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Directional Duality Theory

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Abstract

Shephard (1953, 1970,1974)introduced radial distance functions as representations of a firm's technology and developed a number of dual representations that have been widely applied in empirical work. A systematic exposition of Shephard's work can be found in Färe and Primont (1995). More recently, work by Luenberger (1992, 1995) has provided some new technology representations, the benefit and the shortage functions. Exploiting these results, Chambers, Chung, and Färe (1996, 1998a) introduced directional distance functions; these can be thought of as additive alternatives to the corresponding radial concepts. In this paper, the radial approach is further extended by introducing and characterizing indirect directional distance functions; these are directional versions of their radial counterparts. This, in turn, leads to a new set of duality results that will be of use in applied work.

1 Introduction

Shephard (1953, 1970,1974) introduced radial distance functions as representations of a firm's technology and developed a number of dual representations that have been widely applied in empirical work. A systematic exposition of Shephard's work can be found in Färe and Primont (1995). More recently, work by Luenberger (1992,1995) has provided some new technology representations, the benefit and the shortage functions. Exploiting these results, Chambers, Chung, and Färe (1996, 1998a) introduced directional distance functions; these can be thought of as additive alternatives to the corresponding radial concepts. In this paper, the radial approach is further extended by introducing and characterizing indirect directional distance functions; these are directional versions of their radial counterparts. This, in turn, leads to a new set of duality results that will be of use in applied work.

Before proceeding we need to ask the question: Does the world need more duality theorems? We think that the answer is Yes. In Färe and Primont (1995) we made the argument that a variety of models are useful in explaining the behavior of firms.

This arises from the consideration that it is useful to study different firms using different behavioral assumptions[1]. These different behavioral assumptions lead us to different dual representations of the production technology. These dual functions have natural counterparts, called distance functions, that represent either direct or indirect technologies. Here is a list of these dual function-distance functions pairs along with their related behavioral assumption.

- Revenue function - Output distance function: the firm maximizes the revenue from outputs given inputs and output prices.
- Cost function - Input distance function: the firm minimizes the cost of inputs given outputs and input prices.
- Indirect revenue function - Indirect output distance function: the firm maximizes the revenue from outputs given input prices, total input cost, and output prices.
- Indirect cost function - Indirect input distance function: the firm minimizes the cost of inputs given output prices, total output revenue, and input prices.
- Profit function: the firm maximizes profit given output prices and input prices.

(The profit function is dual to the technology set; a distance function was not used to represent this set.)

Now that Luenberger (1992,1995) and Chambers, Chung, and Färe (1996, 1998a) have introduced the directional technology distance function we are motivated to update our 1995 book to account for directional distance functions. The arguments for why we should use directional distance functions has been advanced in the above references and won't be repeated here. Additional references with applications include Chambers and Färe (1998b), Färe and Grosskopf (2000), Färe and Primont (2003), and Hudgins and Primont (2003).

2 Some Basic Concepts

Let $x \in \mathbb{R}^N_+$ be the input vector and let $y \in \mathbb{R}^M_+$ be the output vector. The technology T is given by

$$T = \{(x, y) : \ x \text{ can produce } y\} . \tag{1}$$

[1] Since the different dual functions have different arguments they dictate the use of different data. In practice this may be reversed; the availability of data may dictate the model choice.

Certain assumptions about the technology enable us to establish duality properties for functional representations of T. They are[2]

$$\begin{aligned}&T \text{ is a nonempty, closed, convex set}\\&\text{and both inputs and outputs are strongly disposable.}\end{aligned} \tag{T}$$

When working with duality[3] relationships in output price/quantities spaces it is sometimes convenient to represent the technology with *direct output sets* defined by

$$P(x) = \{y : (x, y) \in T\}. \tag{2}$$

We assume that

$$\begin{aligned}&\text{For all } x \text{ in } R_+^N,\ P(x) \text{ is a nonempty, compact, convex set}\\&\text{and outputs are strongly disposable.}\end{aligned} \tag{P}$$

At other times it is convenient to represent the technology by *indirect output sets* defined by

$$IP(w/C) = \{y : y \in P(x), wx \leq C\}, \tag{3}$$

where $w \geq 0_N$ is a vector of input prices and $C > 0$ is the total input cost. We assume that

$$\begin{aligned}&\text{For all } w/C \text{ in } R_+^N,\ IP(w/C) \text{ is a nonempty, compact, convex set}\\&\text{and outputs are strongly disposable.}\end{aligned} \tag{IP}$$

For duality relationships in input spaces it is sometimes convenient to represent technology with direct input sets defined by

$$L(y) = \{x : (x, y) \in T\}. \tag{4}$$

Duality theory requires that

$$\begin{aligned}&\text{For all } y \text{ in } R_+^M,\ L(y) \text{ is a nonempty, closed, convex set}\\&\text{and inputs are freely disposable.}\end{aligned} \tag{L}$$

At other times it is convenient to work with indirect inputs sets defined by

$$IL(p/R) = \{x : x \in L(y), py \geq R\}, \tag{5}$$

where p is a vector of output prices and R is total revenue. We also assume that

$$\begin{aligned}&\text{For all } p/R \text{ in } R_+^M,\ IL(p/R) \text{ is a nonempty, closed, convex set}\\&\text{and inputs are freely disposable.}\end{aligned} \tag{IL}$$

In closing, we note that the above definitions of (2) and (4) imply that

$$x \in L(y) \Leftrightarrow (x, y) \in T \Leftrightarrow y \in P(x). \tag{6}$$

[2]Inputs are strongly disposable if $(x', y) \in T$ whenever $x' \geq x$ and $(x, y) \in T$. Outputs are strongly disposable if $(x, y') \in T$ whenever $y' \leq y$ and $(x, y) \in T$.

[3]Duality will not be defined in this paper.

3 Function Characterization of Technology

The five different technology sets, $T, P(x), IP(w/C), L(y)$, and $IL(p/R)$ can all be represented by the directional distance functions that are defined in this section. Their derivation involves some form of technical efficiency, i.e., movements to frontiers of technically efficient input and/or output vectors. As we will soon see, two of these five representations also have some economic efficiency embedded in them. In addition to these five directional distance functions, one can define five dual functions that are explicitly derived by some form of economic optimization, e.g., revenue maximization, cost minimization, and profit maximization.

3.1 Directional Distance Functions

We start with T. We must first choose a directional vector, $g = (-g_x, g_y)$ where $g_x \in R_+^N, g_y \in R_+^M$, and $g \neq 0_{M+N}$. The *directional technology distance function on T* is defined by[4]

$$\vec{D}_T(x, y; -g_x, g_y) = \sup_{\beta} \{\beta : (x - \beta g_x, y + \beta g_y) \in T\}. \tag{7}$$

We illustrate this distance function for the one-input, one-output case.

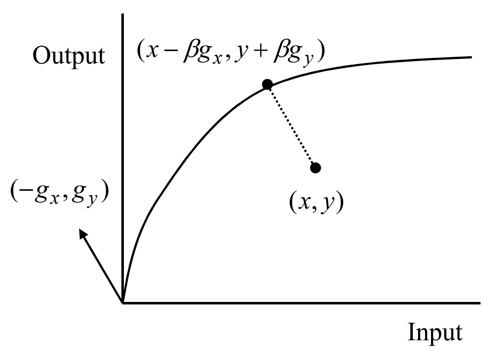


Figure 1

The directional technology distance function has a number of useful properties. Two of them are:

$T-$Indication:

$$\vec{D}_T(x, y; -g_x, g_y) \geq 0 \Leftrightarrow (x, y) \in T. \tag{8}$$

[4]This function was introduced by Luenberger (1992) who named it the shortage function. We follow Chambers, Chung, and Färe (1998).

Translation:

$$\vec{D}_T(x - \alpha g_x, y + \alpha g_y; -g_x, g_y) = \vec{D}_T(x, y; -g_x, g_y) - \alpha \text{ for all } \alpha \in R. \tag{9}$$

The indication property follows directly from the strong disposability assumptions. The translation property follows directly from the definition. To see this note that

$$\begin{aligned}
& \vec{D}_T(x - \alpha g_x, y + \alpha g_y; -g_x, g_y) \\
= & \sup_{\beta} \{\beta : (x - \alpha g_x - \beta g_x, y + \alpha g_y + \beta g_y) \in T\} \\
= & \sup_{\beta} \{\beta : (x - (\alpha + \beta) g_x, y + (\alpha + \beta) g_y) \in T\} \\
= & \sup_{\alpha+\beta} \{\alpha + \beta : (x - (\alpha + \beta) g_x, y + (\alpha + \beta) g_y) \in T\} - \alpha \\
= & \vec{D}_T(x, y; -g_x, g_y) - \alpha.
\end{aligned}$$

In addition, it has been shown that $\vec{D}_T(\cdot)$ is homogeneous of degree minus one in $(-g_x, g_y)$, nondecreasing in x, nonincreasing in y, and concave in (x, y). See Chambers, Chung, and Färe (1998).

We now move on to $P(x)$. The directional output distance function, defined on the output sets, in the direction g_x, is given by

$$\vec{D}_o(x, y; g_y) = \sup_{\beta} \{\beta : (y + \beta g_y) \in P(x)\} \tag{10}$$

The following property is easily established using strong disposability of outputs.

$\vec{D}_o$-Indication

$$\vec{D}_o(x, y; g_y) \geq 0 \Leftrightarrow y \in P(x). \tag{11}$$

Thus,

$$(y + \beta g_y) \in P(x) \Leftrightarrow \vec{D}_o(x, y + \beta g_y; g_y) \geq 0. \tag{12}$$

Now note that

$$(y + \beta g_y) \in P(x) \Leftrightarrow (x, y + \beta g_y) \in T \quad \text{by (6)}$$

Hence, we have

$$\begin{aligned}
\vec{D}_o(x, y; g_y) &= \sup_{\beta} \{\beta : (y + \beta g_y) \in P(x)\} \\
&= \sup_{\beta} \{\beta : (x, y + \beta g_y) \in T\} \\
&= \sup_{\beta} \{\beta : (x - \beta 0_N, y + \beta g_y) \in T\} \\
&= \vec{D}_T(x, y; 0_N, g_y)
\end{aligned}$$

This gives us the relationship between $\vec{D}_T(x, y; -g_x, g_y)$ and $\vec{D}_o(x, y; g_y)$, viz.,

$$\vec{D}_o(x, y; g_y) = \vec{D}_T(x, y; 0_N, g_y) \tag{13}$$

Moreover, using (11) we may recover the directional technology distance function $\vec{D}_T$ from the output directional distance function $\vec{D}_o$ by

$$\vec{D}_T(x, y; -g_x, g_y) = \sup_\beta \left\{\beta : \vec{D}_o(x - \beta g_x, y + \beta g_y; g_y) \geq 0\right\} \tag{14}$$

It is important to stress that the pair of equations (13) and (14) *do not* constitute a duality relationship. Instead, we refer to this pair of equations as an *inverse relationship.*

We now turn our attention to $L(y)$. The *directional input distance function* for the direction $-g_x$ is defined by

$$\vec{D}_i(x, y, -g_x) = \sup_\beta \{\beta : (x - \beta g_x) \in L(y)\}. \tag{15}$$

Strong disposability of inputs implies D_i- Indication:

$$\vec{D}_i(x, y, -g_x) \geq 0 \Leftrightarrow x \in L(y). \tag{16}$$

Note that

$$(x - \beta g_x) \in L(y) \Leftrightarrow (x - \beta g_x, y) \in T \quad \text{by (6)}$$

and thus

$$\begin{aligned}
\vec{D}_i(x, y; -g_x) &= \sup_\beta \{\beta : (x - \beta g_x) \in L(y)\} \\
&= \sup_\beta \{\beta : (x - \beta g_x, y) \in T\} \\
&= \sup_\beta \{\beta : (x - \beta g_x, y + \beta 0_M) \in T\} \\
&= \vec{D}_T(x, y; -g_x, 0_M).
\end{aligned}$$

i.e., we can compute $\vec{D}_i$ from $\vec{D}_T$ by

$$\vec{D}_i(x, y; -g_x) = \vec{D}_T(x, y; -g_x, 0_M). \tag{17}$$

Moreover, using (16), we may compute $\vec{D}_T$ from $\vec{D}_i$ by

$$\vec{D}_T(x, y, -g_x, g_y) = \sup_\beta \left\{\beta : \vec{D}_i(x - \beta g_x, y + \beta g_y, -g_x) \geq 0\right\} \tag{18}$$

It is important to stress that the pair of equations (17) and (18) *do not* constitute a duality relationship. It is, rather, an inverse relationship.

We may also consider the relationship between $\vec{D}_o$ and $\vec{D}_i$. We compute $\vec{D}_o$ in several steps,

$$\begin{aligned}
\vec{D}_o(x, y; g_y) &= \sup_\beta \{\beta : (y + \beta g_y) \in P(x)\} && (19)\\
&= \sup_\beta \{\beta : x \in L(y + \beta g_y)\} \quad \text{by (6)} && (20)\\
&= \sup_\beta \left\{\beta : \vec{D}_i(x, y + \beta g_y; -g_x) \geq 0\right\} \quad \text{by (16).} && (21)
\end{aligned}$$

and we compute $\vec{D}_i$ in several steps,

$$\begin{aligned}
\vec{D}_i(x, y; -g_x) &= \sup_\beta \{\beta : (x - \beta g_x) \in L(y)\} && (22)\\
&= \sup_\beta \{\beta : y \in P(x - \beta g_x)\} \quad \text{by (6)} && (23)\\
&= \sup_\beta \left\{\beta : \vec{D}_o(x - \beta g_x, y; g_y) \geq 0\right\} \quad \text{by (11)} && (24)
\end{aligned}$$

We get the pair of relationships:

$$\vec{D}_o(x, y; g_y) = \sup_\beta \left\{\beta : \vec{D}_i(x, y + \beta g_y; -g_x) \geq 0\right\} \tag{25}$$

and

$$\vec{D}_i(x, y, -g_x) = \sup_\beta \left\{\beta : \vec{D}_o(x - \beta g_x, y; g_y) \geq 0\right\}. \tag{26}$$

Equations (25) and (26) form an inverse relationship.[5]

We now take up the functional representation of the indirect output sets:

$$IP(w/C) = \{y : y \in P(x), wx \leq C\},$$

and the indirect input sets:

$$IL(p/R) = \{x : x \in L(y), py \geq R\}.$$

They are given by *indirect directional output distance function*:

$$I\vec{D}_o(w/C, y; g_y) = \sup_\beta \{\beta : (y + \beta g_y) \in IP(w/C)\}, \tag{27}$$

and the *indirect directional input distance function*:

$$I\vec{D}_i(x, p/R; -g_x) = \sup_\beta \{\beta : (x - \beta g_x) \in IL(p/R)\}. \tag{28}$$

[5]The reader may be bothered by the fact that the righthand side of (25) seemingly depends on $-g_x$ while the lefthand side does not. However, the effect of $-g_x$ is eliminated by the optimization over β. Put another way, since (19) and (20) do not depend on $-g_x$ then neither does (21). An analogous argument can be made for equation (26).

Note that

$$\begin{aligned}
I\vec{D}_o(w/C, y + \alpha g_y; g_y) &= \sup_{\beta} \{\beta : (y + \alpha g_y + \beta g_y) \in IP(w/C)\} \\
&= \sup_{\beta} \{\alpha + \beta : (y + (\alpha + \beta)\, g_y) \in IP(w/C)\} - \alpha \\
&= \sup_{\alpha+\beta} \{\alpha + \beta : (y + (\alpha + \beta)\, g_y) \in IP(w/C)\} - \alpha \\
&= I\vec{D}_o(w/C, y; g_y) - \alpha.
\end{aligned}$$

Hence, we have the translation property:

$$I\vec{D}_o(w/C, y + \alpha g_y; g_y) = I\vec{D}_o(w/C, y; g_y) - \alpha. \tag{29}$$

Similarly, one can show that

$$I\vec{D}_i(x - \alpha g_x, p/R; -g_x) = I\vec{D}_i(x, p/R; -g_x) - \alpha \tag{30}$$

This completes our catalogue of directional distance functions.

3.2 Dual Functions

Two rather standard dual functions are the *revenue function*, defined by

$$R(x, p) = \sup_{y} \{py : y \in P(x)\}, \tag{31}$$

and the *cost function*, defined by:

$$C(y, w) = \inf_{x} \{wx : x \in L(y)\}. \tag{32}$$

In addition, there are indirect versions; the first is the *indirect revenue function* defined by

$$IR(w/C, p) = \sup_{y} \{py : y \in IP(w/C)\}, \tag{33}$$

and the second is the *indirect cost function* defined by

$$IC(p/R, w) = \inf_{x} \{wx : x \in IL(p/R)\}. \tag{34}$$

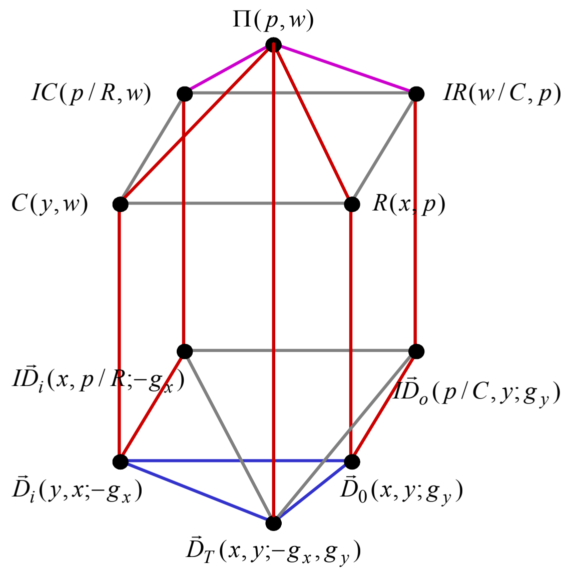
Finally, we define the *profit function*:

$$\Pi(p, w) = \sup_{x,y} \{py - wx : (x, y) \in T\}. \tag{35}$$

4 The Duality Diamond

The ten functional representations of technology can be displayed in a three dimensional figure that has the shape of a diamond. It is given below.

Duality Diamond



4.1 Inverse Relationships

Moving from one node to another in the above figure involves either an "inverse" operation or a "dual" operation. The inverse operations are those between two-member subsets (pairs) of

$$\left\{ \vec{D}_T(x,y;-g_x,g_y), \vec{D}_o\left(x,y;g_y\right), \vec{D}_i(x,y;-g_x) \right\}.$$

These representations are functions of quantities (and directional vectors) only.

We have already introduced the inverse relationships for the three possible pairs in the previous section. From equations (13) and (14) above we have

$$\boxed{\begin{aligned}&\vec{D}_o(x,y;g_y) = \vec{D}_T(x,y;0_N,g_y)\\&\vec{D}_T(x,y;-g_x,g_y) = \sup_\beta\left\{\beta : \vec{D}_o(x-\beta g_x, y+\beta g_y; g_y) \geq 0\right\}\end{aligned}} \tag{I}$$

from equations (17) and (18) we have

$$\boxed{\begin{aligned}&\vec{D}_i(x,y;-g_x) = \vec{D}_T(x,y;-g_x,0_M)\\&\vec{D}_T(x,y;-g_x,g_y) = \sup_\beta\left\{\beta : \vec{D}_i(x-\beta g_x, y+\beta g_y; -g_x) \geq 0\right\}\end{aligned}} \tag{II}$$

and from equations (25) and (26) we have

$$\boxed{\begin{aligned}&\vec{D}_o(x,y;g_y) = \sup_\beta\left\{\beta : \vec{D}_i(x, y+\beta g_y; -g_x) \geq 0\right\}\\&\vec{D}_i(x,y;-g_x) = \sup_\beta\left\{\beta : \vec{D}_o(x-\beta g_x, y; g_y) \geq 0\right\}\end{aligned}} \tag{III}$$

4.2 Duality Relationships

The duality relationship between the technology directional distance function and the profit function involves the bottom and the top of the duality diamond. It is given by the pair of optimization problems:

$$\boxed{\begin{aligned}&\Pi(p,w) = \sup_{x,y}\left\{py - wx + \vec{D}_T(x,y;-g_x,g_y)(pg_y + wg_x)\right\}\\&\vec{D}_T(x,y;-g_x,g_y) = \inf_{p,w}\left\{\frac{\Pi(p,w) - (py - wx)}{pg_y + wg_x}\right\}.\end{aligned}} \tag{IV}$$

Sketch of Proof:[6] From the definition of $\vec{D}_T$,

$$(\bar{x},\bar{y}) = \left(x - \vec{D}_T(x,y;-g_x,g_y)g_x, y + \vec{D}_T(x,y;-g_x,g_y)g_y\right) \in T$$

[6] See Luenberger (1992) for a complete proof.

and

$$\vec{D}_T(\bar{x}, \bar{y}; -g_x, g_y) = 0. \tag{36}$$

The condition (36) is clearly necessary for any profit maximizing choice of an input-output vector. Thus

$$\begin{aligned}
\Pi(p, w) &= \sup_{x,y} \{py - wx : (x, y) \in T\} \\
&= \sup_{x,y} \left\{ p \left[y + \vec{D}_T(x, y; -g_x, g_y) g_y \right] - w \left[x - \vec{D}_T(x, y; -g_x, g_y) g_x \right] \right\} \\
&= \sup_{x,y} \left\{ py - wx + \vec{D}_T(x, y; -g_x, g_y)(pg_y + wg_x) \right\}.
\end{aligned}$$

This yields the following inequality:

$$\Pi(p, w) \geq py - wx + \vec{D}_T(x, y; -g_x, g_y)(pg_y + wg_x),$$

for all $(x, y) \in R_+^N \times R_+^M$ and for all $(p, w) \in R_+^M \times R_+^N$. Rearranging,

$$\vec{D}_T(x, y; -g_x, g_y) \leq \frac{\Pi(p, w) - (py - wx)}{pg_y + wg_x}$$

for all $(x, y) \in R_+^N \times R_+^M$ and for all $(p, w) \in R_+^M \times R_+^N$. Thus,

$$\vec{D}_T(x, y; -g_x, g_y) \leq \inf_{p,w} \left\{ \frac{\Pi(p, w) - (py - wx)}{pg_y + wg_x} \right\}. \tag{37}$$

The rest of the proof simply showing that (37) holds with equality. This is done by using strong disposability of inputs and outputs, convexity of T, and a separating hyperplane theorem.

We now consider the duality relationship between the directional input distance function and the cost function. It is given by the following pair of optimization problems.

$$\boxed{\begin{aligned}
&C(y, w) = \inf_x \left\{ wx - \vec{D}_i(x, y; -g_x) \, wg_x \right\} \\
&\vec{D}_i(x, y; -g_x) = \inf_w \left\{ \frac{wx - C(y, w)}{wg_x} \right\}
\end{aligned}} \tag{V}$$

From the first optimization problem we see that

$$C(y, w) \leq wx - \vec{D}_i(x, y; -g_x) \, wg_x \tag{38}$$

for all $x \in R_+^N, w \in R_+^N$ since

$$C(y,w) \leq wx \text{ for all } x \in L(y)$$

and

$$x - \vec{D}_i\left(x, y; -g_x\right) g_x \in L(y).$$

Rearranging (38) yields

$$\vec{D}_i\left(x, y; -g_x\right) \leq \frac{wx - C(y,w)}{wg_x},$$

for all $x \in R_+^N, w \in R_+^N$, an inequality that leads us to the second optimization problem.[7]

An analogous result holds for the directional output distance function and the revenue function.

$$\boxed{\begin{array}{l} R(x,p) = \sup\limits_{y} \left\{ py + \vec{D}_o\left(x, y; g_y\right) pg_y \right\} \\[2ex] \vec{D}_o\left(x, y; g_y\right) = \inf\limits_{p} \left\{ \dfrac{R(x,p) - py}{pg_y} \right\} \end{array}} \tag{VI}$$

The first optimization problem is justified by starting with the following two conditions:

$$R(x,p) \geq py \text{ for all } y \in P(x),$$

and

$$y + \vec{D}_o\left(x, y; g_y\right) g_y \in P(x),$$

which lead us to the inequality

$$R(x,p) \geq py + \vec{D}_o\left(x, y; g_y\right) pg_y,$$

for all $y \in R_+^M, p \in R_+^M$. Then, rearranging this inequality we get

$$\vec{D}_o\left(x, y; g_y\right) \leq \frac{R(x,p) - py}{pg_y},$$

which leads us to the second optimization problem.

We now consider dualities between direct and indirect directional distance functions. On the output side we start with some particular input vector, $x^0 \in R_+^N$, and write the definition of the directional output distance function:

$$\vec{D}_o\left(x^0, y; g_y\right) = \sup_{\beta} \left\{ \beta : y + \beta g_y \in P(x^0) \right\}. \tag{39}$$

[7] For a complete proof see Luenberger (1992) and Chambers, Chung, and Färe (1996).

Now, choose any normalized input price vector, w/C, such that $(w/C)\,x^0 \leq 1$. If we enlarge the feasible set in (39), the supremal value cannot decrease. Hence,

$$\sup_{\beta}\left\{\beta : y + \beta g_y \in P(x^0)\right\} \leq \sup_{\beta,x}\left\{\beta : y + \beta g_y \in P(x), (w/C)\,x \leq 1\right\} \tag{40}$$

since x^0 is in the enlarged feasible set but it is not necessarily optimal. Moreover,

$$\begin{aligned} \sup_{\beta,x}\left\{\beta : y + \beta g_y \in P(x), (w/C)\,x \leq 1\right\} &= \sup_{\beta}\left\{\beta : y + \beta g_y \in IP(w/C)\right\} & (41)\\ &= I\vec{D}_o\,(w/C, y; g_y) & (42) \end{aligned}$$

Then (39) - (42) imply the inequality:

$$\vec{D}_o\,(x, y; g_y) \leq I\vec{D}_o\,(w/C, y; g_y)\,,$$

for all $x \in R_+^N, (w/C) \in R_+^N$ such that $(w/C)\,x \leq 1$. This leads us to the duality relationship:

$$\boxed{\begin{aligned} I\vec{D}_o\,(w/C, y; g_y) &= \sup_{x}\left\{\vec{D}_o\,(x, y; g_y) : (w/C)\,x \leq 1\right\} \\ \vec{D}_o\,(x, y; g_y) &= \inf_{w/C}\left\{I\vec{D}_o\,(w/C, y; g_y) : (w/C)\,x \leq 1\right\} \end{aligned}} \tag{VII}$$

Proof: The proof is similar to the proof of (IV) in Färe and Primont (1995, page 88). In Färe and Primont (1995, page 97) it is first proved that

$$IP(w/C) = \left\{y \in R_+^M : C(y, w) \leq C\right\}.$$

Hence

$$\begin{aligned} I\vec{D}_o\,(w/C, y; g_y) &= \sup_{\beta}\left\{\beta : y + \beta g_y \in IP(w/C)\right\} \\ &= \sup_{\beta}\left\{\beta : C\,(y + \beta g_y, w) \leq C\right\} \\ &= \sup_{\beta}\left\{\beta : C\,(y + \beta g_y, w/C) \leq 1\right\}, \end{aligned}$$

where the last equality follows from the homogeneity of C in w. Thus, if

$$I\vec{D}_o\,(w/C, y; g_y) = \beta^* \geq 0$$

then $C\,(y + \beta^* g_y, w/C) \leq 1$ and hence $C\,(y, w/C) \leq 1$ since C is nondecreasing in y. (Lowering outputs cannot increase cost since outputs are strongly disposable.) Conversely, if $C\,(y + \beta g_y, w/C) \leq 1$ then $I\vec{D}_o\,(w/C, y; g_y) \geq \beta$. But then

$I\vec{D}_o(w/C, y; g_y) - \beta \geq 0 \Rightarrow I\vec{D}_o(w/C, y + \beta g_y; g_y) \geq 0$ using the translation property (29). Setting β equal to zero we get the result that if $C(y, w/C) \leq 1$ then $I\vec{D}_o(w/C, y; g_y) \geq 0$. Hence we have shown that

$$I\vec{D}_o(w/C, y; g_y) \geq 0 \text{ if and only if } C(y, w/C) \leq 1. \tag{43}$$

Now, since

$$\vec{D}_o(x, y; g_y) \leq I\vec{D}_o(w/C, y; g_y),$$

for all $x \in R_+^N, (w/C) \in R_+^N$ such that $(w/C)x \leq 1$ it must be the case that

$$\vec{D}_o(x, y; g_y) \leq \inf_{w/C} \left\{ I\vec{D}_o(w/C, y; g_y) : (w/C)x \leq 1 \right\}. \tag{44}$$

We want to show that (44) holds with equality. Suppose it does not, i.e., suppose that

$$\vec{D}_o(x, y; g_y) < \beta^* = \inf_{w/C} \left\{ I\vec{D}_o(w/C, y; g_y) : (w/C)x \leq 1 \right\}.$$

Then $\vec{D}_o(x, y; g_y) - \beta^* < 0$ which implies that, using the translation properties of $\vec{D}_o$ and $I\vec{D}_o$,

$$\vec{D}_o(x, \bar{y}; g_y) < 0 = \inf_{w/C} \left\{ I\vec{D}_o(w/C, \bar{y}; g_y) : (w/C)x \leq 1 \right\}$$

where $\bar{y} = y + \beta^* g_y$. Thus,

$$(w/C)x \leq 1 \Rightarrow I\vec{D}_o(w/C, \bar{y}; g_y) \geq 0,$$

which is equivalent to

$$(w/C)x \leq 1 \Rightarrow C(\bar{y}, w/C) \leq 1 \tag{45}$$

because of (43).

Now, $\vec{D}_o(x, \bar{y}; g_y) < 0$ implies that $x \notin L(\bar{y})$ by (6) and (11). Since $L(\bar{y})$ is closed and convex and satisfies strong disposability, the strongly separating hyperplane theorem (see ()) implies that there is an input price vector, $\hat{w} > 0$, such that $\hat{w}x < C(\bar{y}, \hat{w})$. Let $\hat{C} = \hat{w}x$. Then $1 = \left(\hat{w}/\hat{C}\right)x < C(\bar{y}, \hat{w}/\hat{C})$, i.e., $\left(\hat{w}/\hat{C}\right)x = 1$ and $C(\bar{y}, \hat{w}/\hat{C}) > 1$. This contradicts (45). QED

On the input side we proceed in a similar fashion. For any $y^0 \in R_+^M$ such that $(p/R)y^0 \geq 1$ we have

$$\begin{aligned}
\vec{D}_i\left(x, y^0; -g_x\right) &= \sup_{\beta} \left\{ \beta : x - \beta g_x \in L(y^0) \right\} \\
&\leq \sup_{\beta, y} \left\{ \beta : x - \beta g_x \in L(y), (p/R)y \geq 1 \right\} \quad (\text{since } (p/R)y^0 \geq 1) \\
&= \sup_{\beta} \left\{ \beta : x - \beta g_x \in IL(p/R) \right\} \\
&= I\vec{D}_i(x, p/R; -g_x).
\end{aligned}$$

Hence we get the inequality,

$$\vec{D}_i\left(x,y;-g_x\right) \leq I\vec{D}_i\left(x,p/R;-g_x\right)$$

for all $y \in R_+^M, p/R \in R_+^M$ such that $(p/R)\,y \geq 1$. We conclude that

$$\boxed{\begin{aligned} I\vec{D}_i\left(x,p/R;-g_x\right) &= \sup_y \left\{ \vec{D}_i\left(x,y;-g_x\right) : (p/R)\,y \geq 1 \right\} \\ \vec{D}_i\left(x,y;-g_x\right) &= \inf_{p/R} \left\{ I\vec{D}_i\left(x,p/R;-g_x\right) : (p/R)\,y \geq 1 \right\} \end{aligned}} \qquad \text{(VIII)}$$

The dualities between the indirect input and output directional distance functions and the indirect cost and revenue functions are now considered. From (34) we have

$$IC(p/R,w) = \inf_x \left\{ wx : x \in IL(p/R) \right\}.$$

This implies that

$$IC(p/R,w) \leq wx \text{ for all } x \in IL(p/R).$$

Since

$$x - I\vec{D}_i\left(x,p/R;-g_x\right) g_x \in IL(p/R),$$

we have

$$IC(p/R,w) \leq w\left(x - I\vec{D}_i\left(x,p/R;-g_x\right) g_x\right)$$

or

$$IC(p/R,w) \leq wx - I\vec{D}_i\left(x,p/R;-g_x\right) wg_x,$$

for all $x \in R_+^N, w \in R_+^N$. This inequality leads us to the duality result:

$$\boxed{\begin{aligned} IC(p/R,w) &= \inf_x \left\{ wx - I\vec{D}_i\left(x,p/R;-g_x\right) wg_x \right\} \\ I\vec{D}_i\left(x,p/R;-g_x\right) &= \inf_w \left\{ \frac{wx - IC(p/R,w)}{wg_x} \right\} \end{aligned}} \qquad \text{(IX)}$$

On the output side, the analogous inequality is:

$$IR(w/C,p) \geq py + I\vec{D}_o\left(w/C,y;g_y\right) pg_y,$$

for all $y \in R^M_+, p \in R^M_+$. We are thus lead to

$$\boxed{\begin{aligned} &IR(w/C,p) = \sup_y \left\{ py + I\vec{D}_o\left(w/C, y; g_y\right) pg_y \right\} \\ &I\vec{D}_o\left(w/C, y; g_y\right) = \inf_p \left\{ \frac{IR(w/C,p) - py}{pg_y} \right\} \end{aligned}} \tag{X}$$

One can also find duality relationships between the profit function and the indirect directional output and input distance functions. For example, we may compute the profit function by

$$\begin{aligned} \Pi(p,w) &= \sup_{y,C} \left\{ py - C : I\vec{D}_o\left(w/C, y; g_y\right) \geq 0 \right\} \\ &= \sup_{y,C} \left\{ p\left(y + I\vec{D}_o\left(w/C, y; g_y\right) g_y \right) - C \right\} \\ &= \sup_{y,C} \left\{ py - C + I\vec{D}_o\left(w/C, y; g_y\right) pg_y \right\} \end{aligned}$$

This leads to the inequality

$$\Pi(p,w) \geq py - C + I\vec{D}_o\left(w/C, y; g_y\right) pg_y$$

for all $y \in R^M_+, p \in R^M_+, C > 0$. Hence,

$$I\vec{D}_o\left(w/C, y; g_y\right) \leq \frac{\Pi(p,w) - (py - C)}{pg_y}$$

for all $y \in R^M_+, p \in R^M_+, C > 0$. This suggests that

$$I\vec{D}_o\left(w/C, y; g_y\right) = \inf_{p,C} \left\{ \frac{\Pi(p,w) - (py - C)}{pg_y} \right\}.$$

We conclude that

$$\boxed{\begin{aligned} &\Pi(p,w) = \sup_{y,C} \left\{ py - C + I\vec{D}_o\left(w/C, y; g_y\right) pg_y \right\} \\ &I\vec{D}_o\left(w/C, y; g_y\right) = \inf_{p,C} \left\{ \frac{\Pi(p,w) - (py - C)}{pg_y} \right\} \end{aligned}} \tag{XI}$$

In an analogous fashion,

$$\Pi(p,w) = \sup_{x,R} \left\{ R - wx + I\vec{D}_i(x, p/R; -g_x) wg_x \right\}$$

leads to the inequality

$$\Pi(p, w) \geq R - wx + I\vec{D}_i(x, p/R; -g_x)wg_x.$$

We conclude that

$$\boxed{\begin{aligned} \Pi(p, w) &= \sup_{x,R} \left\{ R - wx + I\vec{D}_i(x, p/R; -g_x)wg_x \right\} \\ I\vec{D}_i(x, p/R; -g_x) &= \inf_{w,R} \left\{ \frac{\Pi(p, w) - (R - wx)}{wg_x} \right\} \end{aligned}} \quad \text{(XII)}$$

The remaining duality relationships in the duality diamond do not involve $\vec{D}_o$, $\vec{D}_i$, $I\vec{D}_o$ or $I\vec{D}_i$. Therefore, they are the same as those presented in Färe and Primont (1995) and will not be repeated here.

5 Concluding Remarks

The ten representations of the technology illustrated in the duality diamond can be separated into two types, those that possess a homogeneity property and those that possess the translation property. This observation has implications for the appropriate parametric forms of these representations. The property of homogeneity is easily accomodated by translog functions while the translation property is easily modelled with quadratic functional forms.

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