10-2004

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John P. McSorley
Southern Illinois University Carbondale, mcsorley60@hotmail.com

Thomas D. Porter
Southern Illinois University Carbondale

Published in *Discrete Mathematics*, 287(1-3), 85-91.

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John P. McSorley
Dept. of CCTM
London Metropolitan University
100 Minories
London, EC3N 1JY
mcsorley60@hotmail.com

Thomas D. Porter
Department of Mathematics
Southern Illinois University
Carbondale, IL 62901-4408
tporter@math.siu.edu

Abstract

Let $\{G_{p1}, G_{p2}, \ldots\}$ be an infinite sequence of graphs with G_{pn} having pn vertices. This sequence is called K_p-removable if $G_{p1} \cong K_p$, and $G_{pn} - S \cong G_{p(n-1)}$ for every $n \geq 2$ and every vertex subset S of G_{pn} that induces a K_p. Each graph in such a sequence has a high degree of symmetry: every way of removing the vertices of any fixed number of disjoint K_p's yields the same subgraph. Here we construct such sequences using componentwise Eulerian digraphs as generators. The case in which each G_{pn} is regular is also studied, where Cayley digraphs based on a finite group are used.

Keywords: Cayley, clique, digraph, Eulerian, reconstruction, removal, symmetric, uniform

1 K_p-removable sequences

In general we follow the notation in [5]. In particular, if $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S. Let p be a positive integer and n be a variable running from one to infinity. We use $[p] = \{1, \ldots, p\}$, and i for an element in $[p]$.

An infinite sequence of graphs $\{G_{pn}\} = \{G_{p1}, G_{p2}, \ldots\}$, with G_{pn} having pn vertices, is K_p-*removable* if it satisfies the following two properties:

P1 $G_{p1} \cong K_p$,

P2 for every $n \geq 2$, the graph G_{pn} contains at least one K_p and $G_{pn} - S \cong G_{p(n-1)}$ for every S for which $G_{pn}[S] \cong K_p$.

Each graph in such a sequence has a high degree of symmetry: every way of removing the vertices of any fixed number of disjoint K_p's yields the same subgraph. We call this property *clique-symmetric*.

We often write $G = G'$ in place of $G \cong G'$, and refer to K_p as a p-clique.

Let $\vec{D}$ be a digraph without loops and multiple arcs, and with vertex set $[p]$. Let $\vec{ii'}$ denote an arc in $A(\vec{D})$, then i' is an out-neighbour to vertex i. Let i have $d^+(i)$ out-neighbours and $d^-(i)$ in-neighbours.

The following graph construction is central to this paper:

Consider a copy of K_p with vertices labelled $\{(1,1), \ldots, (p,1)\} = \{(i,1) \mid i \in [p]\}$; call these vertices *vertices at level* 1, and call this graph $D_1(K_p)$. Now consider another K_p with vertices labelled $\{(i,2) \mid i \in [p]\}$, these are vertices at level 2. For any vertex $(i,2)$ join it to vertices $\{(i',1) \mid \vec{ii'} \in A(\vec{D})\}$ at level 1; call this graph $D_2(K_p)$. Now consider a third K_p with vertices labelled $\{(i,3) \mid i \in [p]\}$, at level 3. Join any vertex $(i,3)$ to vertices $\{(i',2) \mid \vec{ii'} \in A(\vec{D})\}$ at level 2 and to vertices $\{(i',1) \mid \vec{ii'} \in A(\vec{D})\}$ at level 1; this is $D_3(K_p)$.

Now, for any $n \geq 1$, consider the graph which has been constructed level by level, up to n levels, according to this definition; call this graph $D_n(K_p)$ or simply D_n when p is clear. We say that the digraph $\vec{D}$ *generates* the sequence $\{D_n\} = \{D_1, D_2, \ldots\}$.

In D_n the vertices are of the form (i,j) for every $i \in [p]$ and every j, $1 \leq j \leq n$, (where j is their level); and the edges are of two types:

(i) *fixed-level* edges, say at level j

$$((i_1, j), (i_2, j)) \text{ is an edge for all } i_1, i_2 \in [p] \text{ where } i_1 \neq i_2; \text{ and}$$

(ii) *cross-level* edges, for $j > j'$

$$((i,j),(i',j')) \text{ is an edge if and only if } \vec{ii'} \in A(\vec{D}).$$

Call digraph $\vec{D}$ *uniform* if $d^+(i) = d^-(i)$ for every vertex i in $\vec{D}$. Note that $\vec{D}$ need not be connected. Then $\vec{D}$ is an Eulerian digraph if it has one component, otherwise $\vec{D}$ is Eulerian on each of its components.

In this paper we study the sequences $\{D_n\}$. In Section 2 our main result (Theorem 2.3) states that if $\vec{D}$ is uniform then its generated sequence $\{D_n\}$ is K_p-removable. In Section 3 we construct sequences in which each graph is regular. We use λ-uniform digraphs; these satisfy $\lambda = d^+(i) = d^-(i)$ for every vertex i in $\vec{D}$. They can be constructed in a similar manner to Cayley digraphs. We count the exact number of K_p's in the graphs in these sequences. Many examples are given throughout the paper, as well as indications for further research.

2 $\{D_n\}$ is K_p-removable for uniform $\vec{D}$

In this section we consider $\{D_n\}$, the sequence of graphs generated by digraph $\vec{D}$. Often $\vec{D}$ will be uniform. In order to prove that $\{D_n\}$ is K_p-removable in this case, we are interested in the K_p's in such D_n. The next theorem gives necessary and sufficient conditions for their existence.

For each $i \in [p]$, let $I_i = \{(i,1),\ldots,(i,n)\} = \{(i,j) \mid 1 \le j \le n\}$ be the set of vertices in D_n in 'column i'. Then, because $\vec{D}$ is loopless, *i.e.*, $\vec{ii} \notin A(\vec{D})$, this is an independent set of vertices, the *i-th independent set*.

Now let $V = \{(1,v_1),\ldots,(p,v_p)\}$ be an arbitrary vertex subset in D_n with exactly one vertex from each independent set I_i. Let V have vertices at m different levels: $\ell_1,\ldots,\ell_m$ where $\ell_1 < \cdots < \ell_m$. For each k, $1 \le k \le m$, let $V_k = \{i \mid v_i = \ell_k\} \ne \emptyset$ be the set of first coordinates of all vertices of V at level ℓ_k . Then the sets $V_1,\ldots,V_m$ form a *level-partition* of $[p] = \{1,\ldots,p\}$.

Now $D_n[V]$ contains the cross-level edge $((i,\ell_k),(i',\ell_{k'}))$ where $\ell_{k'} < \ell_k$ if and only if $\vec{ii'}$ is an arc in $\vec{D}$. We call $\vec{ii'}$ a *V-skew arc*. Hence a V-skew arc in $\vec{D}$ 'joins' different levels of V.

Let $\overrightarrow{AB}$ denote the set of arcs in $\vec{D}$ from A to B, *i.e.*, all arcs $\vec{ab}$ with $a \in A$ and $b \in B$.

Theorem 2.1 *Let $\vec{D}$ be a uniform digraph with p vertices. Then $D_n[V]$ is a p-clique in D_n if and only if the associated V-skew arcs form a complete symmetric m-partite subdigraph in $\vec{D}$.*

Proof. Suppose that $D_n[V]$ is a p-clique with level-partition $V_1, \ldots, V_m$. The digraph $\vec{D}$ is uniform so the number of arcs entering any vertex subset equals the number of arcs outgoing from it. Now $D_n[V]$ is a p-clique so, in $\vec{D}$, $\overrightarrow{V_k V_{k'}}$ is complete for each k' and k, $1 \leq k' < k \leq m$; in particular $\overrightarrow{V_k V_1}$ is complete for each k, $2 \leq k \leq m$. The number of arcs entering V_1 is $|V_1|(|V_2| + \cdots + |V_m|)$ which equals the number of outgoing arcs, hence $\overrightarrow{V_1 V_k}$ is also complete for each k, $2 \leq k \leq m$.

So $\overrightarrow{V_1 V_2}$ is complete, and we can apply a similar argument to V_2 to show that $\overrightarrow{V_2 V_k}$ is complete for each k, $3 \leq k \leq m$, then to V_3,..., and so on. Consequently, $\overrightarrow{V_{k'} V_k}$ is complete for each k' and k, $1 \leq k' < k \leq m$, *i.e.*, the V-skew arcs form a complete symmetric m-partite subdigraph in $\vec{D}$.

The converse is straightforward. ■

We usually refer to a p-clique in D_n as W. From the construction of D_n, for vertex (i, j) in D_n its degree is given by

$$\deg(i, j) = d^+(i)(j-1) + d^-(i)(n-j) + p - 1.$$

Corollary 2.2 *Let $\vec{D}$ be a uniform digraph with p vertices. If $D_n[W]$ is a p-clique then the number of edges in $D_n - W$ equals the number of edges in D_{n-1}.*

Proof. Now $D_n[W] = K_p$ so the number of edges 'inside' W equals the number of edges inside the K_p at level n of D_n. For any vertex (i, j) in D_n we have by uniformity that $\deg(i, j) = d^+(i)(n-1) + p - 1$. So, if (i, j) is in W then its degree 'outside' W is $d^+(i)(n-1)$, which is independent of its level j. This outside degree is the same as the degree outside the K_p at level n of the level n vertex (i, n). Hence the removal of W from D_n removes the same number of edges as the removal of the K_p at level n, and so the result. ■

Now for our main result.

Theorem 2.3 *Let $\vec{D}$ be a uniform digraph with p vertices. Then its generated sequence of graphs $\{D_n\}$ is K_p-removable.*

Proof. Suppose W induces a p-clique in D_n. Let the vertices of W be $\{(i, w_i) \,|\, 1 \leq i \leq p\}$. Now we construct a bijection ϕ between the vertices of $D_n - W$ and the vertices of D_{n-1}. Under ϕ, for a fixed $i \in [p]$, the vertices in the i-th independent set of $D_n - W$, namely in the set $I_i \backslash \{(i, w_i)\}$, are mapped to the vertices in the i-th independent set of D_{n-1}, namely to the set $\{(i, 1), \ldots, (i, n-1)\}$, as follows:

$$\phi(i,j) = \begin{cases} (i,j), & \text{for } 1 \leq j < w_i \\ (i, j-1), & \text{for } w_i < j \leq n. \end{cases}$$

Clearly ϕ is a bijection. It is straightforward to show that ϕ moves edges in $D_n - W$ to edges in D_{n-1}.

Now, from Corollary 2.2, the graphs $D_n - W$ and D_{n-1} have the same number of edges, and so ϕ is an isomorphism. Hence $\{D_n\}$ satisfies **P2**. Clearly $\{D_n\}$ satisfies **P1**, which gives the result. ∎

Example 1 $p = 3$, $V(\vec{D}) = \{1, 2, 3\}$, $A(\vec{D}) = \{\vec{12}, \vec{21}, \vec{23}, \vec{32}\}$. Then $\vec{D}$ is uniform with 3 vertices. The first three graphs in the K_3-removable sequence $\{D_n\}$ are shown in Figure 1 on page 7. Notice the level-partition $V_1 = \{1, 3\}$, $V_2 = \{2\}$ which illustrates Theorem 2.1.

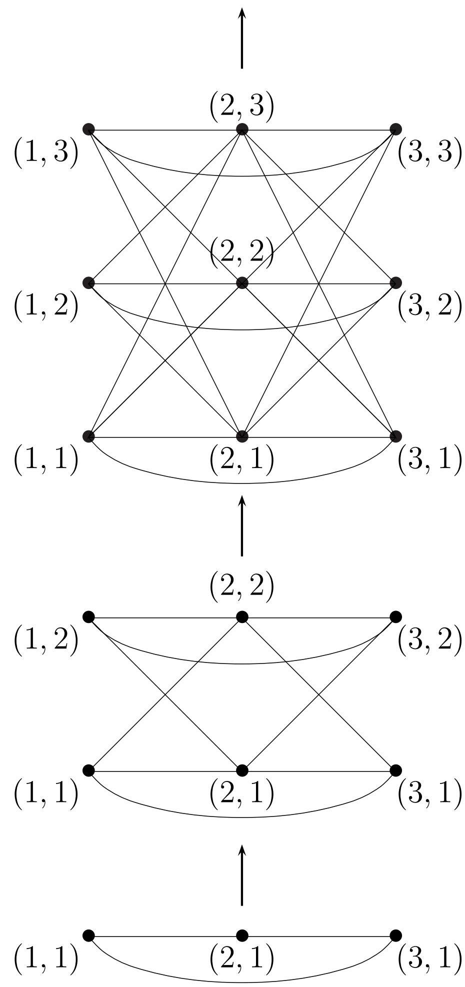

Figure 1

The converse of Theorem 2.3 is not true:

Example 2 $p = 3$, $V(\vec{D}) = \{1,2,3\}$, $A(\vec{D}) = \{\vec{12}\}$. Then $\{D_n\}$ is K_3-removable, but $\vec{D}$ is not uniform.

Question Is every K_p-removable sequence isomorphic to the generated sequence of some digraph $\vec{D}$? (From Example 2 we know that $\vec{D}$ need not be uniform.)

The K_p-removable sequence $\{G_{pn}\}$ is *regular* if every graph G_{pn} is regular, and *irregular* otherwise. In general, the sequence $\{D_n\}$ is irregular, see Example 1. It is straightforward to show that all K_p-removable sequences with $p = 1$ or 2 are regular; they will given in Theorem 3.3 below. However, for every $p \geq 3$ an irregular K_p-removable sequence exists:

Example 3 $p \geq 3$, $V(\vec{D}) = [p]$, $A(\vec{D}) = \{\vec{12}, \vec{21}, \vec{23}, \vec{32}\}$. Then $\vec{D}$ is uniform with p vertices, so $\{D_n\}$ is K_p-removable. However the graph D_2 is irregular because $\deg(1,2) = p$ but $\deg(2,2) = p+1$, so $\{D_n\}$ is irregular.

Call two K_p-removable sequences $\{G_{pn}\}$ and $\{G'_{pn}\}$ *isomorphic*, denoted by $\{G_{pn}\} \cong \{G'_{pn}\}$, if $G_{pn} \cong G'_{pn}$ for every $n \geq 1$.

Let $\theta : \vec{D} \to \vec{D}'$ be an isomorphism between uniform digraphs $\vec{D}$ and $\vec{D}'$. For every fixed $n \geq 1$, θ induces an isomorphism Θ between D_n and D'_n given by: $\Theta(i,j) = (\theta(i), j)$, for every $i \in [p]$ and j with $1 \leq j \leq n$. Hence, for every $n \geq 1$, $D_n \cong D'_n$ and so $\{D_n\} \cong \{D'_n\}$. We conjecture that the converse is true:

Conjecture Let $\{D_n\}$ and $\{D'_n\}$ be two K_p-removable sequences generated by uniform digraphs $\vec{D}$ and $\vec{D}'$, respectively. If $\{D_n\} \cong \{D'_n\}$ then $\vec{D} \cong \vec{D}'$.

As a final remark we note that the above construction of a K_p-removable sequence needs a uniform digraph with vertex set $[p]$. One way to construct such a uniform digraph is to take an undirected graph H with vertex set $[p]$ and 'double-orientate' each edge in H, *i.e.*, replace each edge (i, i') with two arcs $\vec{ii'}$ and $\vec{i'i}$. Indeed, $\vec{D}$ in Example 1 was obtained from double-orientating the path on 3 vertices.

3 Generating regular (K_p, λ)-removable sequences using finite groups

Recall the definition of a regular K_p-removable sequence given above.

A uniform digraph $\vec{D}$ is called λ-*uniform* if there is a natural number λ such that $\lambda = d^+(i) = d^-(i)$ for every vertex i in $\vec{D}$. Note that $0 \le \lambda \le p-1$ when $\vec{D}$ has p vertices.

We noted in the proof of Corollary 2.2 that, for a uniform digraph $\vec{D}$ with p vertices, the degree of any vertex (i,j) in D_n is $\deg(i,j) = d^+(i)(n-1) + p - 1$. If $\vec{D}$ is λ-uniform, then $\deg(i,j) = \lambda(n-1) + p - 1$, which does not depend on i or j. Hence D_n is regular of degree $\lambda(n-1)+p-1$, and $\{D_n\}$ is a regular K_p-removable sequence. We call $\{D_n\}$ a regular (K_p, λ)-removable sequence.

So, from Theorem 2.3, we have

Theorem 3.1 *Let $\vec{D}$ be a λ-uniform digraph with p vertices. Then its generated sequence of graphs $\{D_n\}$ is regular (K_p, λ)-removable.* ■

In this section we study such regular sequences $\{D_n\}$. To generate such a sequence we need a λ-uniform digraph. For this we can double-orientate a λ-regular graph H. However, this is only sufficient when such a λ-regular graph exists. Instead, we use a Cayley-type digraph which we obtain from an arbitrary finite group. See Biggs [2] and Grossman and Magnus [4].

Let $p \ge 1$ and let $\mathcal{G}_p = \{g_1, \ldots, g_p\}$ be a finite group with p elements, where e is the identity element. Let $\Lambda \subseteq \mathcal{G}_p$ be a subset of $\mathcal{G}_p$ with $e \notin \Lambda$ and with $|\Lambda| = \lambda$, where clearly $0 \le \lambda \le p-1$.

We form a digraph $\vec{D} = \overrightarrow{(\mathcal{G}_p, \Lambda)}$ from $\mathcal{G}_p$ and Λ as follows:

the vertices of $\vec{D}$ are $\{g_1, \ldots, g_p\}$ and

$\overrightarrow{g_i g_{i'}}$ is an arc in $\vec{D}$ if and only if $g_{i'} g_i^{-1} \in \Lambda$.

We see that $d^+(g_i) = d^-(g_i) = |\Lambda| = \lambda$ for every vertex g_i, hence $\vec{D}$ is λ-uniform. Consequently, using Theorem 3.1 above, $\{D_n\}$ is a regular (K_p, λ)-removable sequence. (Note that Λ need not be a generating set for $\mathcal{G}_p$; this is why we call $\overrightarrow{(\mathcal{G}_p, \Lambda)}$ a Cayley-type digraph rather than a Cayley digraph.)

Now for every $p \ge 1$ there is a cyclic group with p elements, $\mathcal{C}_p$, and a $\Lambda \subseteq \mathcal{C}_p$ with $e \notin \Lambda$ and $|\Lambda| = \lambda$ for each $0 \le \lambda \le p-1$; and, permitting henceforth $\lambda = p$ corresponding to loops in $\vec{D}$, for every $p \geq 1$ there is a regular (K_p, p)-removable sequence, namely $\{K_{pn}\}$. So we have the following existence result for regular (K_p, λ)-removable sequences:

Theorem 3.2 *For every $p \geq 1$ and every λ, $0 \leq \lambda \leq p$, there exists a regular (K_p, λ)-removable sequence.* ∎

The cases corresponding to $\lambda = 0$, $p - 1$, and p are especially interesting; they result in sequences that are unique up to isomorphism. Let $K_{p\times n} = K_{\underbrace{n, \ldots, n}_{p}}$ be the complete p-partite graph on pn vertices. The proof of the following Theorem is straightforward.

Theorem 3.3 *For every $p \geq 1$ there is a unique regular (K_p, λ)-removable sequence for $\lambda = 0$, $p - 1$, or p:*
(i) $\{nK_1\}$ *is the unique regular $(K_1, 0)$-removable sequence,*
(ii) $\{K_n\}$ *is the unique regular $(K_1, 1)$-removable sequence.*
and, for every $p \geq 2$,
(iii) $\{nK_p\}$ *is the unique regular $(K_p, 0)$-removable sequence,*
(iv) $\{K_{p\times n}\}$ *is the unique regular $(K_p, p-1)$-removable sequence,*
(v) $\{K_{pn}\}$ *is the unique regular (K_p, p)-removable sequence.* ∎

The λ-uniform digraphs needed to generate the last three sequences in Theorem 3.3 are: (iii) the 0-uniform digraph with p vertices and no arcs; (iv) the $(p-1)$-uniform digraph obtained by double-orientating the complete undirected graph K_p; and (v) the p-uniform digraph obtained by attaching one loop to each vertex to the digraph in (iv). (Note that in (v) the digraph is not loopless, but the construction still works.)

Example 4 Let $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ be the additive group (mod p). For $\lambda = 0$ set $\Lambda = \emptyset$, and for $1 \leq \lambda \leq p - 1$ set $\Lambda = \{1, 2, \ldots, \lambda\}$, and for $\lambda = p$ set $\Lambda = \mathbb{Z}_p$. Note that in this last case $0 \in \Lambda$, contrary to our previous assumption that $e \notin \Lambda$, but this causes no problems. Then $\overrightarrow{(\mathbb{Z}_p, \Lambda)}$ generates a regular (K_p, λ)-removable sequence for each λ, $0 \leq \lambda \leq p$. So $\overrightarrow{(\mathbb{Z}_p, \Lambda)}$ generates a spectrum of graph sequences among which are the three sequences of Theorem 3.3$(iii) - (v)$, namely $\{nK_p\}, \ldots, \{K_{p\times n}\}$, and $\{K_{pn}\}$.

As usual let $\{D_n\}$ be the regular (K_p, λ)-removable sequence obtained from a generating digraph $\vec{D} = (\overrightarrow{\mathcal{G}_p, \Lambda})$. Analogous to Theorem 2.1, we describe the structure induced on $\vec{D}$ from p-cliques in D_n.

Let $\overline{\Lambda}$ denote the complement of Λ in $\mathcal{G}_p$ and let $\langle\overline{\Lambda}\rangle$ be the subgroup generated by $\overline{\Lambda}$, also let $\langle\overline{\Lambda}\rangle g$ denote a typical coset of this subgroup.

Let $V = \{(g_1, v_1), \ldots, (g_p, v_p)\}$ be an arbitrary vertex subset in D_n with exactly one vertex from each independent set $I_i = \{(g_i, j) \,|\, 1 \leq j \leq n\}$. As in Section 2, let V have vertices at m different levels: $\ell_1, \ldots, \ell_m$ where $\ell_1 < \cdots < \ell_m$. For each k, $1 \leq k \leq m$, let $V_k = \{g_i \,|\, v_i = \ell_k\} \neq \emptyset$ be the set of first coordinates of all vertices of V at level ℓ_k . Then the sets $V_1, \ldots, V_m$ form a level-partition of $\mathcal{G}_p$, and we have:

Theorem 3.4 *Let $\vec{D} = (\overrightarrow{\mathcal{G}_p, \Lambda})$ be a λ-uniform digraph with generated sequence $\{D_n\}$. Then $D_n[V]$ is a p-clique in D_n if and only if each V_k is a union of cosets of $\langle\overline{\Lambda}\rangle$.*

Proof. For any $r \geq 1$ let $\prod(r) = h_1 \cdots h_r$ denote a product of r arbitrary elements $h_1, \ldots, h_r$ from $\overline{\Lambda}$. Clearly for any $a \in \langle\overline{\Lambda}\rangle$ we can express a as $\prod(r)$ for some fixed $r \geq 1$ and some suitably chosen r elements $h_1, \ldots, h_r$ from $\overline{\Lambda}$.

Suppose $D_n[V]$ is a p-clique in D_n with level partition $V_1, \ldots, V_m$. Consider any V_k and let $g_i \in V_k$. Then $\prod(1)g_i \in V_k$ for any $\prod(1)$. For suppose otherwise. Then there exists a $\prod(1) = h_1$, say, with $h_1 g_i \in V_{k'}$ for some $k' \neq k$. However, this implies from Theorem 2.1 that $\overrightarrow{g_i(h_1 g_i)}$ is an arc in $\vec{D}$, *i.e.*, $(h_1 g_i)g_i^{-1} = h_1 \in \Lambda$, a contradiction.

Now we show that if any $\prod(r)g_i \in V_k$ then any $\prod(r+1)g_i \in V_k$. For suppose that there is a $\prod(r+1) = a(r+1) = h_1 \cdots h_{r+1}$ with $a(r+1)g_i \notin V_k$. Then, by similar reasoning to the above, we must have $a(r+1)g_i \in V_{k''}$ for some $k'' \neq k$. Let $a(r) = h_2 \cdots h_{r+1}$; then, by the induction hypothesis, $a(r)g_i \in V_k$. Hence $\overrightarrow{a(r)g_i(a(r+1)g_i)}$ is an arc in $\vec{D}$, and, as above, $h_1 \in \Lambda$, a contradiction.

Hence the induction goes through, and, for any $a \in \langle\overline{\Lambda}\rangle$ we have $ag_i \in V_k$, *i.e.*, we have $\langle\overline{\Lambda}\rangle g_i \subseteq V_k$. Hence V_k is a union of cosets of $\langle\overline{\Lambda}\rangle$.

For the converse, let each V_k be a union of cosets of $\langle\overline{\Lambda}\rangle$. Let (g_i, ℓ_k) and $(g_{i'}, \ell_{k'})$ be two arbitrary vertices in V. We show that $((g_i, \ell_k), (g_{i'}, \ell_{k'}))$ is an edge in D_n. If $\ell_k = \ell_{k'}$ then, certainly, $((g_i, \ell_k), (g_{i'}, \ell_{k'}))$ is an edge by construction of D_n. Otherwise, without loss of generality, let $\ell_k > \ell_{k'}$. Then g_i and $g_{i'}$ are in different cosets of $\langle\overline{\Lambda}\rangle$, so $g_{i'}g_i^{-1} \notin \langle\overline{\Lambda}\rangle$, so $g_{i'}g_i^{-1} \in \overline{\langle\overline{\Lambda}\rangle} \subseteq \Lambda$, and again $((g_i, \ell_k), (g_{i'}, \ell_{k'}))$ is an edge. Thus $D_n[V] = K_p$, as required. ∎

Theorem 3.4 enables us to count the exact number of K_p's in D_n. Let $|\mathcal{G}_p : \langle\overline{\Lambda}\rangle|$ be the index of $\langle\overline{\Lambda}\rangle$ in $\mathcal{G}_p$, *i.e.*, the number of cosets of $\langle\overline{\Lambda}\rangle$ in $\mathcal{G}_p$.

Corollary 3.5 *The number of K_p's in D_n is $n^{|\mathcal{G}_p:\langle\overline{\Lambda}\rangle|}$.*

Proof. Consider any coset $\langle\overline{\Lambda}\rangle g$, let us 'place' the elements of this coset at any fixed level j, where $1 \leq j \leq n$, in the graph D_n. Each such placement of every coset of $\langle\overline{\Lambda}\rangle$ gives a K_p and every K_p corresponds to such a placement of every coset of $\langle\overline{\Lambda}\rangle$. Hence, the number of K_p's in D_n equals the number of such placements of all the cosets of $\langle\overline{\Lambda}\rangle$. There are $|\mathcal{G}_p : \langle\overline{\Lambda}\rangle|$ cosets, and n levels to place each, hence $n^{|\mathcal{G}_p:\langle\overline{\Lambda}\rangle|}$ such placements and so $n^{|\mathcal{G}_p:\langle\overline{\Lambda}\rangle|}$ corresponding K_p's. ∎

Finally we briefly consider three more topics: firstly, we discuss pairs (p, λ) for which there is a unique regular (K_p, λ)-removable sequence up to isomorphism; secondly, we prove that if any member of an arbitrary K_p-removable sequence $\{G_{pn}\}$ contains a K_{p+1} then $\{G_{pn}\} = \{K_{pn}\}$; lastly, we list some possibilities for further research.

Let $\mathfrak{U}$ denote the set of pairs (p, λ) for which there is a *unique* regular (K_p, λ)-removable sequence up to isomorphism. Then, from Theorem 3.3, for every $p \geq 1$ we have $(p, 0)$, $(p, p-1)$, and $(p, p) \in \mathfrak{U}$. Now we use Corollary 3.5 to show that for every even $p \geq 4$, we have $(p, p-2) \notin \mathfrak{U}$.

Example 5 For every even $p \geq 4$ there are at least two non-isomorphic regular $(K_p, p-2)$-removable sequences:

For the first let $\mathcal{G}_p = \mathcal{D}_{\frac{p}{2}}$ be the dihedral group with p elements, the group of symmetries of the regular $\frac{p}{2}$-gon. We have $\mathcal{D}_{\frac{p}{2}} = \langle a, b \,|\, a^{\frac{p}{2}} = b^2 = (ab)^2 = e\rangle$. Let $\Lambda = \mathcal{D}_{\frac{p}{2}} \backslash \{e, b\}$ so that $|\Lambda| = p-2$ and $e \notin \Lambda$. So $\langle\overline{\Lambda}\rangle = \{e, b\}$ and $|\mathcal{D}_{\frac{p}{2}} : \langle\overline{\Lambda}\rangle| = \frac{p}{2}$. Thus D_n has $n^{\frac{p}{2}}$ K_p's.

For the second let $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ be the additive group (mod p). Let $\Lambda = \{1, 2, \ldots, p-2\}$, then $|\Lambda| = p-2$ and $0 \notin \Lambda$. But $p-1 \in \overline{\Lambda}$ and $p-1$ generates $\mathbb{Z}_p$ *i.e.*, $\langle\overline{\Lambda}\rangle = \mathbb{Z}_p$, and so $|\mathbb{Z}_p : \langle\overline{\Lambda}\rangle| = 1$ and D'_n has n K_p's.

Thus $D_2 \not\cong D'_2$ and so $\{D_n\} \not\cong \{D'_n\}$, and for every even $p \geq 4$, we have $(p, p-2) \notin \mathfrak{U}$. Note that D_2 is K_{2p} minus the edges of $p/2$ disjoint 4-cycles, while D'_2 is K_{2p} minus the edges of a Hamiltonian cycle.

Now we show that if any member of an arbitrary K_p-removable sequence $\{G_{pn}\}$ contains a K_{p+1} then $\{G_{pn}\} = \{K_{pn}\}$.

Theorem 3.6 *Suppose that for some $n \geq 2$ the n^{th} member, G_{pn}, of the K_p-removable sequence $\{G_{pn}\}$ contains a K_{p+1}. Then $G_{pn} = K_{pn}$ and $\{G_{pn}\} = \{K_{pn}\}$.*

Proof. Now G_{pn} contains a K_{p+1}. Since every K_p in G_{pn} is part of a partition of $V(G_{pn})$ into disjoint p-cliques, we may assume without loss of generality that $V(G_{pn})$ is partitioned into n p-cliques $L_1, \ldots, L_n$ so that some vertex u in L_2 is joined to every vertex of L_1, *i.e.*, $L_1 \cup \{u\} = K_{p+1}$. Let v be any vertex in L_1. Deleting the $n-1$ p-cliques $L_3, L_4, \ldots, L_n, L_1 + \{u\} - \{v\}$ in this order, we obtain the p-clique $L_2 + \{v\} - \{u\}$. Hence v is adjacent to every vertex of L_2 and the union of L_1 and L_2 is K_{2p}. Consequently, the removal of any $n-2$ disjoint K_p's must produce a K_{2p}. This implies that the union of every two levels L_j and $L_{j'}$ is K_{2p}; therefore, G_{pn} is a complete graph. Hence $G_{pn} = K_{pn}$.

Then clearly for every $n' > n$ we have $G_{pn'} = K_{pn'}$. And, by removing the required number of K_p's, for every $n' < n$ we have $G_{pn'} = K_{pn'}$ also. Hence $\{G_{pn}\} = \{K_{pn}\}$. ∎

Some further research possibilities are the following:

(A) Investigate the Question and Conjecture mentioned near the end of Section 2.

(B) Investigate the set $\mathfrak{U}$; in particular, is $(3, 1) \in \mathfrak{U}$?

For other papers on graph sequences see Barefoot, Entringer, and Jackson [1], and the references therein; another somewhat related paper is Duchet, Tuza, and Vestergaard [3].

Acknowledgment The authors thank the referees for providing insight and suggestions that greatly improved this paper.

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