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SFDE's as Dynamical Systems (Symposium 2000/2001, University of Warwick)

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SFDE'S
AS DYNAMICAL SYSTEMS
I: THE LINEAR CASE

Warwick: November 10, 2000

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1. Regular Linear SFDE's-Ergodic Theory.

Linear sfde's on $\mathbf{R}^d$ driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$.

$$\left.\begin{aligned} dx(t) &= H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\ &\qquad + \sum_{i=1}^{m} g_i x(t)\, dW_i(t), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (I)$$

(I) is defined on

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}}, P)$ = canonical complete filtered Wiener space.

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R} \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F}$:=(completed) Borel σ-field of Ω;

$\mathcal{F}_t :=$ (completed) sub-σ-field of $\mathcal{F}$ generated by the evaluations $\omega \to \omega(u), \;\; u \leq t, \quad t \in \mathbf{R}$.

$P :=$ Wiener measure on Ω.

$dW_i(t) =$ Itô stochastic differentials.

Several finite delays $0 < d_1 < d_2 < \cdots < d_N \leq r$ in drift term; *no delays in diffusion coefficient.*

$H : (\mathbf{R}^d)^{N+1} \times L^2([-r,0],\mathbf{R}^d) \to \mathbf{R}^d$ is a fixed continuous linear map, g_i, $i = 1, 2, \ldots, m$, fixed (deterministic) $d \times d$-matrices.

2. Plan

Use state space $M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d)$. For (I) consider the following themes:

I) Existence of a "perfect" cocycle on M_2-a modification of the trajectory field $(x(t), x_t) \in M_2$.

II) Existence of almost sure Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|(x(t), x_t)\|_{M_2}$$

Multiplicative ergodic theorem and *hyperbolicity* of cocycle.

III) *"Random Saddle-Point Property"* in hyperbolic case.

3. Regularity

Say SFDE (I) is *regular* (wrt. M_2) if trajectory $\{(x(t), x_t) : (x(0), x_0) = (v, \eta) \in M_2\}$ admits a measurable modification $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ such that $X(\cdot, \cdot, \omega)$ is continuous for a.a. $\omega \in \Omega$.

Theorem 1.([Mo], 1990])

(I) is regular with respect to state space $M_2 = \mathbf{R}^d \times \mathbf{L}^2([-r, 0], \mathbf{R}^d)$. There is a measurable version $X : \mathbf{R}^+ \times$

$M_2 \times \Omega \to M_2$ of the trajectory field $\{(x(t), x_t) : t \in \mathbf{R}^+,\ (x(0), x_0) = (v, \eta) \in M_2\}$ of (I) with the following properties:

(i) For each $(v, \eta) \in M_2$ and $t \in \mathbf{R}^+$, $X(t, (v, \eta), \cdot) = (x(t), x_t)$ a.s., is $\mathcal{F}_t$-measurable and belongs to $L^2(\Omega, M_2; P)$.

(ii) There exists $\Omega_0 \in \mathcal{F}$ of full measure such that, for all $\omega \in \Omega_0$, the map $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times M_2 \to M_2$ is continuous.

(iii) For each $t \in \mathbf{R}^+$ and every $\omega \in \Omega_0$, the map $X(t, \cdot, \omega) : M_2 \to M_2$ is continuous linear; for each $\omega \in \Omega_0$, the map $\mathbf{R}^+ \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ is measurable and locally bounded in the uniform operator norm on $L(M_2)$. The map $[r, \infty) \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ is continuous for all $\omega \in \Omega_0$.

(iv) For each $t \geq r$ and all $\omega \in \Omega_0$, the map

$$X(t, \cdot, \omega) : M_2 \to M_2$$

is compact.

Compactness of semi-flow for $t \geq r$ will be used to define hyperbolicity for (I) and the associated exponential dichotomies.

Example: $dx(t) = x(t-1)\,dW(t)$ *is not regular (singular).*

4. Lyapunov Exponents. Hyperbolicity

Version X of the trajectory field of (I) (in Theorem 1) is a multiplicative $L(M_2)$-valued linear cocycle over the canonical Brownian shift $\theta : \mathbf{R} \times \Omega \to \Omega$ on Wiener space:

$$\theta(t, \omega)(u) := \omega(t+u) - \omega(t), \quad u, t \in \mathbf{R}, \quad \omega \in \Omega.$$

I.e.

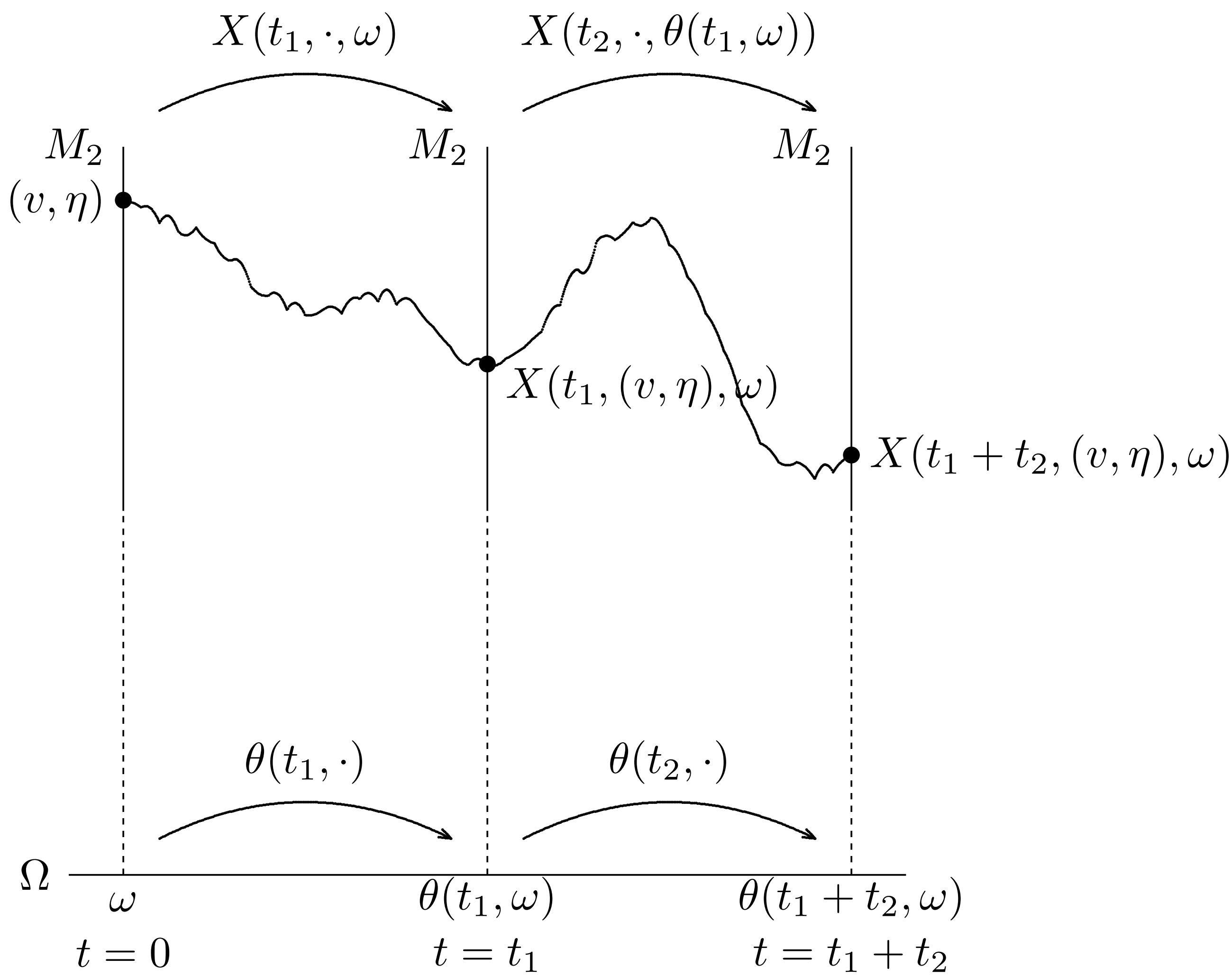
Theorem 2([Mo], 1990)

There is an $\mathcal{F}$-measurable set $\hat{\Omega}$ of full P-measure such that $\theta(t,\cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$ for all $t \geq 0$ and

$$X(t_2, \cdot, \theta(t_1, \omega)) \circ X(t_1, \cdot, \omega) = X(t_1 + t_2, \cdot, \omega)$$

for all $\omega \in \hat{\Omega}$ and t_1, $t_2 \geq 0$.

The Cocycle Property



Vertical solid lines represent random fibers: copies of M_2. (X, θ) is a "vector-bundle morphism".

The a.s. Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v(\omega),\eta(\omega)),\omega)\|_{M_2},$$

(for a.a. $\omega \in \Omega$, $(v,\eta) \in L^2(\Omega, M_2)$) of the system (I) are characterized by the following "spectral theorem". Each $\theta(t,\cdot)$ is ergodic and preserves Wiener measure P. The proof of Theorem 3 below uses compactness of $X(t,\cdot,\omega) : M_2 \to M_2$, $t \geq r$, together with an infinite-dimensional version of Oseledec's multiplicative ergodic theorem due to Ruelle (1982).

Theorem 3. ([Mo], 1990)

Let $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ be the flow of (I) given in Theorem 1. Then there exist

(a) an $\mathcal{F}$-measurable set $\Omega^* \subseteq \Omega$ such that $P(\Omega^*) = 1$ and $\theta(t,\cdot)(\Omega^*) \subseteq \Omega^*$ for all $t \geq 0$,

(b) *a fixed (non-random) sequence of real numbers* $\{\lambda_i\}_{i=1}^\infty$, *and*

(c) *a random family* $\{E_i(\omega) : i \geq 1, \omega \in \Omega^*\}$ *of (closed) finite-codimensional subspaces of* M_2, *with the following properties:*

(i) If the **Lyapunov spectrum** $\{\lambda_i\}_{i=1}^\infty$ *is infinite, then* $\lambda_{i+1} < \lambda_i$ *for all* $i \geq 1$ *and* $\lim_{i\to\infty} \lambda_i = -\infty$*; otherwise there is a fixed (non-random) integer* $N \geq 1$ *such that* $\lambda_N = -\infty < \lambda_{N-1} < \cdots < \lambda_2 < \lambda_1$*;*

(ii) each map $\omega \mapsto E_i(\omega)$, $i \geq 1$, *is* $\mathcal{F}$*-measurable into the Grassmannian of* M_2*;*

(iii) $E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2$, $i \geq 1$, $\omega \in \Omega^*$*;*

(iv) for each $i \geq 1$, *codim* $E_i(\omega)$ *is fixed independently of* $\omega \in \Omega^*$*;*

(v) for each $\omega \in \Omega^*$ *and* $(v,\eta) \in E_i(\omega)\backslash E_{i+1}(\omega)$,

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2} = \lambda_i,\ i \geq 1;$$

(vi) **Top Exponent***:*

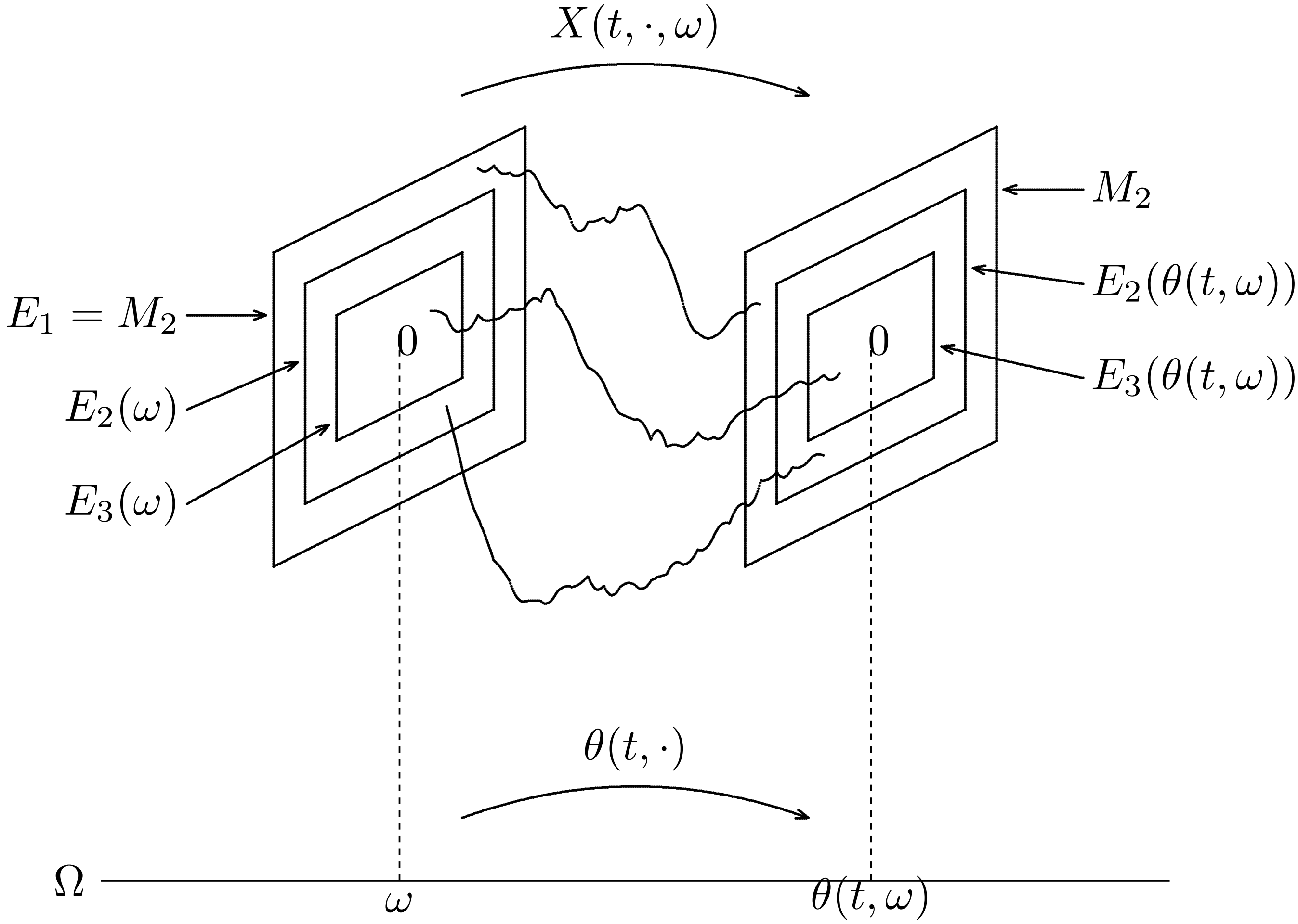
$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log \|X(t,\cdot,\omega)\|_{L(M_2)} \quad \textit{for all } \omega \in \Omega^*;$$

(vii) **Invariance***:*

$$X(t,\cdot,\omega)(E_i(\omega)) \subseteq E_i(\theta(t,\omega))$$

for all $\omega \in \Omega^*,\ t \geq 0,\ i \geq 1.$

Spectral Theorem



Proof of Theorem 3 is based on Ruelle's discrete version of Oseledec's multiplicative ergodic theorem in Hilbert space ([Ru], Ann. of Math. 1982, Theorem (1.1), p. 248 and Corollary (2.2), p. 253):

Theorem 4 ([Ru], 1982)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\tau : \Omega \to \Omega$ a P-preserving transformation. Assume that H is a separable Hilbert space and $T : \Omega \to L(H)$ a measurable map (w.r.t. the Borel field on the space of all bounded linear operators $L(H)$). Suppose that $T(\omega)$ is compact for almost all $\omega \in \Omega$, and $E \log^+ \|T(\cdot)\| < \infty$. Define the family of linear operators $\{T^n(\omega) : \omega \in \Omega,\ n \geq 1\}$ by

$$T^n(\omega) := T(\tau^{n-1}(\omega)) \circ \cdots T(\tau(\omega)) \circ T(\omega)$$

for $\omega \in \Omega,\ n \geq 1$.

Then there is a set $\Omega_0 \in \mathcal{F}$ of full P-measure such that $\tau(\Omega_0) \subseteq \Omega_0$, and for each $\omega \in \Omega_0$, the limit

$$\lim_{n\to\infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm and is a positive compact self-adjoint operator on H. Furthermore, each $\Lambda(\omega)$ has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

where the μ_i's are distinct. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at $\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigen-space $F_i(\omega)$, with $m_i(\omega) := \dim F_i(\omega)$. Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \quad E_\infty(\omega) := \ker \Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = H$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|T^n(\omega)x\|_H = \begin{cases} \mu_i(\omega), & \textit{if} \quad x \in E_i(\omega)\backslash E_{i+1}(\omega) \\ -\infty & \textit{if} \quad x \in \textit{ker}\ \Lambda(\omega). \end{cases}$$

Proof.

[Ru], Ann. of Math., 1982, pp. 248-254.

□

The following "perfect" version of Kingman's subadditive ergodic theorem is also used to construct the shift invariant set Ω^* appearing in Theorem 3 above.

Theorem 5([M], 1990)("Perfect" Subadditive Ergodic Theorem)

Let $f : \mathbf{R}^+ \times \Omega \to \mathbf{R} \cup \{-\infty\}$ *be a measurable process on the complete probability space* $(\Omega, \mathcal{F}, P)$ *such that*

(i) $E \sup_{0 \le u \le 1} f^+(u, \cdot) < \infty$, $E \sup_{0 \le u \le 1} f^+(1-u, \theta(u, \cdot)) < \infty$;

(ii) $f(t_1+t_2, \omega) \le f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ *for all* $t_1, t_2 \ge 0$ *and* **every** $\omega \in \Omega$.

Then there exist a set $\hat{\hat{\Omega}} \in \mathcal{F}$ *and a measurable* $\tilde{f} : \Omega \to \mathbf{R} \cup \{-\infty\}$ *with the properties:*

(a) $P(\hat{\hat{\Omega}}) = 1$, $\theta(t, \cdot)(\hat{\hat{\Omega}}) \subseteq \hat{\hat{\Omega}}$ *for all* $t \ge 0$;

(b) $\tilde{f}(\omega) = \tilde{f}(\theta(t, \omega))$ *for all* $\omega \in \hat{\hat{\Omega}}$ *and all* $t \ge 0$;

(c) $\tilde{f}^+ \in \mathbf{L}^1(\Omega, \mathbf{R}; P)$;

(d) $\lim_{t \to \infty} (1/t) f(t, \omega) = \tilde{f}(\omega)$ *for every* $\omega \in \hat{\hat{\Omega}}$.

If θ *is ergodic, then there exist* $f^* \in \mathbf{R} \cup \{-\infty\}$ *and* $\tilde{\tilde{\Omega}} \in \mathcal{F}$ *such that* *(a)′* $P(\tilde{\tilde{\Omega}}) = 1, \theta(t,\cdot)(\tilde{\tilde{\Omega}}) \subseteq \tilde{\tilde{\Omega}}$, $t \geq 0$;

(b)′ $\tilde{f}(\omega) = f^* = \lim_{t\to\infty}(1/t)f(t,\omega)$ *for every* $\omega \in \tilde{\tilde{\Omega}}$.

Proof.

[Mo], Stochastics, 1990, Lemma 7, pp. 115–117. □

Proof of Theorem 3 is an application of Theorem 4. Requires Theorem 5 and the following sequence of lemmas.

Lemma 1

For each integer $k \geq 1$ *and any* $0 < a < \infty$,

$$E \sup_{0\leq t\leq a} \|\phi(t,\omega)^{-1}\|^{2k} < \infty;$$

$$E \sup_{0\leq t_1,t_2\leq a} \|\phi(t_2,\theta(t_1,\cdot))\|^{2k} < \infty.$$

Proof.

Follows by standard sode estimates, the co-cycle property for ϕ and Hölder's inequality. ([Mo], pp. 106-108). □

The next lemma is a crucial estimate needed to apply Ruelle-Oseledec theorem (Theorem 4).

Lemma 2

$$E \sup_{0\le t_1,t_2\le r} \log^+ \|X(t_2,\cdot,\theta(t_1,\cdot))\|_{L(M_2)} < \infty.$$

Proof.

If $y(t,(v,\eta),\omega)$ is the solution of the fde (8), then using Gronwall's inequality, taking $E \sup_{0\le t_1,t_2\le r} \log^+ \sup_{\|(v,\eta)\|\le 1}$ and applying Lemma 1, gives

$$E \sup_{0\le t_1,t_2\le r} \log^+ \sup_{\|(v,\eta)\|\le 1} \|(y(t_2,(v,\eta),\theta(t_1,\cdot)), y_{t_2}(\cdot,(v,\eta),\theta(t_1,\cdot)))\|_{M_2} < \infty.$$

Conclusion of lemma now follows by replacing ω' with $\theta(t_1, \omega)$ in the formula

$$X(t_2, (v, \eta), \omega')$$
$$= (\phi(t_2, \omega')(y(t_2, (v, \eta), \omega')), \phi_{t_2}(\cdot, \omega') \circ (id_J, y_{t_2}(\cdot, (v, \eta), \omega'))$$

and Lemma 1. $\square$

The existence of the Lyapunov exponents is obtained by interpolating the discrete limit

$$\frac{1}{r} \lim_{k \to \infty} \frac{1}{k} \log \|X(kr, (v(\omega), \eta(\omega)), \omega)\|_{M_2}, \tag{12}$$

a.a. $\omega \in \Omega$, $(v, \eta) \in L^2(\Omega, M_2)$, between delay periods of length r. This requires the next two lemmas.

Lemma 3

Let $h : \Omega \to \mathbf{R}^+$ *be* $\mathcal{F}$*-measurable and suppose* $E \sup_{0 \le u \le r} h(\theta(u, \cdot)$ *is finite. Then*

$$\Omega_1 := \big(\lim_{t\to\infty} \frac{1}{t} h(\theta(t, \cdot) = 0\big)$$

is a sure event and $\theta(t, \cdot)(\Omega_1) \subseteq \Omega_1$ *for all* $t \geq 0$*.*

Proof.

Use interpolation between delay periods and the discrete ergodic theorem applied to the L^1 function

$$\hat{h} := \sup_{0 \le u \le r} h(\theta(u, \cdot).$$

([Mo], Stochastics, 1990, Lemma 5, pp. 111-113.) □

Lemma 4

Suppose there is a sure event Ω_2 *such that* $\theta(t,\cdot)(\Omega_2) \subseteq \Omega_2$ *for all* $t \geq 0$*, and the limit (12) exists (or equal to* $-\infty$*) for all* $\omega \in \Omega_2$ *and all* $(v,\eta) \in M_2$*. Then there is a sure event* Ω_3 *such that* $\theta(t,\cdot)(\Omega_3) \subseteq \Omega_3$ *and*

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2} = \frac{1}{r} \lim_{k\to\infty} \frac{1}{k} \log \|X(kr,(v,\eta),\omega)\|_{M_2}, \tag{13}$$

for all $\omega \in \Omega_3$ *and all* $(v,\eta) \in M_2$*.*

Proof:

Take $\Omega_3 := \hat{\Omega} \cap \Omega_1 \cap \Omega_2$. Use cocycle property for X, Lemma 2 and Lemma 3 to interpolate. ([Mo], Stochastics 1990, Lemma 6, pp. 113-114.)

□

Proof of Theorem 3. (Sketch)

Apply Ruelle-Oseledec Theorem (Theorem 4) with

$T(\omega) := X(r,\omega) \in L(M_2)$, compact linear for $\omega \in \hat{\Omega}$;

$\tau : \Omega \to \Omega; \quad \tau := \theta(r,\cdot)$.

Then cocycle property for X implies

$$X(kr,\omega,\cdot) = T(\tau^{k-1}(\omega)) \circ T(\tau^{k-2}(\omega)) \circ \cdots \circ T(\tau(\omega)) \circ T(\omega)$$
$$:= T^k(\omega)$$

for all $\omega \in \hat{\Omega}$.

Lemma 2 implies

$$E \log^+ \|T(\cdot)\|_{L(M_2)} < \infty.$$

Theorem 4 gives a random family of compact self-adjoint positive linear operators $\{\Lambda(\omega) : \omega \in \Omega_4\}$ such that

$$\lim_{n\to\infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm for $\omega \in \Omega_4$, a (continuous) shift-invariant set of full measure. Furthermore each $\Lambda(\omega)$ has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

where the μ_i's are distinct, with no accumulation points except possibly $-\infty$. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at $\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigenspace $F_i(\omega)$, with $m_i(\omega) := dim F_i(\omega)$. Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \quad E_\infty(\omega) := \ker \Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2.$$

Note that $\operatorname{codim} E_i(\omega) = \sum_{j=1}^{i-1} m_j(\omega) < \infty$. Also

$$\lim_{k\to\infty} \frac{1}{k} \log \|X(kr,(v,\eta),\omega)\|_{M_2} = \begin{cases} \mu_i(\omega), & \text{if } (v,\eta) \in E_i(\omega) \backslash E_{i+1}(\omega) \\ -\infty & \text{if } (v,\eta) \in \ker \Lambda(\omega). \end{cases}$$

The functions

$$\omega \mapsto \mu_i(\omega), \quad \omega \mapsto m_i(\omega), \quad \omega \mapsto N(\omega)$$

are invariant under the ergodic shift $\theta(r,\cdot)$. Hence they take the fixed values $\mu_i,\ m_i,\ N$ almost surely, respectively.

Lemma 4 gives a continuous-shift-invariant sure event $\Omega^* \subseteq \Omega_4$ such that

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2} &= \frac{1}{r} \lim_{k\to\infty} \frac{1}{k} \log \|X(kr,(v,\eta),\omega)\|_{M_2} \\ &= \frac{\mu_i}{r} =: \lambda_i, \end{aligned}$$

for $(v,\eta) \in E_i(\omega)\backslash E_{i+1}(\omega),\ \omega \in \Omega^*, i \geq 1.$

$\{\lambda_i := \frac{\mu_i}{r} : i \geq 1\}$ is the *Lyapunov spectrum* of (I).

Since Lyapunov spectrum is discrete with no finite accumulation points, then $\{\lambda_i : \lambda_i > \lambda\}$ is finite for all $\lambda \in \mathbf{R}$.

To prove invariance of the Oseledec space $E_i(\omega)$ under the cocycle (X, θ) use the random field

$$\lambda((v,\eta),\omega) := \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2},\ (v,\eta) \in M_2,\ \omega \in \Omega^*$$

and the relations

$$E_i(\omega) := \{(v,\eta) \in M_2 : \lambda((v,\eta),\omega) \leq \lambda_i\},$$

$$\lambda(X(t,(v,\eta),\omega), \theta(t,\omega)) = \lambda((v,\eta),\omega), \quad \omega \in \Omega^*,\ t \geq 0$$

([Mo], Stochastics 1990, p. 122). $\square$

Lyapunov exponents $\{\lambda_i\}_{i=1}^{\infty}$ of (I) are non-random because θ is ergodic. Say (I) is *hyperbolic* if $\lambda_i \neq 0$ for all $i \geq 1$. When (I) is hyperbolic the flow satisfies a *stochastic saddle-point property* (or exponential dichotomy) (cf. the deterministic case with $E = C([-r, 0], \mathbf{R}^d)$, $g_i \equiv 0$, $i = 1, \ldots, m$, in Hale [H], Theorem 4.1, p. 181).

Theorem 6 *(Random Saddles)*([Mo], 1990)

Suppose the sfde (I) is hyperbolic. Then there exist

(a) a set $\tilde{\Omega}^* \in \mathcal{F}$ *such that* $P(\tilde{\Omega}^*) = 1$, *and* $\theta(t, \cdot)(\tilde{\Omega}^*) = \tilde{\Omega}^*$ *for all* $t \in \mathbf{R}$,

and

(b) a measurable splitting

$$M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \tilde{\Omega}^*,$$

with the following properties:

(i) $\mathcal{U}(\omega)$, $\mathcal{S}(\omega)$, $\omega \in \tilde{\Omega}^*$, *are closed linear subspaces of* M_2, $\dim \mathcal{U}(\omega)$ *is finite and fixed independently of* $\omega \in \tilde{\Omega}^*$.

(ii) *The maps* $\omega \mapsto \mathcal{U}(\omega)$, $\omega \mapsto \mathcal{S}(\omega)$ *are* $\mathcal{F}$*-measurable into the Grassmannian of* M_2.

(iii) *For each* $\omega \in \tilde{\Omega}^*$ *and* $(v, \eta) \in \mathcal{S}(\omega)$ *there exists* $\tau_1 = \tau_1(v, \eta, \omega) > 0$ *and a positive* δ_1, *independent of* (v, η, ω) *such that*

$$\|X(t, (v, \eta), \omega)\|_{M_2} \leq \|(v, \eta)\|_{M_2} e^{-\delta_1 t}, \quad t \geq \tau_1.$$

(iv) *For each* $\omega \in \tilde{\Omega}^*$ *and* $(v, \eta) \in \mathcal{U}(\omega)$ *there exists* $\tau_2 = \tau_2(v, \eta, \omega) > 0$ *and a positive* δ_2, *independent of* (v, η, ω) *such that*

$$\|X(t, (v, \eta), \omega)\|_{M_2} \geq \|(v, \eta)\|_{M_2} e^{\delta_2 t}, \quad t \geq \tau_2.$$

(v) For each $t \geq 0$ and $\omega \in \tilde{\Omega}^*$,

$$X(t,\omega,\cdot)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t,\omega)),$$

$$X(t,\omega,\cdot)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)).$$

In particular, the restriction

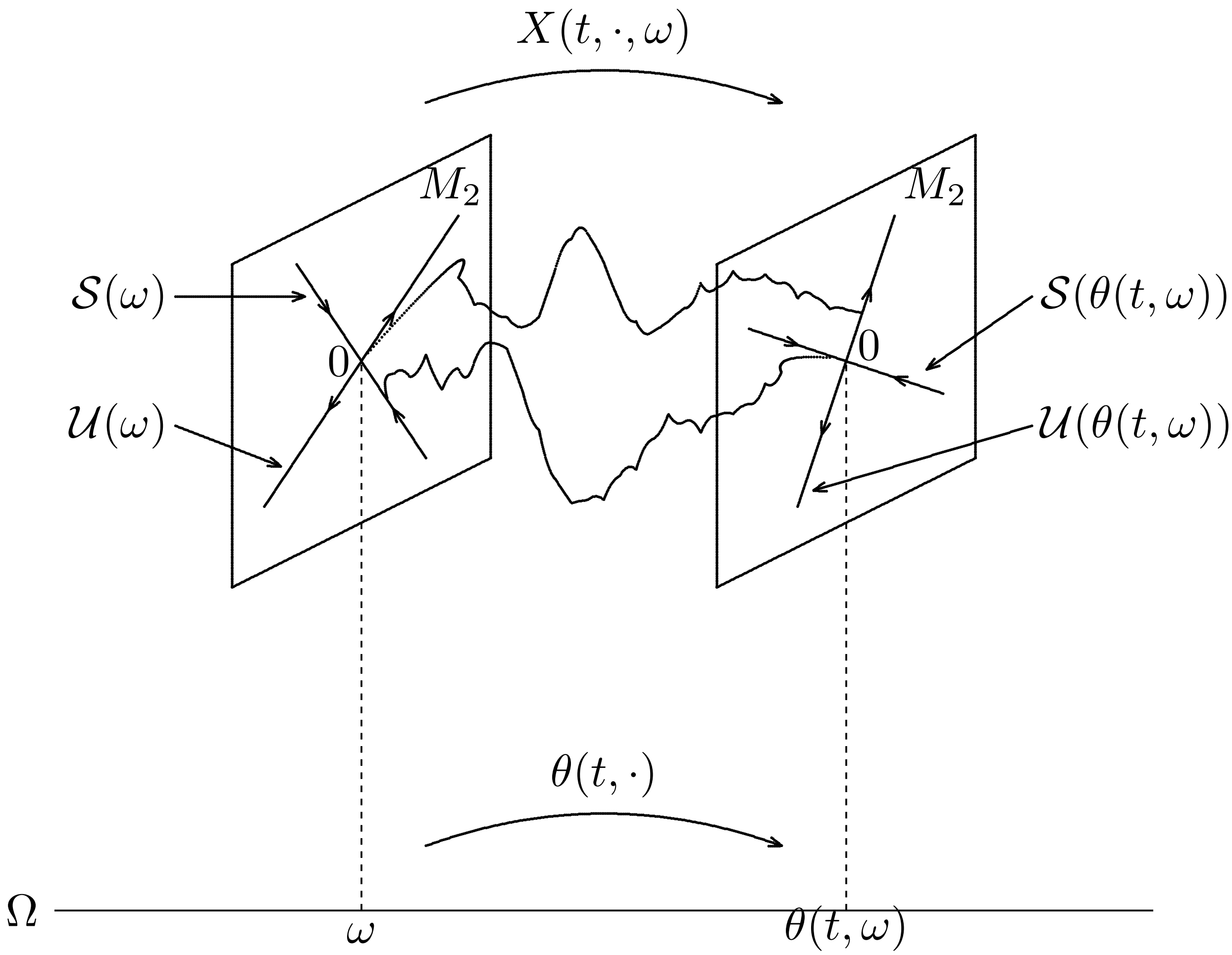
$$X(t,\omega,\cdot)\,|\,\mathcal{U}(\omega) : \mathcal{U}(\omega) \to \mathcal{U}(\theta(t,\omega))$$

is a linear homeomorphism onto.

Proof.

[Mo], Stochastics, 1990, Corollary 2, pp. 127-130. □

The Saddle-Point Property



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Stable Manifolds

Outline

- Smooth cocycles in Hilbert space. Stationary trajectories.
- Linearization of a cocycle along a stationary trajectory.
- Ergodic theory of cocycles in Hilbert space.
- Hyperbolicity of stationary trajectories. Lyapunov exponents.
- Cocycles generated by stochastic systems with memory. Via random diffeomorphism groups.

- *Local Stable Manifold Theorem* for SFDE's: Existence of smooth stable and unstable manifolds in a neighborhood of a hyperbolic stationary trajectory.
- Proof: Ruelle-Oseledec multiplicative ergodic theory+ perfection techniques.

The Cocycle

$(\Omega, \mathcal{F}, P) :=$ complete probability space.

$\theta : \mathbf{R}^+ \times \Omega \to \Omega$ a P-preserving (ergodic) semigroup on $(\Omega, \mathcal{F}, P)$.

$E :=$ real (separable) Hilbert space, norm $\|\cdot\|$, Borel σ-algebra.

Definition.

$k =$ non-negative integer, $\epsilon \in (0, 1]$. A $C^{k,\epsilon}$ *perfect cocycle* (X, θ) on E is a measurable random field $X : \mathbf{R}^+ \times E \times \Omega \to E$ such that:

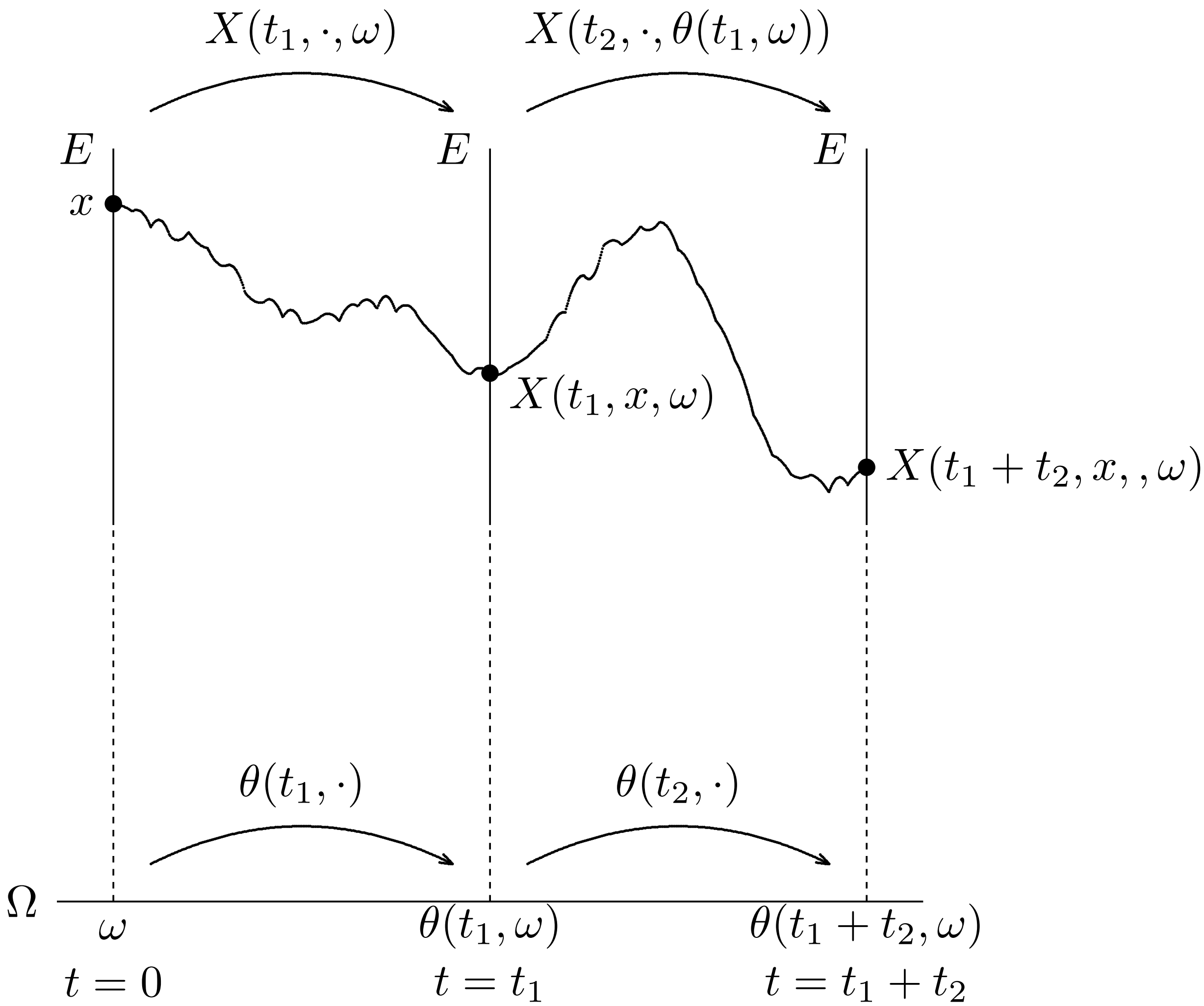
(i) For each $\omega \in \Omega$, the map $\mathbf{R}^+ \times E \ni (t, x) \mapsto X(t, x, \omega) \in E$ is continuous; for fixed $(t, \omega) \in$

$\mathbf{R}^+ \times \Omega$, the map $E \ni x \mapsto X(t,x,\omega) \in E$ is $C^{k,\epsilon}$ $\big(D^k X(t,x,\omega)$ is C^ϵ in $x\big)$.

(ii) $X(t_1+t_2,\cdot,\omega) = X(t_2,\cdot,\theta(t_1,\omega)) \circ X(t_1,\cdot,\omega)$ for all $t_1, t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.

(iii) $X(0,x,\omega) = x$ for all $x \in E, \omega \in \Omega$.

Cocycle Property



Vertical solid lines represent random fibers: copies of E.

Definition

A random variable $Y : \Omega \to E$ is a *stationary point* for the cocycle (X, θ) if

$$X(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \qquad (1)$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $X(t, Y) = Y(\theta(t))$.

Linearization. Hyperbolicity.

Linearize a $C^{k,\epsilon}$ cocycle (X,θ) along a stationary random point Y: Get an $L(E)$-valued cocycle $(DX(t,Y(\omega),\omega),\theta(t,\omega))$. (Follows from cocycle property of X and chain rule.)

Theorem 1. *(Oseledec-Ruelle)*

Let $T:\mathbf{R}^+\times\Omega\to L(E)$ be strongly measurable, such that (T,θ) is an $L(E)$-valued cocycle, with each $T(t,\omega)$ compact. Suppose that

$$E\sup_{0\le t\le 1}\log^+\|T(t,\cdot)\|_{L(E)}<\infty,$$
$$E\sup_{0\le t\le 1}\log^+\|T(1-t,\theta(t,\cdot))\|_{L(E)}<\infty.$$

Then there is a sure event $\Omega_0 \in \mathcal{F}$ *such that* $\theta(t,\cdot)(\Omega_0) \subseteq \Omega_0$ *for all* $t \in \mathbf{R}^+$, *and for each* $\omega \in \Omega_0$,

$$\lim_{t\to\infty}[T(t,\omega)^* \circ T(t,\omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. $\Lambda(\omega)$ *is self-adjoint with a non-random spectrum*

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots$$

where the λ_i*'s are distinct. Each* e^{λ_i} *(with* $\lambda_i > -\infty$*) has a fixed finite non-random multiplicity* m_i *and eigen-space* $F_i(\omega)$, *with* $m_i := dim F_i(\omega)$. *Define*

$$E_1(\omega) := E, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp},\ i > 1,\ E_\infty := ker\Lambda(\omega).$$

Then

$$E_\infty \subset \cdots \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$\lim_{t\to\infty}\frac{1}{t}\log\|T(t,\omega)x\| = \begin{cases}\lambda_i & if \quad x\in E_i(\omega)\backslash E_{i+1}(\omega),\\ -\infty & if \quad x\in E_\infty(\omega),\end{cases}$$

and

$$T(t,\omega)(E_i(\omega))\subseteq E_i(\theta(t,\omega))$$

for all $t\geq 0,\ i\geq 1.$

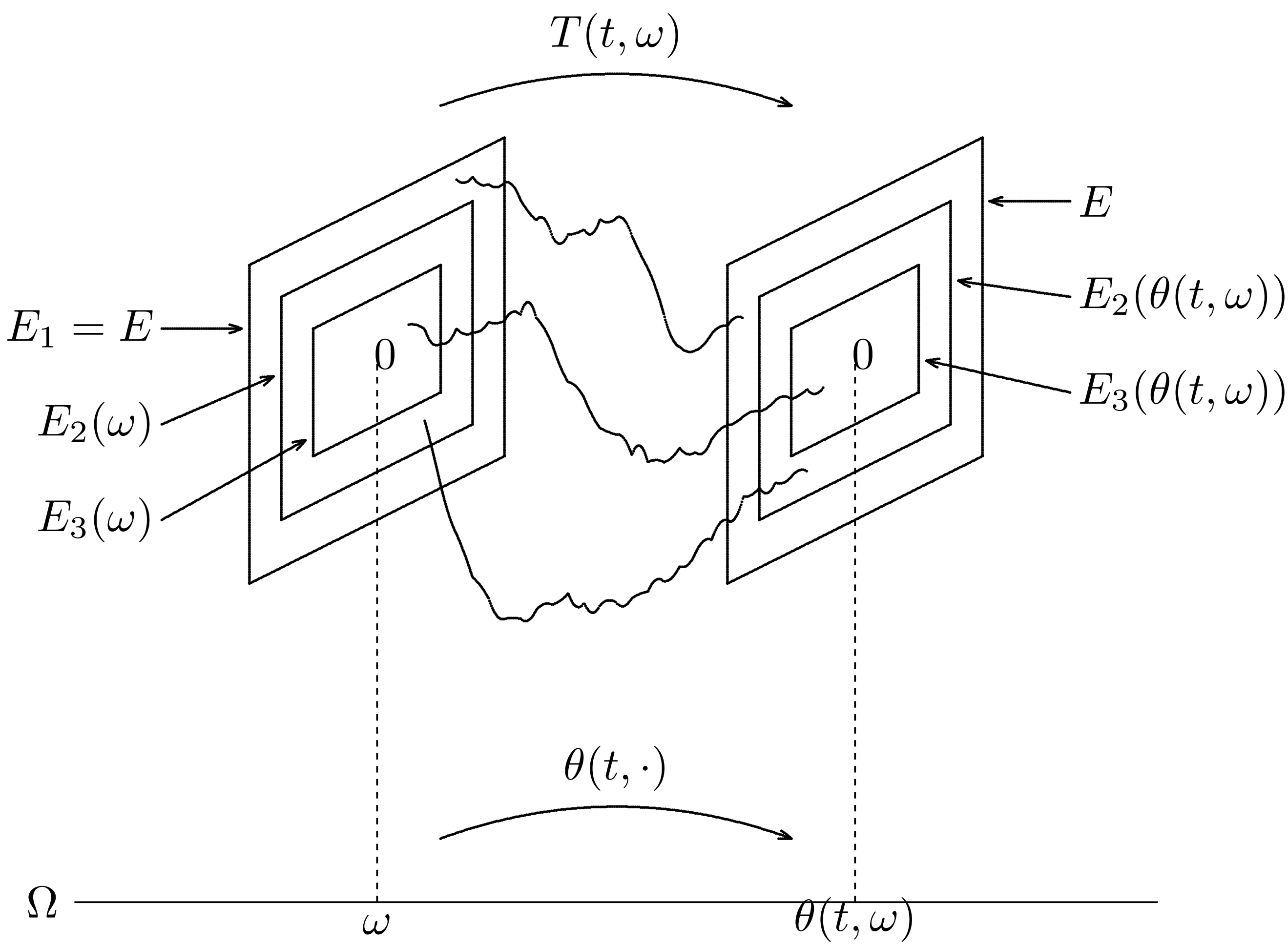
Proof.

Based on discrete version of Oseledec's multiplicative ergodic theorem and the perfect ergodic theorem. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]). □

Lyapunov Spectrum:

$\{\lambda_1,\lambda_2,\lambda_3,\cdots\}$:= *Lyapunov spectrum* of (T,θ).

Spectral Theorem



Definition

A stationary point $Y(\omega)$ of (X, θ) is *hyperbolic* if the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega))$ has a non-vanishing Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, viz. $\lambda_i \neq 0$ for all $i \geq 1$.

Let $i_0 > 1$ be s.t. $\lambda_{i_0} < 0 < \lambda_{i_0 - 1}$.

Assume $X(t, \cdot, \omega)$ *locally compact* and

$$E \log^+ \sup_{0 \leq t_1, t_2 \leq r} \|D_2 X(t_2, Y(\theta(t_1)), \theta(t_1))\|_{L(E)} < \infty.$$

By Oseledec-Ruelle Theorem, there is a sequence of closed finite-codimensional (Oseledec) spaces

$$\cdots E_{i-1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$E_i(\omega) = \{x \in E : \lim_{t \to \infty} \frac{1}{t} \log \|DX(t, Y(\omega), \omega)(x)\| \leq \lambda_i\},$$

$i \geq 1$ and all $\omega \in \Omega^*$, a sure event in $\mathcal{F}$ satisfying $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$.

Let $\{U(\omega), S(\omega) : \omega \in \Omega^*\}$ be the unstable and stable subspaces associated with the linearized cocycle (DX, θ) ([Mo.1], Theorem 4, Corollary 2; [M-S.1], Theorem 5.3). Then get a measurable invariant splitting

$$E = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \Omega^*,$$

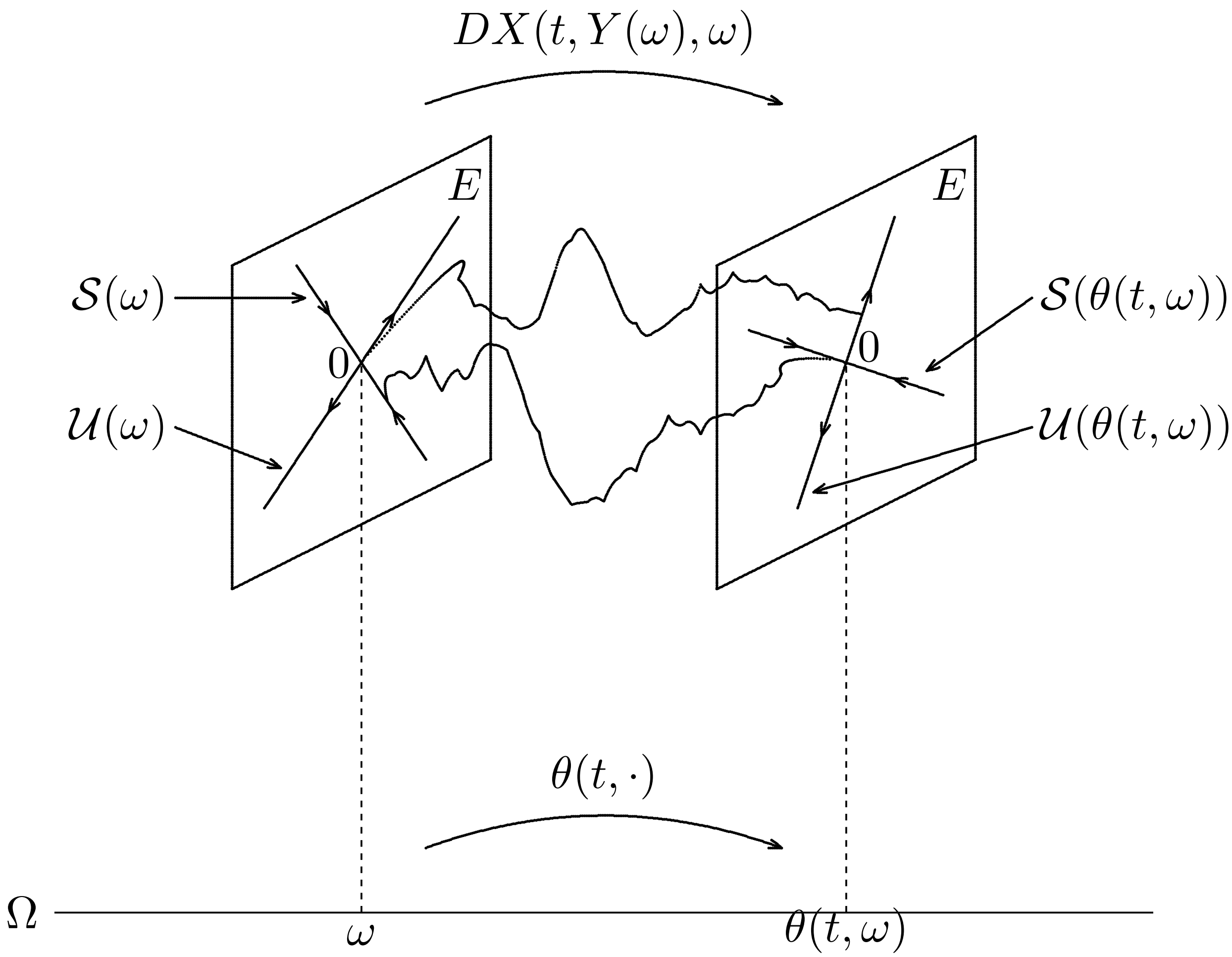
$$DX(t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)), \; DX(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)),$$

for all $t \geq 0$, with *exponential dichotomies*

$$\|DX(t, Y(\omega), \omega)(x)\| \geq \|x\| e^{\delta_1 t} \quad \text{for all} \quad t \geq \tau_1^*, x \in \mathcal{U}(\omega),$$

$$\|DX(t, Y(\omega), \omega)(x)\| \leq \|x\| e^{-\delta_2 t} \quad \text{for all} \quad t \geq \tau_2^*, x \in \mathcal{S}(\omega),$$

with $\tau_i^* = \tau_i^*(x, \omega) > 0, i = 1, 2,$ random times and $\delta_i > 0, i = 1, 2,$ fixed.



$DX(t, Y(\omega), \omega)$
E
E
$\mathcal{S}(\omega)$
0
0
$\mathcal{S}(\theta(t,\omega))$
$\mathcal{U}(\omega)$
$\mathcal{U}(\theta(t,\omega))$
$\theta(t,\cdot)$
Ω
ω
$\theta(t,\omega)$

Nonlinear SFDE's

"Regular" Itô SFDE with finite memory:

$$\left.\begin{aligned} dx(t) = H(x(t), x_t)\,dt + \sum_{i=1}^{m} G_i(x(t))\,dW_i(t), \\ (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \quad (I)$$

Solution segment $x_t(s) := x(t+s),\ t \geq 0, s \in [-r, 0]$.

m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$, $W(0) = 0$.

Ergodic Brownian shift θ on Wiener space $(\Omega, \mathcal{F}, P)$. $\bar{\mathcal{F}} := P-$completion of $\mathcal{F}$.

State space M_2, Hilbert with usual norm $\|\cdot\|$.

Can allow for "smooth memory" in diffusion coefficient.

$H : M_2 \to \mathbf{R}^d$, $C^{k,\delta}$, globally bounded.

$G : \mathbf{R}^d \to L(\mathbf{R}^m, \mathbf{R}^d)$, $C_b^{k+1,\delta}$; $G := (G_1, \cdots, G_m)$.

$B((v,\eta),\rho)$ open ball, radius ρ, center $(v,\eta) \in M_2$;

$\bar{B}((v,\eta),\rho)$ closed ball.

Then (I) has a stochastic semiflow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ with $X(t,(v,\eta),\cdot) = (x(t), x_t)$. X is $C^{k,\epsilon}$ for any $\epsilon \in (0,\delta)$, takes bounded sets into relatively compact sets in M_2. (X,θ) is a perfect cocycle on M_2 ([M-S.4]).

Idea of Proof.

Construction and regularity of the cocycle (X, θ): SFDE is equivalent to the neutral integral equation:

$$\zeta(t, x(t,\omega), \omega) = v + \int_0^t F(u, \zeta(u, x(u,\omega), \omega), x(u,\omega), x_u(\cdot,\omega), \omega)\, du,$$

$0 \leq t \leq T, (v, \eta) \in M_2$;

$F : [0, \infty) \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ is given by

$$F(t, z, v, \eta, \omega) := \{D\psi(t, z, \omega)\}^{-1} H(v, \eta)$$

$t \geq 0, z, v \in \mathbf{R}^d, \eta \in L^2([-r, 0], \mathbf{R}^d), \omega \in \Omega.$

ψ is the $C^{k+1,\epsilon}$ $\big(0 < \epsilon < \delta\big)$ stochastic flow of the sde

$$\left.\begin{aligned} d\psi(t) &= G(\psi(t))\, dW(t), \quad t \geq 0 \\ \psi(0) &= x \in \mathbf{R}^d \end{aligned}\right\}$$

$\zeta : [0, \infty) \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ is defined by

$$\zeta(t, x, \omega) := \psi(t, \cdot, \omega)^{-1}(x), \quad t \geq 0,\, x \in \mathbf{R}^d,\, \omega \in \Omega.$$

Read existence and properties of cocycle from integral equation. $\square$

Example

Consider the affine linear sfde

$$\left. \begin{aligned} dx(t) &= H(x(t), x_t)\, dt + G\, dW(t), \quad t > 0 \\ x(0) &= v \in \mathbf{R}^d, \quad x_0 = \eta \in L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (I')$$

where $H : M_2 \to \mathbf{R}^d$ is a continuous linear map, G is a fixed $(d \times p)$-matrix, and W is p-dimensional Brownian motion. Assume that the linear deterministic $(d \times d)$-matrix-valued FDE

$$dy(t) = H \circ (y(t), y_t)\, dt$$

has a semiflow

$$T_t : L(\mathbf{R}^d) \times L^2([-r, 0], L(\mathbf{R}^d)) \to L(\mathbf{R}^d) \times L^2([-r, 0], L(\mathbf{R}^d)), t \geq 0,$$

which is uniformly asymptotically stable. Set

$$Y := \int_{-\infty}^{0} T_{-u}(I, 0) G \, dW(u) \tag{2}$$

where I is the identity $(d \times d)$-matrix. Integration by parts and

$$W(t, \theta(t_1, \omega)) = W(t + t_1, \omega) - W(t_1, \omega), \quad t, t_1 \in \mathbf{R}, \tag{3}$$

imply that Y has a measurable version satisfying (1). Y is Gaussian and thus has finite moments of all orders. See ([Mo.1], Theorem 4.2, Corollary 4.2.1, pp. 208-217.) More generally, when H is "hyperbolic", one can show that a stationary point of (I') exists ([Mo.1]).

For general white-noise an invariant measure on M_2 for the one-point motion gives a stationary point by enlarging probability space. Conversely, let $Y : \Omega \to M_2$ be a stationary random point independent of the Brownian motion $W(t)$, $t \geq 0$. Let $\rho := P \circ Y^{-1}$ be the distribution of Y. By independence of Y and W, ρ is an invariant measure for the one-point motion

Theorem 2. *([M-S], 2000) (The Stable Manifold Theorem)*

Assume smoothness hypotheses on H and G. Let $Y : \Omega \to M_2$ be a hyperbolic stationary point of the SFDE (I) such that $E(\|Y(\cdot)\|^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$

Suppose the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ of (I) has a Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i <$

$\cdots < \lambda_2 < \lambda_1\}$. Define $\lambda_{i_0} := max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all finite λ_i are positive, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized semiflow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,

(ii) $\bar{\mathcal{F}}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0,1)$, $\beta_i > \rho_i > 0$, $i = 1,2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k,\epsilon}$ ($\epsilon \in (0,\delta)$) submanifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) $\tilde{\mathcal{S}}(\omega)$ *is the set of all* $(v,\eta)\in\bar{B}(Y(\omega),\rho_1(\omega))$ *such that*

$$\|X(nr,(v,\eta),\omega)-Y(\theta(nr,\omega))\|\le\beta_1(\omega)\,e^{(\lambda_{i_0}+\epsilon_1)nr}$$

for all integers $n\ge 0$. *Furthermore,*

$$\limsup_{t\to\infty}\frac{1}{t}\log\|X(t,(v,\eta),\omega)-Y(\theta(t,\omega))\|\le\lambda_{i_0}$$

for all $(v,\eta)\in\tilde{\mathcal{S}}(\omega)$. *Each stable subspace* $\mathcal{S}(\omega)$ *of the linearized semiflow* DX *is tangent at* $Y(\omega)$ *to the submanifold* $\tilde{\mathcal{S}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega)=\mathcal{S}(\omega)$. *In particular, codim* $\tilde{\mathcal{S}}(\omega)=$ *codim* $\mathcal{S}(\omega)$, *is fixed and finite.*

(b) $$\limsup_{t\to\infty}\frac{1}{t}\log\left[\sup\left\{\frac{\|X(t,(v_1,\eta_1),\omega)-X(t,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1)-(v_2,\eta_2)\|}:\right.\right.$$
$$\left.\left.(v_1,\eta_1)\ne(v_2,\eta_2),\ (v_1,\eta_1),\ (v_2,\eta_2)\in\tilde{\mathcal{S}}(\omega)\right\}\right]\le\lambda_{i_0}.$$

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ *such that*

$$X(t,\cdot,\omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t,\omega))$$

for all $t \geq \tau_1(\omega)$. *Also*

$$DX(t,Y(\omega),\omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)), \quad t \geq 0.$$

(d) $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $(v,\eta) \in \bar{B}(Y(\omega),\rho_2(\omega))$ *with the property that there is a unique "history" process* $y(\cdot,\omega) : \{-nr : n \geq 0\} \to M_2$ *such that* $y(0,\omega) = (v,\eta)$ *and for each integer* $n \geq 1$*, one has* $X(r,y(-nr,\omega),\theta(-nr,\omega)) = y(-(n-1)r,\omega)$ *and*

$$\|y(-nr,\omega)-Y(\theta(-nr,\omega))\|_{M_2} \leq \beta_2(\omega)e^{-(\lambda_{i_0-1}-\epsilon_2)nr}.$$

Furthermore, for each $(v,\eta) \in \tilde{\mathcal{U}}(\omega)$*, there is a unique continuous-time "history" process also denoted by* $y(\cdot,\omega) : (-\infty,0] \to M_2$ *such that* $y(0,\omega) = (v,\eta)$,

$X(t, y(s,\omega), \theta(s,\omega)) = y(t+s,\omega)$ *for all* $s \leq 0, 0 \leq t \leq -s$, *and*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|y(-t,\omega) - Y(\theta(-t,\omega))\| \leq -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}(\omega)$ *of the linearized semi-flow* DX *is tangent at* $Y(\omega)$ *to* $\tilde{\mathcal{U}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. *In particular,* $\dim \tilde{\mathcal{U}}(\omega)$ *is finite and non-random.*

(e) *Let* $y(\cdot, (v_i, \eta_i), \omega), i = 1, 2$, *be the history processes associated with* $(v_i, \eta_i) = y(0, (v_i, \eta_i), \omega) \in \tilde{\mathcal{U}}(\omega)$, $i = 1, 2$. *Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log \Big[\sup \Big\{ \frac{\|y(-t, (v_1,\eta_1), \omega) - y(-t, (v_2,\eta_2), \omega)\|}{\|(v_1,\eta_1) - (v_2,\eta_2)\|} : (v_1,\eta_1) \neq (v_2,\eta_2), (v_i,\eta_i) \in \tilde{\mathcal{U}}(\omega), i = 1, 2 \Big\} \Big] \leq -\lambda_{i_0-1}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ *such that*

$$\tilde{\mathcal{U}}(\omega) \subseteq X(t, \cdot, \theta(-t, \omega))(\tilde{\mathcal{U}}(\theta(-t, \omega)))$$

for all $t \geq \tau_2(\omega)$. *Also*

$$DX(t, \cdot, \theta(-t, \omega))(\mathcal{U}(\theta(-t, \omega))) = \mathcal{U}(\omega), \quad t \geq 0;$$

and the restriction

$$DX(t, \cdot, \theta(-t, \omega))|\mathcal{U}(\theta(-t, \omega)) : \mathcal{U}(\theta(-t, \omega)) \to \mathcal{U}(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto.

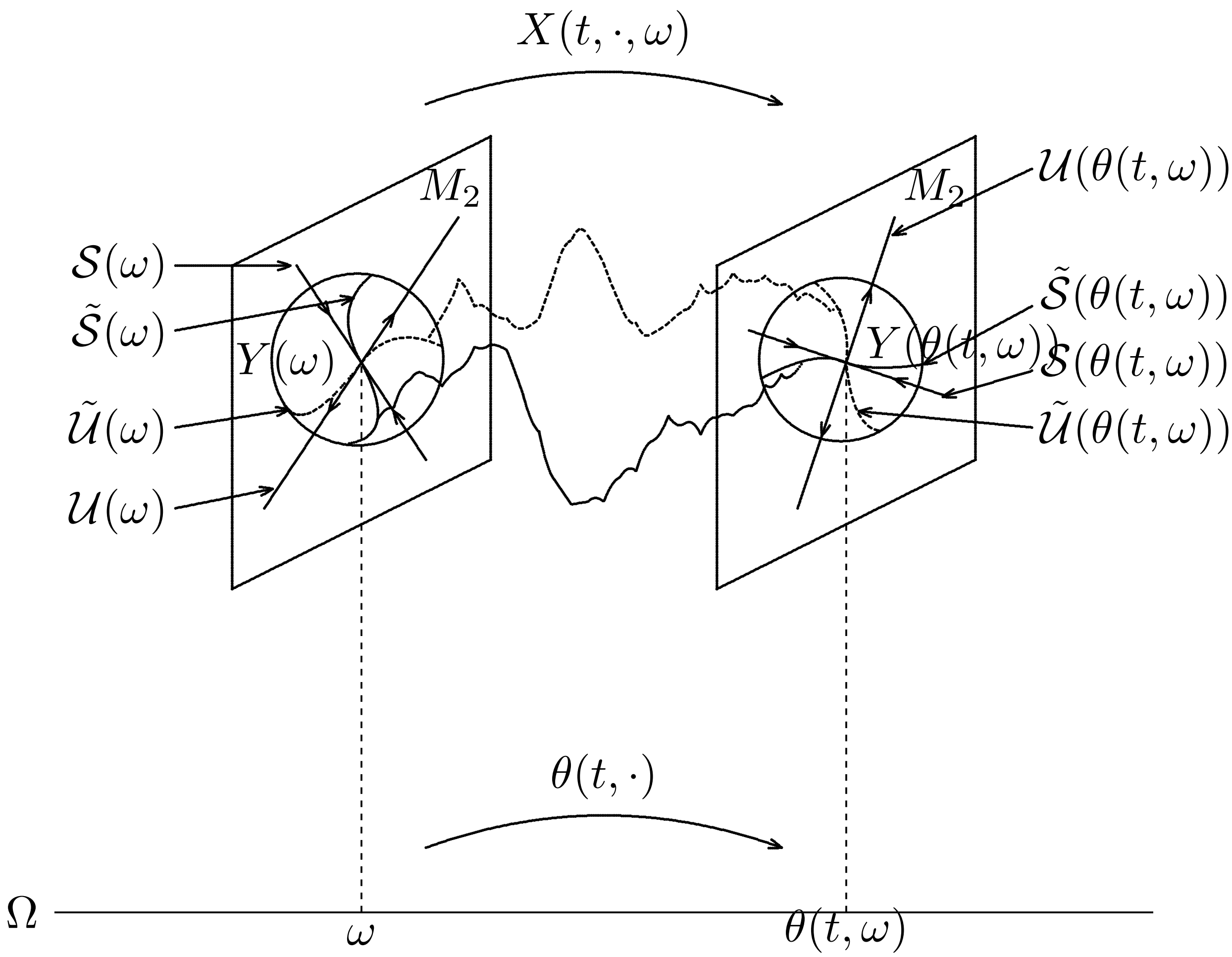
(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ *and* $\tilde{\mathcal{S}}(\omega)$ *are transversal, viz.*

$$M_2 = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

Assume, in addition, that H, G *are* C_b^∞. *Then the local stable and unstable manifolds* $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ *are* C^∞.

Figure summarizes essential features of Stable Manifold Theorem:

Stable Manifold Theorem



$t > \tau_1(\omega)$

A picture is worth a 1000 words!

Outline of Proof of Theorem 2

- By definition, a *stationary* random point $Y(\omega) \in M_2$ is invariant under the semiflow X; viz $X(t, Y) = Y(\theta(t, \cdot))$ for all times t.
- Linearize the semiflow X along the stationary point $Y(\omega)$ in M_2. By stationarity of Y and the cocycle property of X, this gives a linear perfect cocycle $(DX(t, Y), \theta(t, \cdot))$ in $L(M_2)$, where $D =$ spatial (Fréchet) derivatives.
- Ergodicity of θ allows for the notion of *hyperbolicity* of a stationary solution of (I) via Oseledec-Ruelle theorem: Use local compactness of the semiflow for times greater than

the delay r ([M-S.4]), and apply multiplicative ergodic theorem to get a discrete non-random Lyapunov spectrum $\{\lambda_i : i \geq 1\}$ for the linearized cocycle. Y is *hyperbolic* if $\lambda_i \neq 0$ for every i.

- Assume that $\|Y\|^{\epsilon_0}$ is integrable (for small ϵ_0). Variational method of construction of the semiflow shows that the linearized cocycle satisfies hypotheses of "perfect versions" of ergodic theorem and Kingman's subadditive ergodic theorem. These refined versions give invariance of the Oseledec spaces under the continuous-time linearized cocycle. Thus the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear semiflow X.

- Establish continuous-time integrability estimates on the spatial derivatives of the nonlinear cocycle X in a neighborhood of the stationary point Y. Estimates follow from the variational construction of the stochastic semiflow coupled with known global spatial estimates for finite-dimensional stochastic flows.
- Introduce the auxiliary perfect cocycle

$$Z(t,\cdot,\omega) := X(t,(\cdot)+Y(\omega),\omega) - Y(\theta(t,\omega)),\ t \in \mathbf{R}^+, \omega \in \Omega.$$

  Refine arguments in ([Ru.2], Theorems 5.1 and 6.1) to construct local stable/unstable manifolds for the discrete cocycle $(Z(nr,\cdot,\omega),\theta(nr,\omega))$ near 0 and hence (by translation) for $X(nr,\cdot,\omega)$

near $Y(\omega)$ for all ω sampled from a $\theta(t,\cdot)$-invariant sure event in Ω. This is possible because of the continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between delay periods of length r and further refining the arguments in [Ru.2], show that the above manifolds also serve as local stable/unstable manifolds for the *continuous-time* semiflow X near Y.

- Final key step: Establish the asymptotic invariance of the local stable manifolds under the stochastic semiflow X. Use arguments underlying the proofs of Theorems 4.1 and

5.1 in [Ru.2] and some difficult estimates using the continuous-time integrability properties, and the perfect subadditive ergodic theorem. Asymptotic invariance of the local unstable manifolds follows by employing the concept of a *stochastic history process* for X coupled with similar arguments to the above. Existence of history process compensates for the lack of invertibility of the semiflow.

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