \subset \dots \subset E_2(\omega) \subset E_1(\omega) = \mathbf{R}^d$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|T(t, \omega)x\| = \lambda_i(\omega), \quad \text{if } x \in E_i(\omega) \setminus E_{i+1}(\omega),$$

and

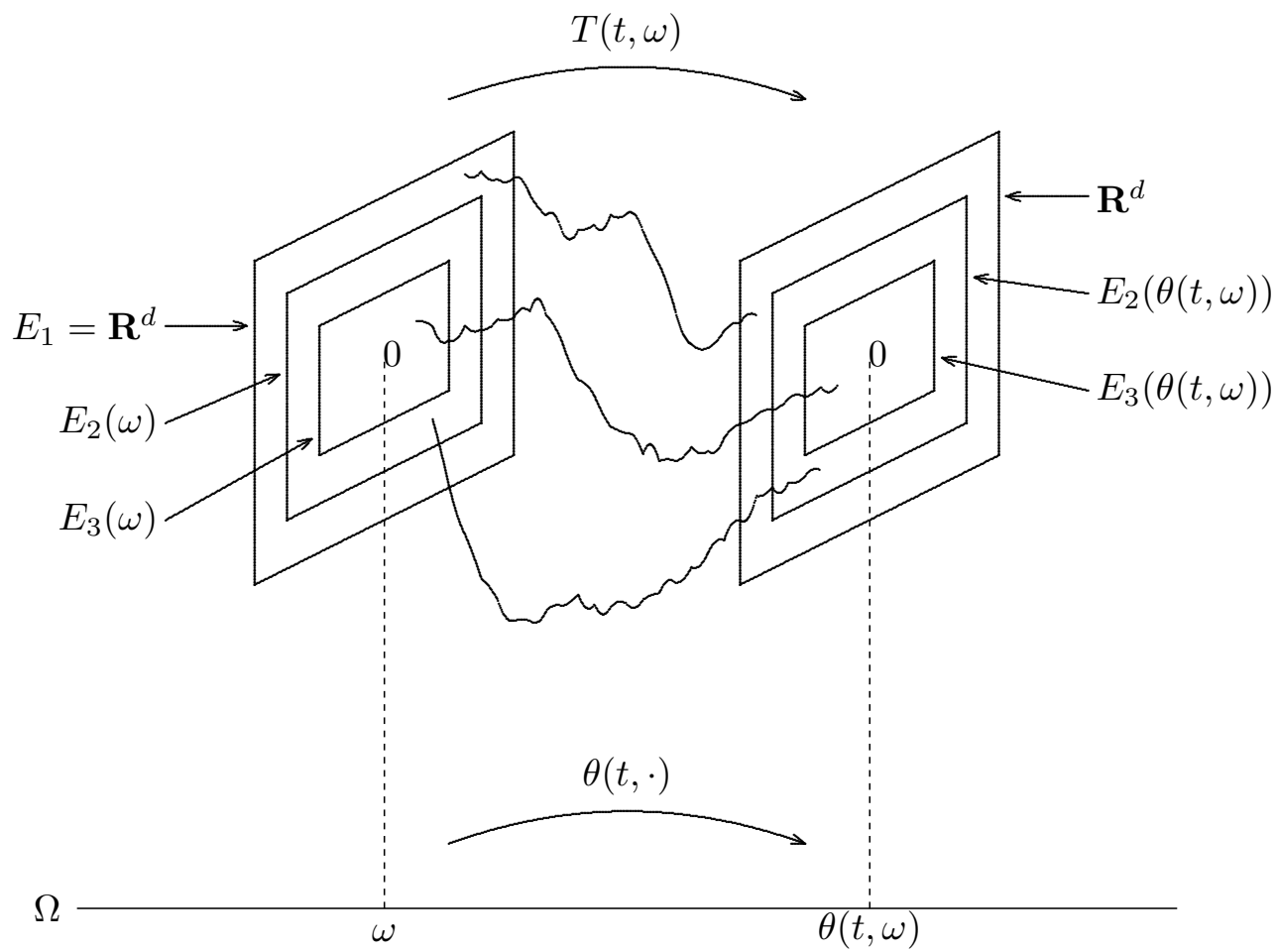
$$T(t, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

for all $t \geq 0$, $1 \leq i \leq p$.

Proof.

Based on the discrete version of Oseledec's multiplicative ergodic theorem and Lemma 1. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]), “perfect” infinite-dimensional version and application to SFDE's. $\square$

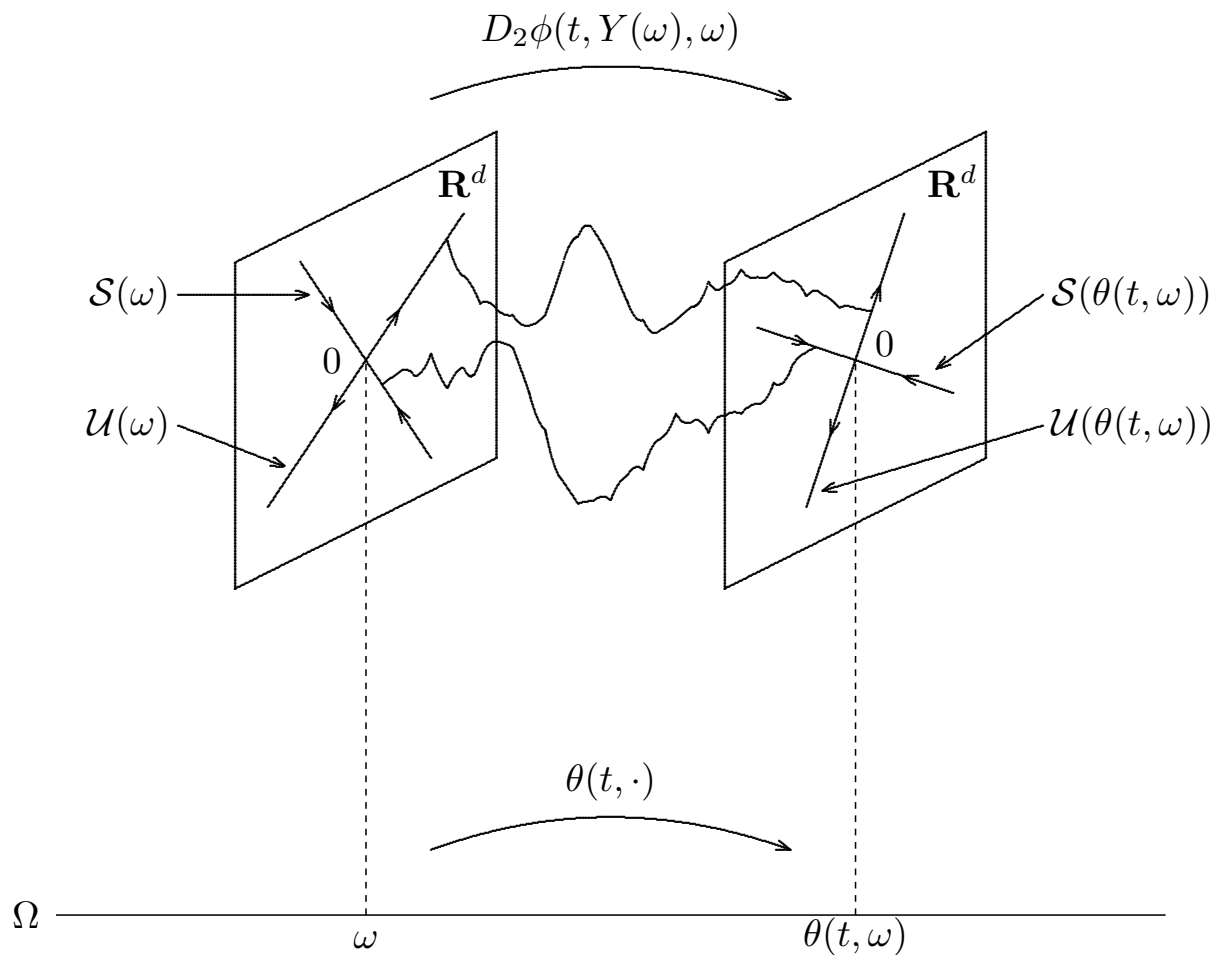
Spectral Theorem



Apply Theorem 2 with $T(t, \omega) := D_2\phi(t, Y(\omega), \omega)$. Then linearized cocycle has random invariant stable and unstable subspaces $\{S(\omega), U(\omega) : \omega \in \Omega\}$:

$$\begin{aligned} D_2\phi(t, Y(\omega), \omega)(S(\omega)) &= S(\theta(t, \omega)), \\ D_2\phi(-t, Y(\omega), \omega)(U(\omega)) &= U(\theta(-t, \omega)), \quad t \geq 0. \end{aligned}$$

[Mo.1].



Estimates on the non-linear cocycle

Theorem 3 (M. + Scheutzow [M-S.2])

There exists a jointly measurable modification of the trajectory random field of (I), denoted by $\{\phi_{s,t}(x) : -\infty < s, t < \infty, x \in \mathbf{R}^d\}$, with the following properties:

Define $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \rightarrow \mathbf{R}^d$ by

$$\phi(t, x, \omega) := \phi_{0,t}(x, \omega), \quad x \in \mathbf{R}^d, \omega \in \Omega, t \in \mathbf{R}.$$

Then the following is true for all $\omega \in \Omega$:

- (i) For each $x \in \mathbf{R}^d$, and $s, t \in \mathbf{R}$, $\phi_{s,t}(x, \omega) = \phi(t - s, x, \theta(s, \omega))$.
- (ii) (ϕ, θ) is a perfect cocycle:

$$\phi(t + s, \cdot, \omega) = \phi(t, \cdot, \theta(s, \omega)) \circ \phi(s, \cdot, \omega),$$

for all $s, t \in \mathbf{R}$.

- (iii) For each $t \in \mathbf{R}$, $\phi(t, \cdot, \omega) : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is a C^k diffeomorphism.
- (iv) The mapping $\mathbf{R}^2 \ni (s, t) \mapsto \phi_{s,t}(\cdot, \omega) \in \text{Diff}^k(\mathbf{R}^d)$ is continuous, where $\text{Diff}^k(\mathbf{R}^d)$ denotes the group of all C^k diffeomorphisms of $\mathbf{R}^d$, given the C^k -topology.
- (v) For every $\epsilon \in (0, \delta)$, $\gamma, \rho, T > 0$, and $1 \leq |\alpha| \leq k$, the quantities

$$\begin{aligned} & \sup_{\substack{0 \leq s, t \leq T, \\ x \in \mathbf{R}^d}} \frac{|\phi_{s,t}(x, \omega)|}{[1 + |x|(\log^+ |x|)^\gamma]}, \quad \sup_{\substack{0 \leq s, t \leq T, \\ x \in \mathbf{R}^d}} \frac{|D_x^\alpha \phi_{s,t}(x, \omega)|}{(1 + |x|^\gamma)}, \\ & \sup_{x \in \mathbf{R}^d} \sup_{\substack{0 \leq s, t \leq T, \\ 0 < |x' - x| \leq \rho}} \frac{|D_x^\alpha \phi_{s,t}(x, \omega) - D_x^\alpha \phi_{s,t}(x', \omega)|}{|x - x'|^\epsilon (1 + |x|)^\gamma}, \end{aligned}$$

are finite. The random variables defined by the above expressions have p -th moments for all $p \geq 1$.

Proof

Cocycle property (ii): approximate the flow using helix mollifiers of Brownian motion:

$$W^k(t) := k \int_{t-1/k}^t W(s) ds - k \int_{-1/k}^0 W(s) ds.$$

$$W^k(t_2, \theta(t_1, \omega)) = W^k(t_1 + t_2, \omega) - W^k(t_1, \omega), \quad k \geq 1$$

([I-W], cf. [Mo.1], [Mo.2] for linear infinite-dimensional case).

(iii) and (iv) are well-known to hold for a.a. $\omega \in \Omega$ ([Ku], Theorem 4.6.5).

A perfect version of $\phi_{s,t}$ satisfying (i)-(iv) for all $\omega \in \Omega$, is obtained in [A-S] by perfection techniques and the diffeomorphism theorem for flows ([Ku], Theorem 4.6.5; cf. also [M-S.1]).

By known estimates (or GRR) ([M-S.2]), the random variables

$$X_1 := \sup_{\substack{0 \leq s \leq t \leq T, \\ x \in \mathbf{R}^d}} \frac{|\phi_{s,t}(x, \cdot)|}{[1 + |x|(\log^+ |x|)^\gamma]},$$

$$X_2 := \sup_{\substack{0 \leq s \leq t \leq T, \\ x \in \mathbf{R}^d}} \frac{|x|}{[1 + |\phi_{s,t}(x, \cdot)|(\log^+ |x|)^\gamma]}$$

have p -th moments for all $p \geq 1$. It is sufficient to show that the random variable

$$\hat{X}_1 := \sup_{\substack{0 \leq s \leq t \leq T, \\ x \in \mathbf{R}^d}} \frac{|\phi_{t,s}(x, \cdot)|}{[1 + |x|(\log^+ |x|)^\gamma]}$$

has p -th moments for all $p \geq 1$. Assume (without loss of generality) that $\gamma \in (0, 1)$. From the definition of X_2 ,

$$|y| \leq X_2[1 + |\phi_{s,t}(y, \cdot)|(\log^+ |y|)^\gamma]$$

for all $0 \leq s \leq t \leq T, y \in \mathbf{R}^d$. Use the substitution

$$y = \phi_{t,s}(x, \omega) = \phi_{s,t}^{-1}(x, \omega), \quad \phi_{s,t}(y, \omega) = x, \quad 0 \leq s \leq t \leq T, \omega \in \Omega, x \in \mathbf{R}^d,$$

to rewrite above inequality as

$$|y| \leq X_2[1 + |x|(\log^+ |y|)^\gamma].$$

Solve above inequality (by taking $\log^+$) for $\log^+ |y|$. Therefore, there exists a non-random constant $K_1 := K_1(\gamma) > 0$ such that

$$|y| \leq K_1 X_2[1 + |x|\{1 + (\log^+ |X_2|)^\gamma + (\log^+ |x|)^\gamma\}].$$

Since X_2 has moments of all orders, the above inequality implies that $\hat{X}_1$ also has p -th moments for all $p \geq 1$.

Complete proof by [Ku], [M-S.2] and GRR. $\square$

$\|\cdot\|_{k,\epsilon} := C^{k,\epsilon}$ -norm on $C^{k,\epsilon}$ mappings $\bar{B}(0,\rho) \rightarrow \mathbf{R}^d$.

Lemma 2

Assume that $\log^+ |Y(\cdot)|$ is integrable. Then the cocycle ϕ satisfies

$$\int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \|\phi(t_2, Y(\theta(t_1, \omega)) + (\cdot), \theta(t_1, \omega))\|_{k,\epsilon} dP(\omega) < \infty \quad (10)$$

for any fixed $0 < T, \rho < \infty$ and any $\epsilon \in (0, \delta)$. Furthermore, the linearized flow $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$, $t \geq 0$, is an $L(\mathbf{R}^d)$ -valued perfect cocycle and

$$\int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \|D_2\phi(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega))\|_{L(\mathbf{R}^d)} dP(\omega) < \infty \quad (11)$$

for any fixed $0 < T < \infty$. The forward cocycle

$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t > 0)$ has a non-random finite Lyapunov spectrum $\{\lambda_m < \dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1\}$. Each Lyapunov

exponent λ_i has a non-random multiplicity q_i , $1 \leq i \leq m$, and $\sum_{i=1}^m q_i =$

d . The backward linearized cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t < 0)$, admits a “backward” non-random finite Lyapunov spectrum:

$$\lim_{t \rightarrow -\infty} \frac{1}{t} \log |D_2\phi(t, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbf{R}^d),$$

taking values in $\{-\lambda_i\}_{i=1}^m$ with non-random multiplicities q_i , $1 \leq i \leq m$, and $\sum_{i=1}^m q_i = d$.

Proof of Lemma 2

We first prove (11). Start with the perfect cocycle property for (ϕ, θ) :

$$\phi(t_1 + t_2, \cdot, \omega) = \phi(t_2, \cdot, \theta(t_1, \omega)) \circ \phi(t_1, \cdot, \omega) \quad (12)$$

for all $t_1, t_2 \in \mathbf{R}$ and all $\omega \in \Omega$. Cocycle property for $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$ follows directly by taking Fréchet derivatives at $Y(\omega)$ on both sides of (12); viz.

$$\begin{aligned} D_2\phi(t_1 + t_2, Y(\omega), \omega) &= D_2\phi(t_2, \phi(t_1, Y(\omega), \omega), \theta(t_1, \omega)) \circ D_2\phi(t_1, Y(\omega), \omega) \\ &= D_2\phi(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega)) \circ D_2\phi(t_1, Y(\omega), \omega) \end{aligned} \quad (13)$$

for all $\omega \in \Omega_0, t_1, t_2 \in \mathbf{R}$. Existence of a fixed discrete spectrum for $D_2\phi(t, Y)$ follows from [Mo.1] and [M-S.1], using the integrability property (11) and the ergodicity of θ . ((11) follows from (13) and Theorem 3 (v)). But (10) implies (11)! Therefore it is sufficient to prove (10).

In view of (1) and the identity

$$\phi_{t_1, t_1+t_2}(x, \omega) = \phi(t_2, x, \theta(t_1, \omega)), \quad x \in \mathbf{R}^d, t_1, t_2 \in \mathbf{R},$$

(Theorem 3(i)), (10) (for $\epsilon = 0$) will follow from

$$\int_{\Omega} \log^+ \sup_{\substack{0 \leq s, t \leq T, \\ |x'| \leq \rho}} |D_x^\alpha \phi_{s,t}(\phi_{0,s}(Y(\omega), \omega) + x', \omega)| dP(\omega) < \infty, \quad 0 \leq |\alpha| \leq k. \quad (14)$$

Denote random “constants” by $K_i, i = 1, 2, 3, 4$. Each $K_i := K_i(\rho, T), i = 1, 2, 3, 4$, has p -th moments for all $p \geq 1$. The following inequalities follow easily from Theorem 3 (v).

$$\begin{aligned}
& \log^+ \sup_{\substack{s, t \in [0, T], \\ |x'| \leq \rho}} |D_x^\alpha \phi_{s,t}(\phi_{0,s}(Y(\omega), \omega) + x', \omega)| \\
& \leq \log^+ \sup_{s \in [0, T]} \{K_1(\omega)[1 + (\rho + |\phi_{0,s}(Y(\omega), \omega)|)^2]\} \\
& \leq \log^+ K_2(\omega) + \log^+[1 + 2\rho^2 + K_3(\omega)(1 + |Y(\omega)|^4)] \\
& \leq \log^+ K_4(\omega) + \log[1 + 2\rho^2] + 4 \log^+ |Y(\omega)| \quad (15)
\end{aligned}$$

for all $\omega \in \Omega$. (15)+ integrability hypothesis on Y imply (14). $\square$

The Auxiliary Cocycle

To apply Ruelle’s discrete non-linear ergodic theorem ([Ru.1], Theorem 5.1, p. 292), introduce the following auxiliary cocycle $Z : \mathbf{R} \times \mathbf{R}^d \times \Omega \rightarrow \mathbf{R}^d$. This a “centering” of the flow ϕ about the stationary solution:

$$Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega)) \quad (16)$$

for $t \in \mathbf{R}, x \in \mathbf{R}^d, \omega \in \Omega$.

Lemma 3

(Z, θ) is a perfect cocycle on $\mathbf{R}^d$ and $Z(t, 0, \omega) = 0$ for all $t \in \mathbf{R}$, and all $\omega \in \Omega$.

Proof of Lemma 3

Let $t_1, t_2 \in \mathbf{R}, \omega \in \Omega, x \in \mathbf{R}^d$.

$$\begin{aligned}
& Z(t_2, Z(t_1, x, \omega), \theta(t_1, \omega)) \\
&= \phi(t_2, Z(t_1, x, \omega) + Y(\theta(t_1, \omega)), \theta(t_1, \omega)) - Y(\theta(t_2, \theta(t_1, \omega))) \\
&= \phi(t_2, \phi(t_1, x + Y(\omega), \omega), \theta(t_1, \omega)) - Y(\theta(t_2 + t_1, \omega)) \\
&= Z(t_1 + t_2, x, \omega), \quad t_1, t_2 \in \mathbf{R}, \omega \in \Omega, x \in \mathbf{R}^d.
\end{aligned}$$

$Z(t, 0, \omega) \equiv 0$ by definition of Z and stationary solution. $\square$

The proof of the local stable-manifold theorem (Theorem 1) uses a discretization argument that requires the following lemma.

Lemma 4

Suppose that $\log^+ |Y(\cdot)|$ is integrable. Then there is a sure event $\Omega_3 \in \mathcal{F}$ with the following properties:

- (i) $\theta(t, \cdot)(\Omega_3) = \Omega_3$ for all $t \in \mathbf{R}$,
- (ii) For every $\omega \in \Omega_3$ and any $x \in \mathbf{R}^d$, the statement

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z(n, x, \omega)| < 0 \quad (17)$$

implies

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |Z(t, x, \omega)| = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z(n, x, \omega)|. \quad (18)$$

Proof

The integrability condition (10) of Lemma 2 implies that

$$\int_{\Omega} \log^+ \sup_{\substack{0 \leq t_1, t_2 \leq 1, \\ x^* \in \bar{B}(0,1)}} \|D_2 Z(t_1, x^*, \theta(t_2, \omega))\|_{L(\mathbf{R}^d)} dP(\omega) < \infty. \quad (19)$$

Therefore by (the perfect version of) the ergodic theorem (Lemma 1(i)), there is a sure event $\Omega_3 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_3) = \Omega_3$ for all $t \in \mathbf{R}$, and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log^+ \sup_{\substack{0 \leq u \leq 1, \\ x^* \in \bar{B}(0,1)}} \|D_2 Z(u, x^*, \theta(t, \omega))\|_{L(\mathbf{R}^d)} = 0 \quad (20)$$

for all $\omega \in \Omega_3$.

Let $\omega \in \Omega_3$ and suppose $x \in \mathbf{R}^d$ satisfies (17). Then (17) implies that there exists a positive integer $N_0(x, \omega)$ such that $Z(n, x, \omega) \in \bar{B}(0, 1)$ for all $n \geq N_0$. Let $n \leq t < n + 1$, $n \geq N_0$. Then by the cocycle property for (Z, θ) and the Mean Value Theorem:

$$\begin{aligned} & \sup_{n \leq t \leq n+1} \frac{1}{t} \log |Z(t, x, \omega)| \\ & \leq \frac{1}{n} \log^+ \sup_{\substack{0 \leq u \leq 1, \\ x^* \in \bar{B}(0,1)}} \|D_2 Z(u, x^*, \theta(n, \omega))\|_{L(\mathbf{R}^d)} + \frac{n}{(n+1)} \frac{1}{n} \log |Z(n, x, \omega)|. \end{aligned}$$

Take $\limsup_{n \rightarrow \infty}$ in the above relation and use (20) to get

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |Z(t, x, \omega)| \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z(n, x, \omega)|.$$

The inequality

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z(n, x, \omega)| \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log |Z(t, x, \omega)|,$$

is obvious. Hence (18) holds. $\square$

Ruelle's Non-linear Ergodic Theorem

Theorem 4 ([Ru.1], 1979)

Let $\Omega \ni \omega \mapsto F_\omega \in C^{k,\delta}(\mathbf{R}^d, 0; \mathbf{R}^d, 0)$ be measurable such that $E \log^+ \|F_\cdot|_{\bar{B}(0,1)}\| < \infty$. Set $F^n(\omega) := F_{\theta(n-1,\omega)} \circ \cdots \circ F_{\theta(1,\omega)} \circ F_\omega$. Suppose $\lambda < 0$ is not in the spectrum of the cocycle $(DF_\omega^n(0), \theta(n, \omega))$. Then there is a sure event $\Omega_0 \in \mathcal{F}$ such that $\theta(1, \cdot)(\Omega_0) \subseteq \Omega_0$, and measurable functions $\beta(\omega) > \alpha(\omega) > 0, \gamma(\omega) > 1$ with the following properties:

(a) If $\omega \in \Omega_0$, the set

$$V_\omega^\lambda := \{x \in \bar{B}(0, \alpha(\omega)) : \|F_\omega^n(x)\| \leq \beta(\omega)e^{n\lambda} \text{ for all } n \geq 0\}$$

is a $C^{k,\delta}$ submanifold of $\bar{B}(0, \alpha(\omega))$.

(b) If $x_1, x_2 \in V_\omega^\lambda$, then

$$\|F_\omega^n(x_1) - F_\omega^n(x_2)\| \leq \gamma(\omega)\|x_1 - x_2\|e^{n\lambda}$$

for all integers $n \geq 0$. If $\lambda' < \lambda$ and $[\lambda', \lambda]$ is disjoint from the spectrum of $(DF_\omega^n(0), \theta(n, \omega))$, then there exists a measurable $\gamma'(\omega) > 1$ such that

$$\|F_\omega^n(x_1) - F_\omega^n(x_2)\| \leq \gamma'(\omega)\|x_1 - x_2\|e^{n\lambda'}$$

for all $x_1, x_2 \in V_\omega^\lambda$ and all integers $n \geq 0$.

Proof

[Ru.1], Theorem 5.1, p. 292.

Construction of the Stable/Unstable Manifolds

Assume the hypotheses of Theorem 1.

Consider the auxiliary cocycle (Z, θ) . Define the family of maps $F_\omega : \mathbf{R}^d \rightarrow \mathbf{R}^d$ by $F_\omega(x) := Z(1, x, \omega)$ for all $\omega \in \Omega$ and $x \in \mathbf{R}^d$. Let $\tau := \theta(1, \cdot) : \Omega \rightarrow \Omega$. Define $F_\omega^n := F_{\tau^{n-1}(\omega)} \circ \cdots \circ F_{\tau(\omega)} \circ F_\omega$. Then cocycle property for Z gives $F_\omega^n = Z(n, \cdot, \omega)$ for each $n \geq 1$. F_ω is $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) and $(DF_\omega)(0) = D_2\phi(1, Y(\omega), \omega)$. By measurability of the flow ϕ , the map $\omega \mapsto (DF_\omega)(0)$ is $\mathcal{F}$ -measurable. By (11) of Lemma 2, the map $\omega \mapsto \log^+ \|D_2\phi(1, Y(\omega), \omega)\|_{L(\mathbf{R}^d)}$ is integrable. The discrete cocycle $((DF_\omega^n)(0), \theta(n, \omega), n \geq 0)$ has a non-random Lyapunov spectrum which coincides with that of the linearized continuous cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$, viz. $\{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, where each λ_i has fixed multiplicity q_i , $1 \leq i \leq m$ (Lemma 2). If $\lambda_i > 0$ for all $1 \leq i \leq m$, then take $\tilde{\mathcal{S}}(\omega) := \{Y(\omega)\}$ for all $\omega \in \Omega$. Theorem is trivial in this case. Suppose that at least one $\lambda_i < 0$.

Use discrete non-linear ergodic theorem of Ruelle (Theorem 4) and its proof to obtain a sure event $\Omega_1^* \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$, $\mathcal{F}$ -measurable positive random variables $\rho_1, \beta_1 : \Omega_1^* \rightarrow (0, \infty)$, $\rho_1 < \beta_1$, and a random family of $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds of $\bar{B}(0, \rho_1(\omega))$ denoted by $\tilde{\mathcal{S}}_d(\omega)$, $\omega \in \Omega_1^*$, and satisfying the following properties for each $\omega \in \Omega_1^*$:

$$\tilde{\mathcal{S}}_d(\omega) = \{x \in \bar{B}(0, \rho_1(\omega)) : |Z(n, x, \omega)| \leq \beta_1(\omega)e^{(\lambda_{i_0} + \epsilon_1)n} \text{ for all } n \in \mathbf{Z}^+\}. \quad (21)$$

$\tilde{\mathcal{S}}_d(\omega)$ is tangent at 0 to the stable subspace $\mathcal{S}(\omega)$ of the linearized flow $D_2\phi$, viz. $T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$. Therefore $\dim \tilde{\mathcal{S}}_d(\omega)$ is non-random by ergodicity of θ . Also

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \left[\sup_{\substack{x_1 \neq x_2, \\ x_1, x_2 \in \tilde{\mathcal{S}}_d(\omega)}} \frac{|Z(n, x_1, \omega) - Z(n, x_2, \omega)|}{|x_1 - x_2|} \right] \leq \lambda_{i_0}. \quad (22)$$

The $\theta(t, \cdot)$ -invariant sure event $\Omega_1^* \in \mathcal{F}$ is constructed using the ideas in Ruelle's proof (of Theorem 5.1 in [Ru.1], p. 293), combined with the estimate (10) of Lemma 2 and the subadditive ergodic theorem (Lemma 1 (ii)).

For each $\omega \in \Omega_1^*$, let $\tilde{\mathcal{S}}(\omega)$ be the set defined in part (a) of the theorem. Then by definition of $\tilde{\mathcal{S}}_d(\omega)$ and Z :

$$\tilde{\mathcal{S}}(\omega) = \tilde{\mathcal{S}}_d(\omega) + Y(\omega). \quad (23)$$

Since $\tilde{\mathcal{S}}_d(\omega)$ is a $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifold of $\bar{B}(0, \rho_1(\omega))$, then $\tilde{\mathcal{S}}(\omega)$ is a $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifold of $\bar{B}(Y(\omega), \rho_1(\omega))$. Furthermore, $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$. Hence $\dim \tilde{\mathcal{S}}(\omega) = \dim \mathcal{S}(\omega) = \sum_{i=i_0}^m q_i$, and is non-random.

Now (22) implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z(n, x, \omega)| \leq \lambda_{i_0} \quad (24)$$

for all ω in Ω_1^* and all $x \in \tilde{\mathcal{S}}_d(\omega)$. Therefore by Lemma 4, there is a sure event $\Omega_2^* \subseteq \Omega_1^*$ such that $\theta(t, \cdot)(\Omega_2^*) = \Omega_2^*$ for all $t \in \mathbf{R}$, and

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |Z(t, x, \omega)| \leq \lambda_{i_0} \quad (25)$$

for all $\omega \in \Omega_2^*$ and all $x \in \tilde{\mathcal{S}}_d(\omega)$. Therefore (2) holds.

To prove (b), let $\omega \in \Omega_1^*$. By (22), there is a positive integer $N_0 := N_0(\omega)$ (independent of $x \in \tilde{\mathcal{S}}_d(\omega)$) such that $Z(n, x, \omega) \in \bar{B}(0, 1)$ for all $n \geq N_0$. Let $\Omega_4^* := \Omega_2^* \cap \Omega_3$, where Ω_3 is the shift-invariant sure event defined in the proof of Lemma 4. Then Ω_4^* is a sure event and $\theta(t, \cdot)(\Omega_4^*) = \Omega_4^*$ for all $t \in \mathbf{R}$. By cocycle property, Mean-Value theorem and the ergodic theorem (Lemma 1(i)), we get (b).

To prove the invariance property (4), apply the Oseledec theorem to the linearized cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$ ([Mo.1], Theorem 4, Corollary 2). This gives a sure $\theta(t, \cdot)$ -invariant event, also denoted by Ω_1^* , such that $D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$. Equality holds because $D_2\phi(t, Y(\omega), \omega)$ is injective and $\dim \mathcal{S}(\omega) = \dim \mathcal{S}(\theta(t, \omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$.

To prove the asymptotic invariance property (3), use the ideas in the proofs of Theorems 5.1 and 4.1 in [Ru.1], pp. 285-297, to pick random variables ρ_1, β_1 and a sure event (also denoted by) Ω_1^* such that $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$, and with the property that for any $\epsilon \in (0, \epsilon_1)$ and every $\omega \in \Omega_1^*$, there exists a positive $K_1^\epsilon(\omega)$ for which the inequalities

$$\rho_1(\theta(t, \omega)) \geq K_1^\epsilon(\omega) \rho_1(\omega) e^{(\lambda_{i_0} + \epsilon)t}, \quad \beta_1(\theta(t, \omega)) \geq K_1^\epsilon(\omega) \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon)t} \quad (26)$$

hold for all $t \geq 0$. Use (b) to obtain a sure event $\Omega_5^* \subseteq \Omega_4^*$ such that $\theta(t, \cdot)(\Omega_5^*) = \Omega_5^*$ for all $t \in \mathbf{R}$, and for any $0 < \epsilon < \epsilon_1$ and $\omega \in \Omega_4^*$, there exists $\beta^\epsilon(\omega) > 0$ (independent of x) with

$$|\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t} \quad (27)$$

for all $x \in \tilde{\mathcal{S}}(\omega)$, $t \geq 0$. Fix $t \geq 0$, $\omega \in \Omega_5^*$ and $x \in \tilde{\mathcal{S}}(\omega)$. Let n be a non-negative integer. Then the cocycle property and (27) imply that

$$\begin{aligned}
& |\phi(n, \phi(t, x, \omega), \theta(t, \omega)) - Y(\theta(n, \theta(t, \omega)))| \\
&= |\phi(n + t, x, \omega) - Y(\theta(n + t, \omega))| \\
&\leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)(n+t)} \\
&\leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t} e^{(\lambda_{i_0} + \epsilon_1)n}. \tag{28}
\end{aligned}$$

If $\omega \in \Omega_5^*$, then it follows from (26), (27), (28) and the definition of $\tilde{\mathcal{S}}(\theta(t, \omega))$ that there exists $\tau_1(\omega) > 0$ such that $\phi(t, x, \omega) \in \tilde{\mathcal{S}}(\theta(t, \omega))$ for all $t \geq \tau_1(\omega)$. This proves asymptotic invariance.

We prove (d), regarding the existence of the local unstable manifolds $\tilde{\mathcal{U}}(\omega)$, by running both the flow ϕ and the shift θ backward in time:

$$\tilde{\phi}(t, x, \omega) := \phi(-t, x, \omega), \quad \tilde{Z}(t, x, \omega) := Z(-t, x, \omega), \quad \tilde{\theta}(t, \omega) := \theta(-t, \omega)$$

for all $t \geq 0$ and all $\omega \in \Omega$. $(\tilde{Z}(t, \cdot, \omega), \tilde{\theta}(t, \omega), t \geq 0)$ is a smooth cocycle, with $\tilde{Z}(t, 0, \omega) = 0$ for all $t \geq 0$. The linearized flow $(D_2 \tilde{\phi}(t, Y(\omega), \omega), \tilde{\theta}(t, \omega), t \geq 0)$ is an $L(\mathbf{R}^d)$ -valued perfect cocycle with a non-random finite Lyapunov spectrum $\{-\lambda_1 < -\lambda_2 < \dots < -\lambda_i < -\lambda_{i+1} < \dots < -\lambda_m\}$ where $\{\lambda_m < \dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1\}$ is the Lyapunov spectrum of the forward linearized flow $(D_2 \phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$. Apply first part of the proof to get *stable manifolds* for the backward flow $\tilde{\phi}$ satisfying assertions (a), (b), (c). This translates into the existence of *unstable manifolds* for the original flow ϕ , and (d), (e), (f) automatically hold. Hence

there is a sure event $\Omega_6^* \in \mathcal{F}$ such that $\theta(-t, \cdot)(\Omega_6^*) = \Omega_6^*$ for all $t \in \mathbf{R}$, and (d), (e) and (f) hold for all $\omega \in \Omega_6^*$.

Define the sure event $\Omega^* := \Omega_6^* \cap \Omega_5^*$. Then $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$. Assertions (a)-(f) hold for all $\omega \in \Omega^*$.

Measurability of the stable manifolds follows from

$$\tilde{\mathcal{S}}(\omega) = Y(\omega) + \tilde{\mathcal{S}}_d(\omega) \quad (29)$$

$$\tilde{\mathcal{S}}_d(\omega) = \lim_{m \rightarrow \infty} \bar{B}(0, \rho_1(\omega)) \cap \bigcap_{i=1}^m f_i(\cdot, \omega)^{-1}(\bar{B}(0, 1)) \quad (30)$$

$$f_n(x, \omega) := \beta_1(\omega)^{-1} e^{-(\lambda_{i_0} + \epsilon_1)n} Z(n, x, \omega), \quad x \in \mathbf{R}^d, \omega \in \Omega_1^*,$$

for all integers $n \geq 0$. (Above limit is taken in the metric d^* on $\mathcal{C}(\mathbf{R}^d)$.) Use joint continuity of translation and measurability of Y , f_i , ρ_1 , finite intersections and the continuity of the maps

$$\mathbf{R}^+ \ni r \mapsto \bar{B}(0, r) \in \mathcal{C}(\mathbf{R}^d).$$

$$\text{Hom}(\mathbf{R}^d) \ni f \mapsto f^{-1}(\bar{B}(0, 1)) \in \mathcal{C}(\mathbf{R}^d).$$

When h, g_i are in C_b^∞ , adapt above argument to give a sure event in $\mathcal{F}$, also denoted by Ω^* such that $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞ for all $\omega \in \Omega^*$. $\square$

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