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On the Autocorrelation of Complex Envelope of White Noise

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***Abstract*—About four decades ago, in an article in this transactions, Thomas Kailath pointed out that the autocorrelation of the complex envelope of white noise is not strictly an impulse function, even though when treated as an impulse in practical problems, it does lead to correct results. However, it is commonly assumed that by simply letting the bandwidth of a flat-bandlimited noise process to go to infinity, one obtains the result that the autocorrelation of the complex envelope of white noise equals an impulse function. In this correspondence, we show that 1) the limit operation has to be done carefully and 2) when done properly, it leads to the result in the Kailath's paper, which is different from a pure impulse function.**

***Index Terms*—Autocorrelation, bandlimited spectrum, complex envelope, impulse function, white noise.**

I. Introduction

Thomas Kailath pointed out that the autocorrelation of the complex envelope of white noise is not strictly an impulse function, even though when treated as an impulse in practical problems, it does lead to correct results [1]. This result was later mentioned in the textbook [2, p. 24]. The widely read textbook on digital communications, by Proakis, incorrectly states that by letting the bandwidth of a flat-bandlimited noise process to go to infinity, one obtains the result that the autocorrelation of the complex envelope of white noise equals an impulse function [3] (Kailath also called (incorrectly) the approach reasonable, see the statement on $\rho(t)$, for large W, between (1) and (2) of [1, p. 397]). The difficulty is that the autocorrelation function 4.1–57 of [3, p. 158] is valid only if the bandwidth B is less than twice the carrier frequency. Hence, if the carrier frequency is finite, B cannot tend to infinity (complex envelope representation of a bandpass process assumes that the carrier frequency is finite). As already mentioned, in practical problems, it is quite all right to consider the autocorrelation function to be an impulse, because such a consideration does not produce incorrect results. However, as a matter of practice, results have to be obtained using correct techniques. Here we show that the limit operation, when done properly, leads to the correct result in Kailath's paper.

II. Result on Complex Envelope of White Noise

Using the complex representation [1], a noise process $n(t)$ can be written as

$$n(t) = \mathrm{Re}\lfloor \eta(t) e^{j2\pi f_0 t} \rfloor \tag{1}$$

where

$$\eta(t) = [n(t) + j\hat{n}(t)] e^{-j2\pi f_0 t} \tag{2}$$

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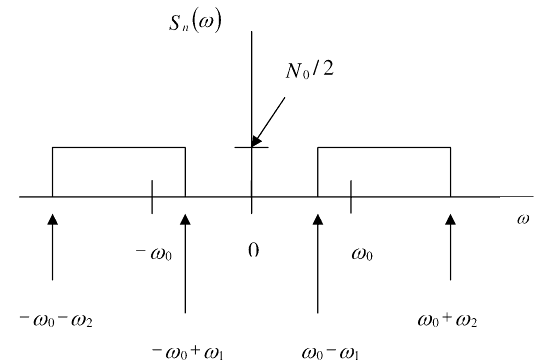

Fig. 1. Power spectral density of flat-bandlimited noise.

and $\hat{n}(t)$ is the Hilbert transform of $n(t)$. It can be shown by direct computation that the autocorrelation of the complex envelope $\eta(t)$ is

$$R_\eta(\tau) = 2[R_n(\tau) + j\hat{R}_n(\tau)] e^{-j\omega_0 \tau} \tag{3}$$

where $\omega_0 = 2\pi f_0$. When $n(t)$ is white, with spectral density $\frac{N_0}{2}$, (3) can be written as

$$R_\eta(\tau) = N_0 \left(\delta(\tau) + j\frac{1}{\pi\tau} \right) e^{-j\omega_0\tau}. \tag{4}$$

It will be shown now that one obtains exactly (4) by using the bandlimited spectrum approach. Fig. 1 shows the power spectral density of a flat-bandlimited process with carrier frequency f_0 Hz and bandwidth $f_2 + f_1$ Hz. We deliberately took the carrier frequency f_0 to be offset from the center frequency of the passband, because, for any finite carrier frequency and a flat power spectrum, an offset carrier frequency is needed, if the bandwidth were to be allowed to tend to infinity. With reference to Fig. 1, a white noise is obtained when both $\omega_1 \to \omega_0$ and $\omega_2 \to \infty$. For the spectrum in Fig. 1, the following equations are obtained:

$$\begin{aligned} R_n(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_n(\omega) e^{j\omega\tau} d\omega \\ &= \frac{N_0}{2\pi\tau} (\sin((\omega_1 - \omega_0)\tau) + \sin((\omega_2 + \omega_0)\tau) \end{aligned} \tag{5}$$

$$\begin{aligned} \hat{R}_n(\tau) &= \frac{1}{2\pi} \left[\int_{-\infty}^{0} jS_n(\omega) e^{j\omega\tau}\, d\omega + \int_{0}^{\infty} -jS_n(\omega) e^{j\omega\tau}\, d\omega \right] \\ &= \frac{N_0}{2\pi\tau} [\cos((\omega_0 - \omega_1)\tau) - \cos((\omega_2 + \omega_0)\tau)]. \end{aligned} \tag{6}$$

Using (3), (5), and (6), we get

$$R_\eta(\tau) = \frac{N_0}{j\pi\tau} \left[-e^{-j(\omega_1 - \omega_0)\tau} + e^{j(\omega_2 + \omega_0)\tau} \right] e^{-j\omega_0\tau}. \tag{7}$$

Now, using the Appendix I of [4],

$$\lim_{\omega_2 \to \infty} \frac{e^{j(\omega_2 + \omega_0)\tau}}{j\pi\tau} = \delta(\tau)$$

and

$$\lim_{\omega_1 \to \omega_0} -e^{-j(\omega_1 - \omega_0)\tau} = -1$$

we arrive at the result

$$\lim_{\substack{\omega_1 \to \omega_0 \\ \omega_2 \to \infty}} R_\eta(\tau) = N_0 \left(\delta(\tau) + \frac{j}{\pi\tau} \right) e^{-j\omega_0\tau}.$$

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Variable-Rate Two-Phase Collaborative Communication Protocols for Wireless Networks

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***Abstract*—The performance of two-phase collaborative communication protocols is studied for wireless networks. All the communication nodes in the cluster are assumed to share the same channel and transmit or receive collaboratively in a quasi-static Rayleigh flat-fading environment. In addition to small-scale fading, the effect of large-scale path loss is also considered. Based on a decode-and-forward approach, we consider various variable-rate two-phase protocols that can achieve full diversity order and analyze the effect of *node geometry* on their performance in terms of the outage probability of mutual information. For the single-relay node case, it is shown that if the collaborator node is close to the source node, a protocol based on space–time coding (STC) can achieve good diversity gain. Otherwise, a protocol based on receiver diversity performs better. These protocols are also compared with one based on fixed-rate repetition coding and their performance tradeoffs with node geometry are studied. The second part deals with multiple relays. It is known that with N relays an asymptotic diversity order of $N+1$ is achievable with STC-based protocols in the two-phase framework. However, in the framework of collaborative STC, those relay nodes which fail to decode remain silent (this event is referred to as a *node erasure*). We show that this node erasure has the potential to considerably reduce the diversity order and point out the importance of designing the STC to be robust against such node erasure.**

***Index Terms*—Collaborative (cooperative) communication, relay channel, space–time coding (STC), spatial diversity, wireless networks.**

I. INTRODUCTION

In many wireless networks, the power consumption of communication nodes is a critical issue. In addition, typical wireless channels suffer from signal fading which, for a given average transmit power, significantly reduces communication capacity and range. If the channel is slow and flat fading, channel coding does not help [1], [2] and spatial diversity may be the only effective option that can either reduce the average transmit power or increase communication range. Results on space–time coding (STC) [3], [4] have shown that the use of antenna arrays at the transmitter and receiver can significantly reduce transmit energy. However, for many applications with low-cost devices such as wireless sensor networks, deployment of multiple antennas at each node is too costly to implement due to severe constraints on both the size and power consumption of analog devices.

The recently proposed collaborative (or cooperative) diversity approaches [5]–[14] demonstrate the potential to achieve diversity or enhance the capacity of wireless systems without deploying multiple antennas at the transmitter. Using nearby collaborators as virtual antennas, significant diversity gains can be achieved. These schemes basically require that the source node shares the information bits with the relay nodes, and this data sharing process is generally achieved at the cost of additional orthogonal channels (in frequency or in time). In a companion paper [15], we have shown that for a given fixed rate and under suitable node geometry conditions, there are collaborative coding schemes that can nearly achieve the same diversity as if all the relay node antennas were connected to the source node, without any additional orthogonal channels or bandwidth. The construction of such codes, however, appears to be challenging.

Among many approaches in the literature, Laneman [5][6] analyzes several low-complexity relaying protocols that can achieve full diversity, under realistic assumptions such as half-duplex constraint and no channel state information (CSI) at the transmitting nodes. It has been shown that in the low-spectral-efficiency regime, the signal-to-noise power ratio (SNR) loss relative to the ideal transmit diversity system with the same information rate is 1.5 dB [5]. Multiple-relay cases are also considered in [6] and bandwidth-efficient STC-based collaborative protocols are proposed.

Collaborative diversity protocols are largely classified into *amplify-and-forward* and *decode-and-forward* schemes [5]. In the following, we will restrict our attention to decode-and-forward schemes since these may provide some salient advantages. First, there is no error propagation if the relay transmits information only when it decodes correctly. Otherwise, the relay remains silent and thus an unnecessary energy transmission can be saved.[1] Second, the information rate per symbol does not need to be the same for each phase. In other words, the relative duration of each phase can be changed according to node geometry.

It is the latter property that we shall focus on in this work. Suppose that we wish to transmit data with information rate R^* bits per second and T is the frame period, also in seconds. Then the total information transmitted during this period is R^*T bits (per frame). The baseline frame design that achieves this is shown in Fig. 1(a). Alternatively, we may split the time interval into two phases of duration T_1 and T_2 where $T = T_1 + T_2$ and each phase is operated with information rate R_1 and R_2, respectively, as depicted in Fig. 1(b). We assume that for both phases, the same information (but with different coding rate) is transmitted. If R_1 and R_2 are chosen such that $R_1T_1 = R_2T_2 = R^*T$, then in principle there is no loss of total transmission rate compared to the baseline system. Let the fraction of the relative time period for each phase be denoted by $\delta_1 \triangleq T_1/T = T_1/(T_1+T_2)$ and $\delta_2 \triangleq T_2/T = 1-\delta_1$. Then, the information rate during each phase is $R_1 = R^*/\delta_1$

[1]Even though perfect detection of the codeword is not feasible in practice, one can design a cyclic redundancy-check (CRC) or error detectable low-density parity-check (LDPC) code such that for a given system outage probability, the effect of error propagation is negligible. Many existing communication networks have this structure.

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