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# The Stable Manifold Theorem for Stochastic Differential Equations (Analysis and Probability Seminar, University of Hull)

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**THE STABLE MANIFOLD  
THEOREM FOR STOCHASTIC  
DIFFERENTIAL EQUATIONS**

**Hull, England : March 15, 2001**

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## SDE's: Stable Manifolds

- Formulate a *Local Stable Manifold Theorem* for SDE's driven by Brownian motion (or general noise with stationary ergodic increments): Stratonovich or Itô type.
- Start with the existence of a stochastic flow for SDE.
- Concept of a hyperbolic stationary trajectory. The stationary trajectory is a solution of the forward/backward anticipating SDE for all time (Stratonovich case).

- Existence of a stationary random family of asymptotically invariant stable and unstable manifolds within a stationary neighborhood of the hyperbolic stationary solution.
- Stable and unstable manifolds dynamically characterized using forward and backward solutions of anticipating versions of the (Stratonovich) SDE.
- Proof based on Ruelle-Oseledec (non-linear) multiplicative ergodic theory and anticipating stochastic calculus.

# Formulation of the Theorem

Stratonovich SDE on  $\mathbf{R}^d$

$$dx(t) = h(x(t)) dt + \sum_{i=1}^m g_i(x(t)) \circ dW_i(t), \quad (I)$$

driven by  $m$ -dimensional Brownian motion  $W := (W_1, \dots, W_m)$ .

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}}, P) :=$  canonical filtered Wiener space.

$\Omega :=$  space of all continuous paths  $\omega : \mathbf{R} \rightarrow \mathbf{R}^m$ ,  $\omega(0) = 0$ , in Euclidean space  $\mathbf{R}^m$ , with compact open topology;

$\mathcal{F} :=$  Borel  $\sigma$ -field of  $\Omega$ ;

$\mathcal{F}_t :=$  sub- $\sigma$ -field of  $\mathcal{F}$  generated by the evaluations  $\omega \rightarrow \omega(u)$ ,  $u \leq t$ ,  $t \in \mathbf{R}$ .

$P :=$  Wiener measure on  $\Omega$ .

$h : \mathbf{R}^d \rightarrow \mathbf{R}^d, 1 \leq i \leq m, C_b^{k,\delta}$  vector fields on  $\mathbf{R}^d$ ; viz.  $h$  has all derivatives  $D^j h, 1 \leq j \leq k$ , continuous and globally bounded,  $D^k h$  Hölder continuous with exponent  $\delta \in (0, 1)$ .

$g_i, 1 \leq i \leq m$ , globally bounded and  $C_b^{k+1,\delta}$ .

$\theta : \mathbf{R} \times \Omega \rightarrow \Omega$  is the (ergodic) Brownian shift

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in \Omega.$$

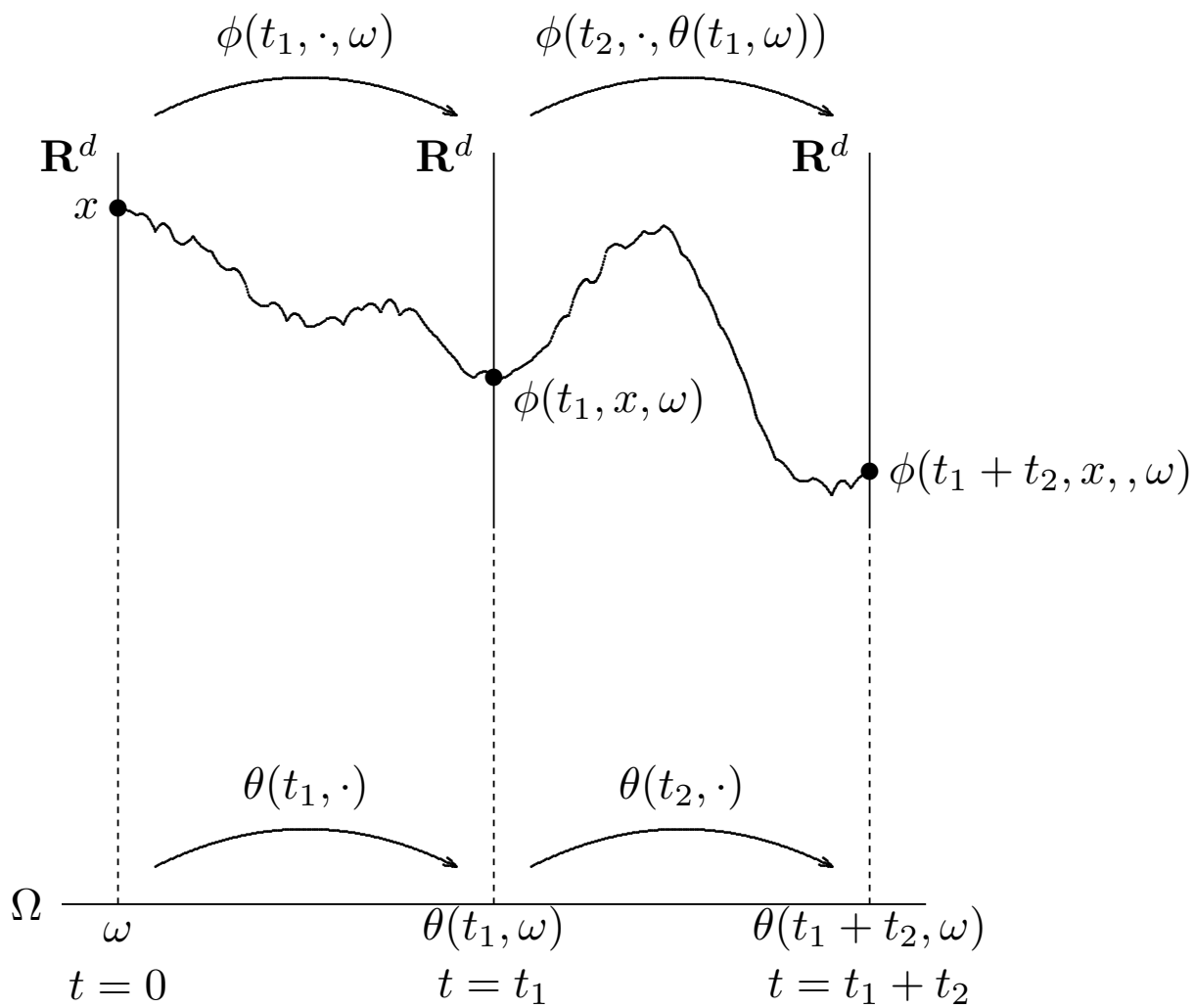
Let  $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \rightarrow \mathbf{R}^d$  be the stochastic flow generated by (I) ( $\phi(t, \cdot, \omega) = [\phi(-t, \cdot, \theta(t, \omega))]^{-1}$ ,  $t < 0$ ). Then  $\phi$  is a perfect *cocycle*:

$$\phi(t_1 + t_2, \cdot, \omega) = \phi(t_2, \cdot, \theta(t_1, \omega)) \circ \phi(t_1, \cdot, \omega),$$

for all  $t_1, t_2 \in \mathbf{R}$  and all  $\omega \in \Omega$  ([I-W], [A-S], [A]).

Figure illustrates the cocycle property. Vertical solid lines represent random fibers consisting of copies of  $\mathbf{R}^d$ .  $(\phi, \theta)$  is a “random vector-bundle morphism” over the “base” probability space  $\Omega$ .

# The Cocycle



## Definition

The SDE (I) has a *stationary trajectory* if there exists an  $\mathcal{F}$ -measurable random variable  $Y : \Omega \rightarrow \mathbf{R}^d$  such that

$$\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{1}$$

for all  $t \in \mathbf{R}$  and every  $\omega \in \Omega$ . Denote stationary trajectory (1) by  $\phi(t, Y) = Y(\theta(t))$ .

## Examples of Stationary Solutions

1. Fixed points:

$$d\phi(t) = h(\phi(t)) dt + \sum_{i=1}^m g_i(\phi(t)) \circ dW_i(t)$$

$$h(x_0) = g_i(x_0) = 0, \quad 1 \leq i \leq m$$

Take  $Y(\omega) = x_0$  for all  $\omega \in \Omega$ .

2. Linear affine case  $d = 1$ :

$$d\phi(t) = \lambda\phi(t) dt + dW(t)$$

$\lambda > 0$  fixed,  $W(t) \in \mathbf{R}$ . Take

$$\phi(t, x, \omega) = e^{\lambda t} \left[ x + \int_0^t e^{-\lambda u} dW(u) \right],$$

$$Y(\omega) := - \int_0^\infty e^{-\lambda u} dW(u),$$

$$\theta(t, \omega)(s) = \omega(t + s) - \omega(t).$$

Check that  $\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega))$ , using integration by parts and variation of parameters.

3. Affine linear SDE in  $d = 2$ :

$$d\phi(t) = A\phi(t) dt + GdW(t)$$

with  $A$  a fixed hyperbolic  $2 \times 2$ -diagonal matrix;  $G$  a constant  $2 \times 2$ -matrix, and  $W$  2-dimensional Brownian motion.

4. Non-linear transforms of (3) under a global diffeomorphism.

5. Invariant measure for SDE: Enlarge probability space ([M-S.3]).

Let  $\phi(t, Y)$  be a stationary solution of (I). Cocycle property of  $\phi$  implies that the linearization

$$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$$

along the stationary solution is also a  $d \times d$ -matrix-valued cocycle. Using Kolmogorov's theorem, the random variables

$$\sup_{x \in \mathbf{R}^d} \frac{\|D_2\phi(t, x)\|}{(1 + |x|^\gamma)}, \quad \gamma > 0,$$

have moments of all orders. If  $E \log^+ |Y| < \infty$ , then  $E \log^+ \|D_2\phi(1, Y)\| < \infty$ . Apply Oseledec's Theorem to get a *non-random* finite Lyapunov spectrum:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |D_2\phi(n, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbf{R}^d).$$

Spectrum takes finitely many *fixed* values  $\{\lambda_i\}_{i=1}^p$  with non-random multiplicities  $q_i$ ,  $1 \leq i \leq p$ , and  $\sum_{i=1}^p q_i = d$  ([Ru.1], Theorem I.6).

### Definition

Stationary trajectory  $\phi(t, Y)$  of (I) is *hyperbolic* if  $E \log^+ |Y(\cdot)| < \infty$ , and if the linearized cocycle  $(D_2\phi(n, Y(\omega), \omega), \theta(n, \omega))$  has a non-vanishing Lyapunov spectrum

$$\{\lambda_p < \cdots < \lambda_{i_0+1} < \lambda_{i_0} < 0 < \lambda_{i_0-1} < \cdots < \lambda_2 < \lambda_1\}$$

i.e.  $\lambda_i \neq 0$  for all  $1 \leq i \leq p$ .

Define  $\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}$  if at least one  $\lambda_i < 0$ . If all  $\lambda_i > 0$ , set  $\lambda_{i_0} = -\infty$ . (This implies that  $\lambda_{i_0-1}$  is the smallest

positive Lyapunov exponent of the linearized flow, if at least one  $\lambda_i > 0$ ; in case all  $\lambda_i$  are negative, set  $\lambda_{i_0-1} = \infty$ .)

Let  $\rho \in \mathbf{R}^+$ ,  $x \in \mathbf{R}^d$ .

$B(x, \rho) :=$  open ball in  $\mathbf{R}^d$ , center  $x$  and radius  $\rho$ ;

$\bar{B}(x, \rho) :=$  corresponding closed ball;

$\mathcal{K}(\mathbf{R}^d) :=$  the class of all non-empty compact subsets of  $\mathbf{R}^d$  with Hausdorff metric  $d^*$ :

$d^*(A_1, A_2) := \sup\{d(x, A_1) : x \in A_2\} \vee \sup\{d(y, A_2) : y \in A_1\}$  where  $A_1, A_2 \in \mathcal{K}(\mathbf{R}^d)$ ;

$d(x, A_i) := \inf\{|x - y| : y \in A_i\}$ ,  $x \in \mathbf{R}^d$ ,  $i = 1, 2$ ;

$\mathcal{B}(\mathcal{K}(\mathbf{R}^d)) :=$  Borel  $\sigma$ -algebra on  $\mathcal{K}(\mathbf{R}^d)$  with respect to the metric  $d^*$ .

**Theorem 1** (The Stable Manifold Theorem) (M.+ Scheutzow, AOP '99)

*Assume that the coefficients of SDE (I) satisfy the given hypotheses. Suppose  $\phi(t, Y)$  is a hyperbolic stationary trajectory of (I) with  $E \log^+ |Y| < \infty$ .*

*Fix  $\epsilon_1 \in (0, -\lambda_{i_0})$  and  $\epsilon_2 \in (0, \lambda_{i_0-1})$ . Then there exist*

*(i) a sure event  $\Omega^* \in \mathcal{F}$  with  $\theta(t, \cdot)(\Omega^*) = \Omega^*$  for all*

*$t \in \mathbf{R}$ ,*

*(ii)  $\mathcal{F}$ -measurable random variables  $\rho_i, \beta_i : \Omega^* \rightarrow (0, 1)$ ,  $\beta_i >$*

*$\rho_i > 0$ ,  $i = 1, 2$ , such that for each  $\omega \in \Omega^*$ , the following is true:*

*There are  $C^{k, \epsilon}$  ( $\epsilon \in (0, \delta)$ ) submanifolds  $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$  of  $\bar{B}(Y(\omega), \rho_1(\omega))$  and  $\bar{B}(Y(\omega), \rho_2(\omega))$  (resp.) with the following properties:*

(a)  $\tilde{\mathcal{S}}(\omega)$  is the set of all  $x \in \bar{B}(Y(\omega), \rho_1(\omega))$  such that

$$|\phi(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers  $n \geq 0$ . Furthermore,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0} \quad (2)$$

for all  $x \in \tilde{\mathcal{S}}(\omega)$ . Each stable subspace  $\mathcal{S}(\omega)$  of the linearized flow  $D_2\phi$  is tangent at  $Y(\omega)$  to the submanifold  $\tilde{\mathcal{S}}(\omega)$ , viz.  $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$ . In particular,  $\dim \tilde{\mathcal{S}}(\omega) = \dim \mathcal{S}(\omega)$  and is non-random.

$$(b) \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[ \sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{\mathcal{S}}(\omega)}} \left\{ \frac{|\phi(t, x_1, \omega) - \phi(t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq \lambda_{i_0}.$$

(c) (Cocycle-invariance of the stable manifolds):

There exists  $\tau_1(\omega) \geq 0$  such that

$$\phi(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega)), \quad t \geq \tau_1(\omega). \quad (3)$$

Also

$$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t, \omega)), \quad t \geq 0. \quad (4)$$

(d)  $\tilde{\mathcal{U}}(\omega)$  is the set of all  $x \in \bar{B}(Y(\omega), \rho_2(\omega))$  with the property that

$$|\phi(-n, x, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega) e^{(-\lambda_{i_0}-1+\epsilon_2)n} \quad (5)$$

for all integers  $n \geq 0$ . Also

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |\phi(-t, x, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0}-1. \quad (6)$$

for all  $x \in \tilde{\mathcal{U}}(\omega)$ . Furthermore, the unstable subspace  $\mathcal{U}(\omega)$  of  $D_2\phi$  is the tangent space to  $\tilde{\mathcal{U}}(\omega)$  at  $Y(\omega)$ , viz.  $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$ . In particular,  $\dim \tilde{\mathcal{U}}(\omega) = \dim \mathcal{U}(\omega)$  and is non-random.

$$(e) \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[ \sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{\mathcal{U}}(\omega)}} \left\{ \frac{|\phi(-t, x_1, \omega) - \phi(-t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq$$

$$-\lambda_{i_0-1}.$$

(f) *(Cocycle-invariance of the unstable manifolds):*

*There exists  $\tau_2(\omega) \geq 0$  such that*

$$\phi(-t, \cdot, \omega)(\tilde{\mathcal{U}}(\omega)) \subseteq \tilde{\mathcal{U}}(\theta(-t, \omega)), \quad t \geq \tau_2(\omega). \quad (7)$$

*Also*

$$D_2\phi(-t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(-t, \omega)), \quad t \geq 0. \quad (8)$$

(g) *The submanifolds  $\tilde{\mathcal{U}}(\omega)$  and  $\tilde{\mathcal{S}}(\omega)$  are transversal, viz.*

$$\mathbf{R}^d = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega). \quad (9)$$

(h) *The mappings*

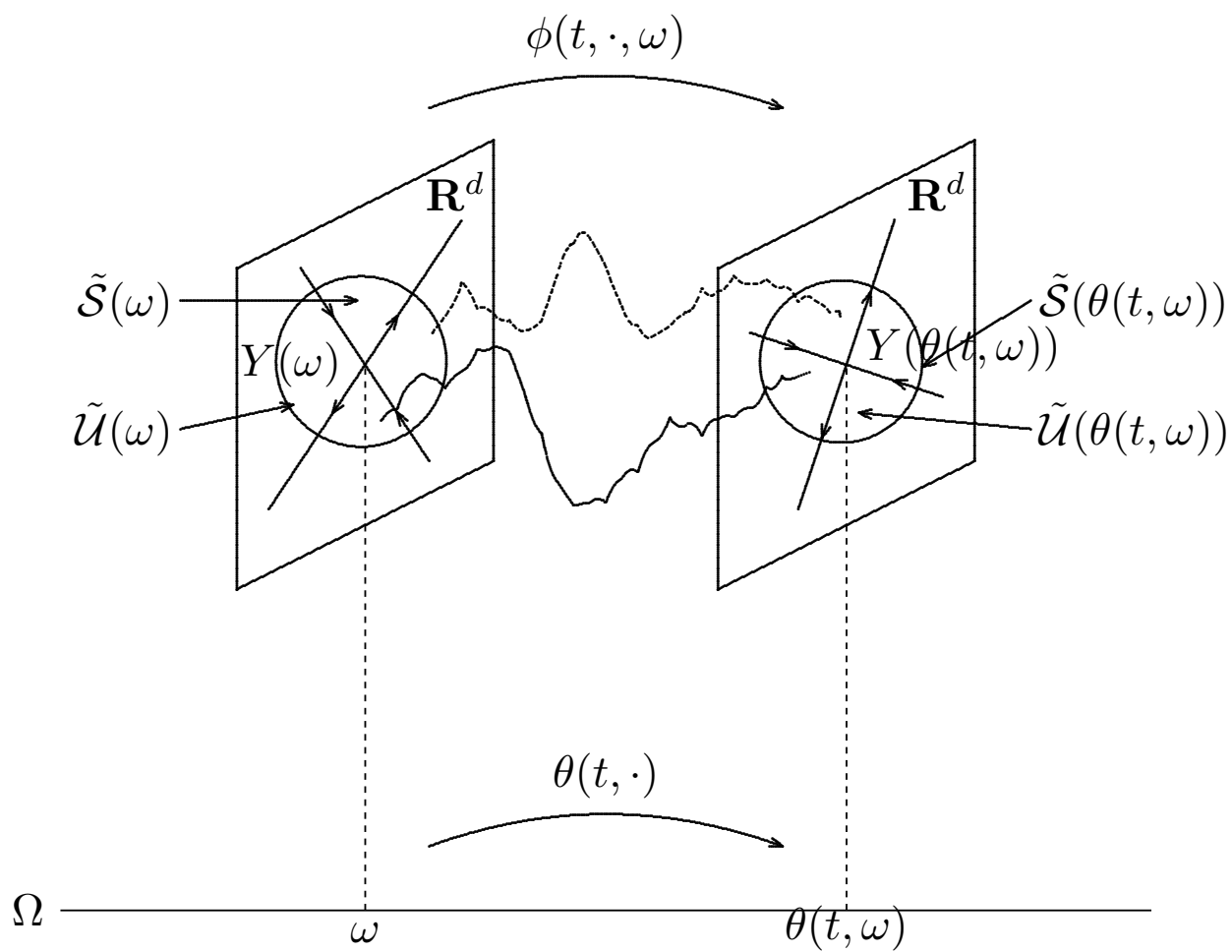
$$\Omega \rightarrow \mathcal{K}(\mathbf{R}^d), \quad \Omega \rightarrow \mathcal{K}(\mathbf{R}^d),$$

$$\omega \mapsto \tilde{\mathcal{S}}(\omega) \quad \omega \mapsto \tilde{\mathcal{U}}(\omega)$$

are  $(\mathcal{F}, \mathcal{B}(\mathcal{K}(\mathbf{R}^d)))$ -measurable.

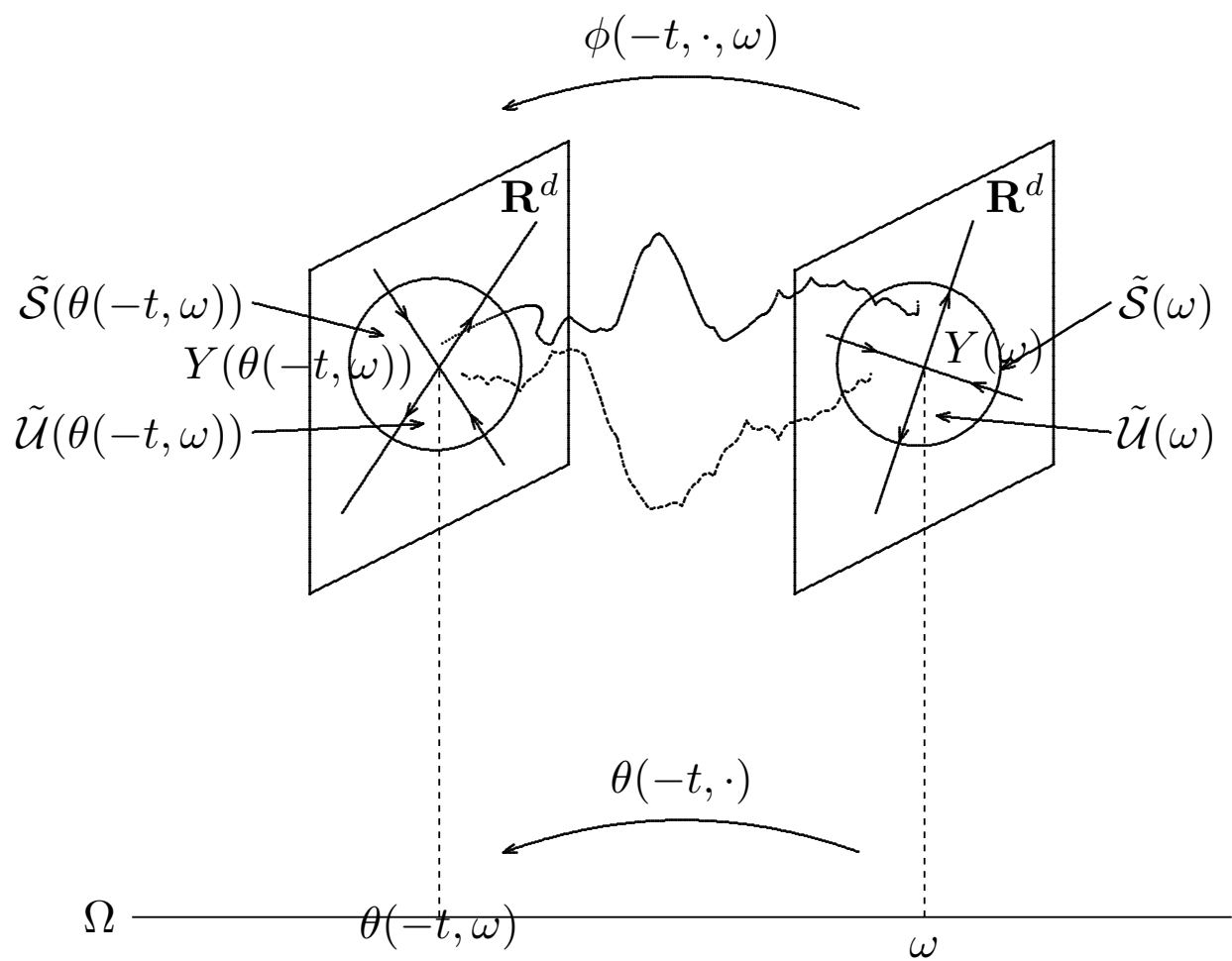
Assume, further, that  $h, g_i, 1 \leq i \leq m$ , are  $C_b^\infty$ .

Then the local stable and unstable manifolds  $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$  are  $C^\infty$ .



$$t > \tau_1(\omega)$$

*A picture is worth a 1000 words!*



$$t > \tau_2(\omega)$$

## Sketch of Proof

### Linearization and Substitution

Assume regularity conditions on the coefficients  $h, g_i$ . By the *Substitution Rule*,  $\phi(t, Y(\omega), \omega)$  is a stationary *solution* of the *anticipating* Stratonovich SDE

$$\left. \begin{aligned} d\phi(t, Y) &= h(\phi(t, Y)) dt + \sum_{i=1}^m g_i(\phi(t, Y)) \circ dW_i(t), & t > 0 \\ \phi(0, Y) &= Y. \end{aligned} \right\} \quad (II)$$

([N-P]).

Linearize the SDE (I) along the stationary trajectory. By substitution, match the solution of the linearized equation

with the linearized cocycle  $D_2\phi(t, Y(\omega), \omega)$ .

Hence  $D_2\phi(t, Y(\omega), \omega)$ ,  $t \geq 0$ , solves the SDE:

$$\left. \begin{aligned} dD_2\phi(t, Y) &= Dh(\phi(t, Y))D_2\phi(t, Y) dt \\ &\quad + \sum_{i=1}^m Dg_i(\phi(t, Y))D_2\phi(t, Y) \circ dW_i(t), \quad t > 0 \\ D_2\phi(0, Y) &= I. \end{aligned} \right\} \quad (III)$$

$D_2, D$  denotes spatial (Fréchet) derivatives.

Similarly, the backward trajectories

$$\phi(t, Y), D_2\phi(t, Y), \quad t < 0,$$

solve the corresponding backward Stratonovich SDE's:

$$\left. \begin{aligned} d\phi(t, Y) &= -h(\phi(t, Y)) dt - \sum_{i=1}^m g_i(\phi(t, Y)) \circ \hat{d}W_i(t), \quad t < 0 \\ \phi(0, Y) &= Y. \end{aligned} \right\} \quad (II^-)$$

$$\left. \begin{aligned}
dD_2\phi(t, Y) &= -Dh(\phi(t, Y))D_2\phi(t, Y) dt \\
&\quad - \sum_{i=1}^m Dg_i(\phi(t, Y))D_2\phi(t, Y) \circ \hat{d}W_i(t), \quad t < 0 \\
D_2\phi(0, Y) &= I.
\end{aligned} \right\} \quad (III^-)$$

Above SDE's (II)-(III)<sup>-</sup> give dynamic characterizations of the stable and unstable manifolds.

The following lemma is used to construct the shift-invariant sure event appearing in the statement of the local stable manifold theorem. Gives “perfect versions” of the ergodic theorem and Kingman’s subadditive ergodic theorem.

## Lemma 1

(i) Let  $h : \Omega \rightarrow \mathbf{R}^+$  be  $\mathcal{F}$ -measurable and such that

$$\int_{\Omega} \sup_{0 \leq u \leq 1} h(\theta(u, \omega)) dP(\omega) < \infty.$$

Then there is a sure event  $\Omega_1 \in \mathcal{F}$  such that  $\theta(t, \cdot)(\Omega_1) =$

$\Omega_1$  for all  $t \in \mathbf{R}$ , and

$$\lim_{t \rightarrow \infty} \frac{1}{t} h(\theta(t, \omega)) = 0$$

for **all**  $\omega \in \Omega_1$ .

(ii) Suppose  $f : \mathbf{R}^+ \times \Omega \rightarrow \mathbf{R} \cup \{-\infty\}$  is a measurable process on  $(\Omega, \mathcal{F}, P)$  satisfying the following conditions

$$(a) \quad E \sup_{0 \leq u \leq 1} f^+(u) < \infty, \quad E \sup_{0 \leq u \leq 1} f^+(1 - u, \theta(u)) < \infty$$

$$(b) \quad f(t_1 + t_2, \omega) \leq f(t_1, \omega) + f(t_2, \theta(t_1, \omega)) \text{ for all } t_1, t_2 \geq 0$$

and **all**  $\omega \in \Omega$ .

Then there is sure event  $\Omega_2 \in \mathcal{F}$  such that  $\theta(t, \cdot)(\Omega_2) = \Omega_2$  for all  $t \in \mathbf{R}$ , and a fixed number  $f^* \in \mathbf{R} \cup \{-\infty\}$  such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} f(t, \omega) = f^*$$

for all  $\omega \in \Omega_2$ .

**Proof**

[Mo.1], Lemma 7.  $\square$

## Theorem 2 ([O], 1968)

Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $\theta : \mathbf{R}^+ \times \Omega \rightarrow \Omega$  a measurable family of ergodic  $P$ -preserving transformations. Let  $T : \mathbf{R}^+ \times \Omega \rightarrow L(\mathbf{R}^d)$  be measurable, such that  $(T, \theta)$  is an  $L(\mathbf{R}^d)$ -valued cocycle. Suppose that

$$E \sup_{0 \leq t \leq 1} \log^+ \|T(t, \cdot)\| < \infty, \quad E \sup_{0 \leq t \leq 1} \log^+ \|T(1-t, \theta(t, \cdot))\| < \infty.$$

Then there is a set  $\Omega_0 \in \mathcal{F}$  of full  $P$ -measure such that  $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$  for all  $t \in \mathbf{R}^+$ , and for each  $\omega \in \Omega_0$ , the limit

$$\lim_{t \rightarrow \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. Each  $\Lambda(\omega)$  has a discrete non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \dots > e^{\lambda_p}$$

where the  $\lambda_i$ 's are distinct. Each  $e^{\lambda_i}$  has an eigen-space  $F_i(\omega)$  and a fixed non-random multiplicity  $m_i := \dim F_i(\omega)$ .

Define

$$E_1(\omega) := \mathbf{R}^d, \quad E_i(\omega) := \left[ \bigoplus_{j=1}^{i-1} F_j(\omega) \right]^\perp, \quad 1 < i \leq p.$$

Then

$$E_p(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = \mathbf{R}^d$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|T(t, \omega)x\| = \lambda_i(\omega), \quad \text{if } x \in E_i(\omega) \setminus E_{i+1}(\omega),$$

and

$$T(t, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

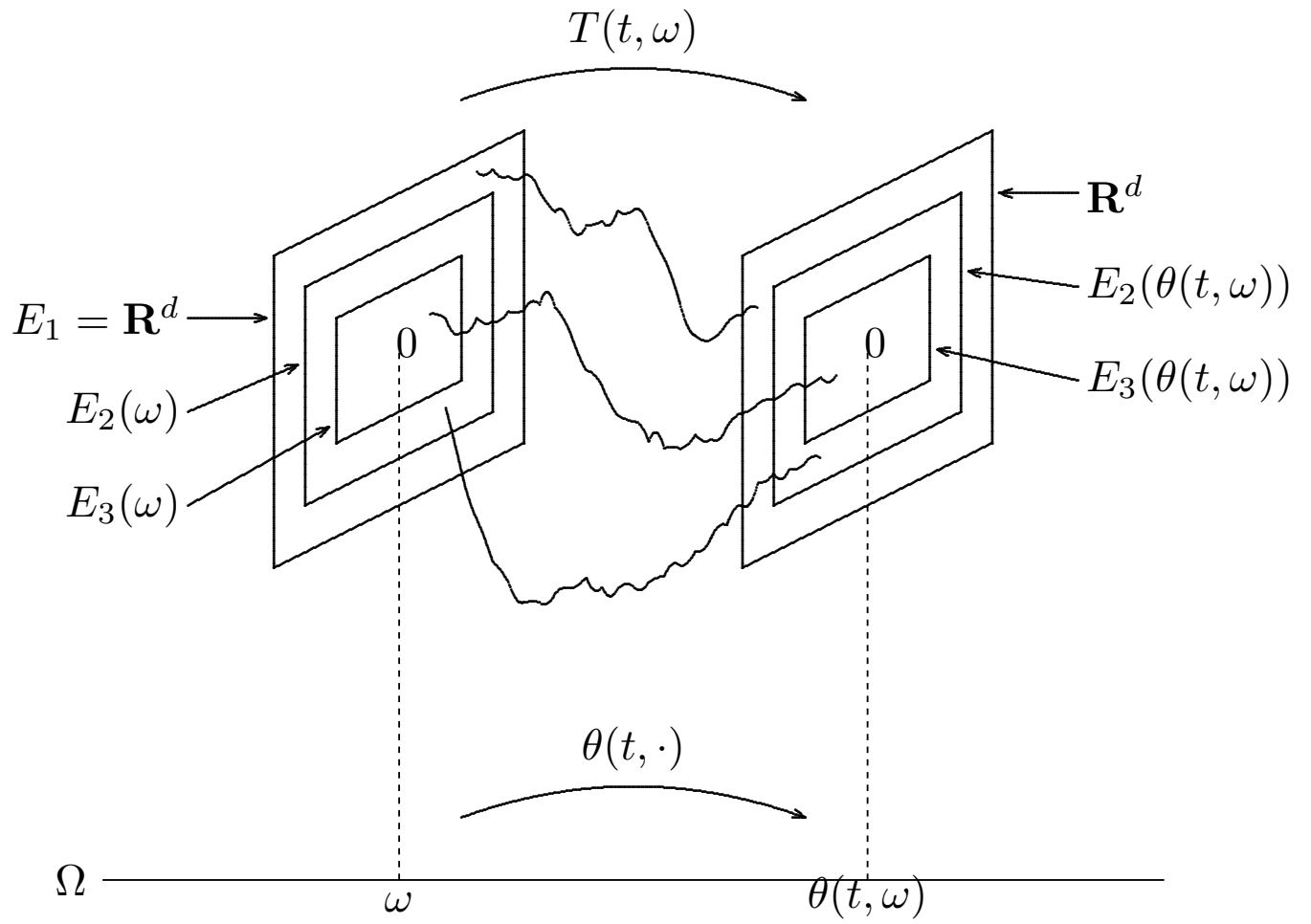
for all  $t \geq 0$ ,  $1 \leq i \leq p$ .

**Proof.**

Based on the discrete version of Oseledec's multiplicative ergodic theorem

and Lemma 1. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]), “perfect” infinite-dimensional version and application to SFDE’s. □

# Spectral Theorem



Apply Theorem 2 with

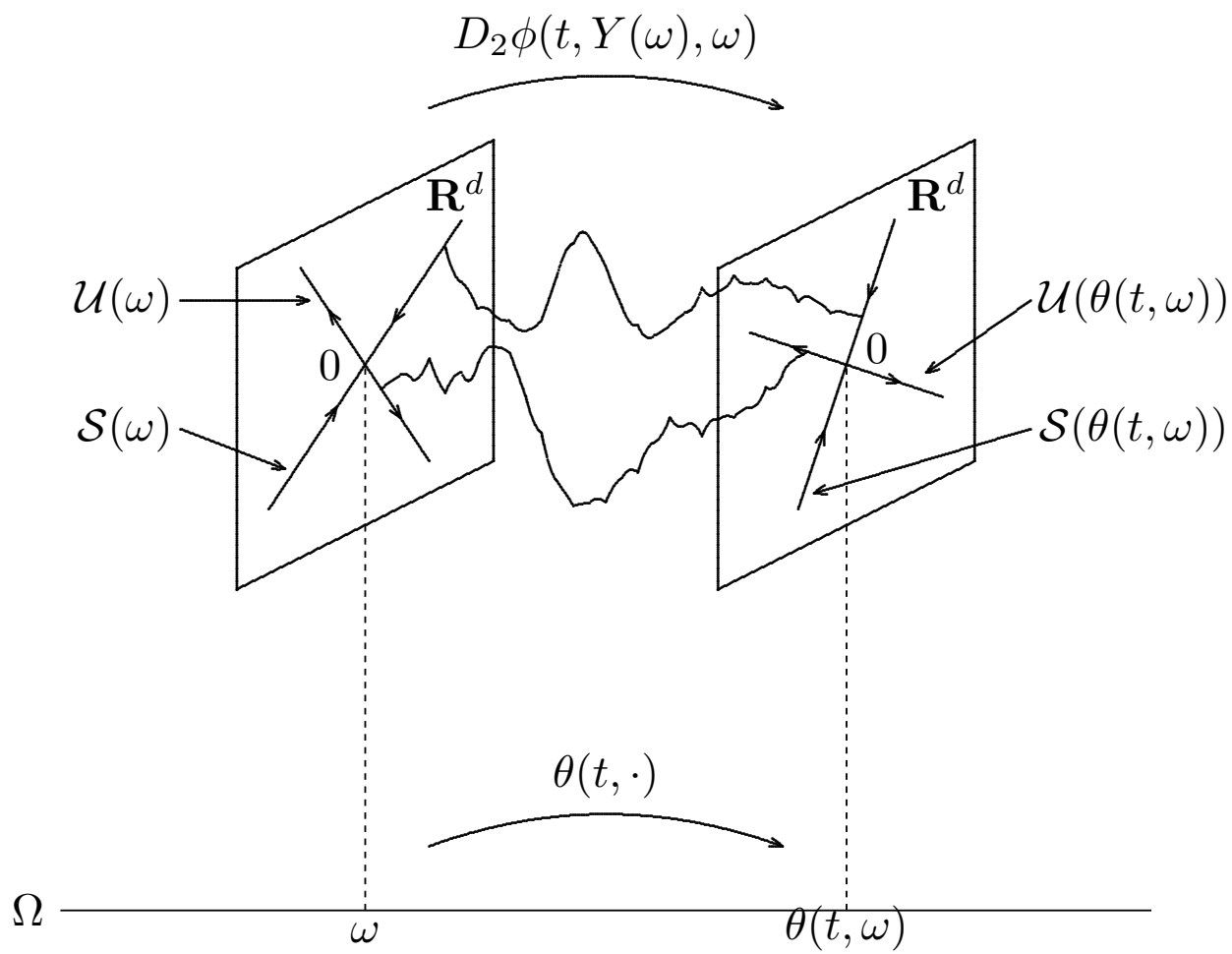
$$T(t, \omega) := D_2\phi(t, Y(\omega), \omega)$$

Then linearized cocycle has random  
invariant stable and unstable subspaces  
 $\{S(\omega), U(\omega) : \omega \in \Omega\}$ :

$$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t, \omega)),$$

$$D_2\phi(-t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(-t, \omega)), \quad t \geq 0.$$

[Mo.1].



## Estimates on the non-linear cocycle

### Theorem 3 (M. + Scheutzow [M-S.2])

There exists a jointly measurable modification of the trajectory random field of (I) (with initial condition  $x$  at  $t = s$ ), denoted by  $\{\phi_{s,t}(x) : -\infty < s, t < \infty, x \in \mathbf{R}^d\}$ , having the following properties:

The cocycle  $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \rightarrow \mathbf{R}^d$  is given by

$$\phi(t, x, \omega) := \phi_{0,t}(x, \omega), \quad x \in \mathbf{R}^d, \omega \in \Omega, t \in \mathbf{R}.$$

Then for all  $\omega \in \Omega$ ,  $\epsilon \in (0, \delta)$ ,  $\gamma, \rho, T > 0$ ,  $1 \leq |\alpha| \leq k$ ,  $\phi(t, \cdot, \omega)$  is  $C^{k,\epsilon}$ ,  $0 < \epsilon < \delta$ , and the quantities

$$\begin{aligned} & \sup_{\substack{0 \leq s, t \leq T, \\ x \in \mathbf{R}^d}} \frac{|\phi_{s,t}(x, \omega)|}{[1 + |x|(\log^+ |x|)^\gamma]}, \quad \sup_{\substack{0 \leq s, t \leq T, \\ x \in \mathbf{R}^d}} \frac{\|D_x^\alpha \phi_{s,t}(x, \omega)\|}{(1 + |x|^\gamma)}, \\ & \sup_{x \in \mathbf{R}^d} \sup_{\substack{0 \leq s, t \leq T, \\ 0 < |x' - x| \leq \rho}} \frac{\|D_x^\alpha \phi_{s,t}(x, \omega) - D_x^\alpha \phi_{s,t}(x', \omega)\|}{|x - x'|^\epsilon (1 + |x|)^\gamma}, \end{aligned}$$

are finite. The random variables defined by the above expressions have  $p$ -th moments for all  $p \geq 1$ .

# Ruelle's Non-linear Ergodic Theorem

## Theorem 4 ([Ru.1], 1979)

Let  $\Omega \ni \omega \mapsto F_\omega \in C^{k,\epsilon}(\mathbf{R}^d, 0; \mathbf{R}^d, 0)$  be measurable such that  $E \log^+ \|F_\cdot|_{\bar{B}(0,1)}\|_{k,\epsilon} < \infty$ . Set  $F^n(\omega) := F_{\theta(n-1,\omega)} \circ \dots \circ F_{\theta(1,\omega)} \circ F_\omega$ . Suppose  $\lambda < 0$  is not in the spectrum of the cocycle  $(DF_\omega^n(0), \theta(n,\omega))$ . Then there is a sure event  $\Omega_0 \in \mathcal{F}$  such that  $\theta(1,\cdot)(\Omega_0) \subseteq \Omega_0$ , and measurable functions  $0 < \alpha(\omega) < \beta(\omega) < 1, \gamma(\omega) > 1$  with the following properties:

(a) If  $\omega \in \Omega_0$ , the set

$$V_\omega^\lambda := \{x \in \bar{B}(0, \alpha(\omega)) : |F_\omega^n(x)| \leq \beta(\omega)e^{n\lambda} \text{ for all } n \geq 0\}$$

is a  $C^{k,\epsilon}$  submanifold of  $\bar{B}(0, \alpha(\omega))$ .

(b) If  $x_1, x_2 \in V_\omega^\lambda$ , then

$$|F_\omega^n(x_1) - F_\omega^n(x_2)| \leq \gamma(\omega)|x_1 - x_2|e^{n\lambda}$$

for all integers  $n \geq 0$ . If  $\lambda' < \lambda$  and  $[\lambda', \lambda]$  is disjoint from the spectrum of  $(DF_\omega^n(0), \theta(n, \omega))$ , then there exists a measurable  $\gamma'(\omega) > 1$  such that

$$|F_\omega^n(x_1) - F_\omega^n(x_2)| \leq \gamma'(\omega)|x_1 - x_2|e^{n\lambda'}$$

for all  $x_1, x_2 \in V_\omega^\lambda$  and all integers  $n \geq 0$ .

## **Proof**

[Ru.1], Theorem 5.1, p. 292.

## Construction of the Stable/Unstable Manifolds

- Use auxiliary cocycle  $(Z, \theta)$ :

$$Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega)) \quad (16)$$

for  $t \in \mathbf{R}, x \in \mathbf{R}^d, \omega \in \Omega$ . Set  $\tau := \theta(1, \cdot) : \Omega \rightarrow \Omega$ . Define maps  $F_\omega, F_\omega^n : \mathbf{R}^d \rightarrow \mathbf{R}^d$ :

$$F_\omega(x) := Z(1, x, \omega) \quad x \in \mathbf{R}^d$$

$$F_\omega^n := F_{\tau^{n-1}(\omega)} \circ \cdots \circ F_{\tau(\omega)} \circ F_\omega$$

for all  $\omega \in \Omega$ . Then cocycle property for  $Z$  gives  $F_\omega^n = Z(n, \cdot, \omega)$  for each  $n \geq 1$ .

$F_\omega$  is  $C^{k, \epsilon}$  ( $\epsilon \in (0, \delta)$ ) and  $(DF_\omega)(0) = D_2\phi(1, Y(\omega), \omega)$ .

- Integrability of the map

$$\omega \mapsto \log^+ \|D_2\phi(1, Y(\omega), \omega)\|_{L(\mathbf{R}^d)}$$

(Lemma 2) implies discrete cocycle  $((DF_\omega^n)(0), \theta(n, \omega), n \geq 0)$  has same non-random Lyapunov spectrum as that of linearized continuous cocycle

$$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0),$$

viz.  $\{\lambda_m < \dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1\}$ , where each  $\lambda_i$  has fixed multiplicity  $q_i$ ,  $1 \leq i \leq m$  (Lemma 2).

- If  $\lambda_i > 0$  for all  $1 \leq i \leq m$ , then take  $\tilde{S}(\omega) := \{Y(\omega)\}$  for all  $\omega \in \Omega$ . Theorem is trivial in this case. Hence assume there is at least one  $\lambda_i < 0$ .
- Use discrete non-linear ergodic theorem of Ruelle (Theorem 4) and its proof to obtain a sure event  $\Omega_1^* \in \mathcal{F}$

such that  $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$  for all  $t \in \mathbf{R}$ ,  $\mathcal{F}$ -measurable positive random variables  $\rho_1, \beta_1 : \Omega_1^* \rightarrow (0, \infty)$ ,  $\rho_1 < \beta_1$ , and a random family of  $C^{k, \epsilon}$  ( $\epsilon \in (0, \delta)$ ) submanifolds of  $\bar{B}(0, \rho_1(\omega))$  denoted by  $\tilde{\mathcal{S}}_d(\omega)$ ,  $\omega \in \Omega_1^*$ , and satisfying the following properties for each  $\omega \in \Omega_1^*$ :  $\tilde{\mathcal{S}}_d(\omega)$  is the set of all  $x \in \bar{B}(0, \rho_1(\omega))$  such that

$$|Z(n, x, \omega)| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}, \quad n \in \mathbf{Z}^+ \quad (21)$$

$\tilde{\mathcal{S}}_d(\omega)$  is tangent at 0 to the stable subspace  $\mathcal{S}(\omega)$  of the linearized flow  $D_2\phi$ , viz.  $T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$ . Therefore  $\dim \tilde{\mathcal{S}}_d(\omega)$  is non-random by ergodicity of  $\theta$ . Also

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \left[ \sup_{\substack{x_1 \neq x_2, \\ x_1, x_2 \in \tilde{\mathcal{S}}_d(\omega)}} \frac{|Z(n, x_1, \omega) - Z(n, x_2, \omega)|}{|x_1 - x_2|} \right] \leq \lambda_{i_0}. \quad (22)$$

The  $\theta(t, \cdot)$ -invariant sure event  $\Omega_1^* \in \mathcal{F}$  is constructed using the ideas in Ruelle's proof (of Theorem 5.1 in [Ru.1], p. 293), combined with the estimate (10) of Lemma 2 and the subadditive ergodic theorem (Lemma 1 (ii)).

- For each  $\omega \in \Omega_1^*$ , let  $\tilde{\mathcal{S}}(\omega)$  be as defined in part (a) of the theorem. Then by definition of  $\tilde{\mathcal{S}}_d(\omega)$  and  $Z$ :

$$\tilde{\mathcal{S}}(\omega) = \tilde{\mathcal{S}}_d(\omega) + Y(\omega). \quad (23)$$

Since  $\tilde{\mathcal{S}}_d(\omega)$  is a  $C^{k,\epsilon}$  ( $\epsilon \in (0, \delta)$ ) submanifold of  $\bar{B}(0, \rho_1(\omega))$ , then  $\tilde{\mathcal{S}}(\omega)$  is a  $C^{k,\epsilon}$  ( $\epsilon \in (0, \delta)$ ) submanifold of  $\bar{B}(Y(\omega), \rho_1(\omega))$ . Furthermore,  $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$ .

Hence  $\dim \tilde{\mathcal{S}}(\omega) = \dim \mathcal{S}(\omega) = \sum_{i=i_0}^m q_i$ , and is non-random.

- (22) implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z(n, x, \omega)| \leq \lambda_{i_0} \quad (24)$$

for all  $\omega$  in  $\Omega_1^*$  and all  $x \in \tilde{\mathcal{S}}_d(\omega)$ . Lemma 4 implies there is a sure event  $\Omega_2^* \subseteq \Omega_1^*$  such that  $\theta(t, \cdot)(\Omega_2^*) = \Omega_2^*$  for all  $t \in \mathbf{R}$ , and

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |Z(t, x, \omega)| \leq \lambda_{i_0} \quad (25)$$

for all  $\omega \in \Omega_2^*$  and all  $x \in \tilde{\mathcal{S}}_d(\omega)$ . Therefore (2) holds.

- To prove (b), let  $\omega \in \Omega_1^*$ . By (22), there is a positive integer  $N_0 := N_0(\omega)$  (independent of  $x \in \tilde{\mathcal{S}}_d(\omega)$ ) such that

$Z(n, x, \omega) \in \bar{B}(0, 1)$  for all  $n \geq N_0$ . Let  $\Omega_4^* := \Omega_2^* \cap \Omega_3$ , where  $\Omega_3$  is the shift-invariant sure event defined in the proof of Lemma 4. Then  $\Omega_4^*$  is a sure event and  $\theta(t, \cdot)(\Omega_4^*) = \Omega_4^*$  for all  $t \in \mathbf{R}$ . By cocycle property, Mean-Value theorem and the ergodic theorem (Lemma 1(i)), we get (b).

- To prove the invariance property (4), apply the Oseledec theorem to  $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$ . Get a sure  $\theta(t, \cdot)$ -invariant event, also denoted by  $\Omega_1^*$ , such that

$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega))$  for all  $t \geq 0$  and all  $\omega \in \Omega_1^*$ . Equality holds because  $D_2\phi(t, Y(\omega), \omega)$  is injective and  $\dim \mathcal{S}(\omega) = \dim \mathcal{S}(\theta(t, \omega))$  for all  $t \geq 0$  and all  $\omega \in \Omega_1^*$ .

- To prove the asymptotic invariance property (3), use ideas from Ruelle's Theorems 5.1 and 4.1 in [Ru.1], to pick random variables  $\rho_1, \beta_1$  and a sure event (also denoted by)  $\Omega_1^*$  such that  $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$  for all  $t \in \mathbf{R}$ , and for any  $\epsilon \in (0, \epsilon_1)$  and every  $\omega \in \Omega_1^*$ , there exists a positive  $K_1^\epsilon(\omega)$  for which the inequalities

$$\begin{aligned}\rho_1(\theta(t, \omega)) &\geq K_1^\epsilon(\omega) \rho_1(\omega) e^{(\lambda_{i_0} + \epsilon)t}, \\ \beta_1(\theta(t, \omega)) &\geq K_1^\epsilon(\omega) \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon)t}\end{aligned}\tag{26}$$

hold for all  $t \geq 0$ . Use (b) to obtain a sure event  $\Omega_5^* \subseteq \Omega_4^*$  such that  $\theta(t, \cdot)(\Omega_5^*) = \Omega_5^*$  for all  $t \in \mathbf{R}$ , and for any  $0 < \epsilon < \epsilon_1$

and  $\omega \in \Omega_4^*$ , there exists  $\beta^\epsilon(\omega) > 0$  (independent of  $x$ ) with

$$|\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t} \quad (27)$$

for all  $x \in \tilde{\mathcal{S}}(\omega)$ ,  $t \geq 0$ . Fix  $t \geq 0$ ,  $\omega \in \Omega_5^*$  and  $x \in \tilde{\mathcal{S}}(\omega)$ . Let  $n$  be a non-negative integer. Then the cocycle property and (27) imply that

$$\begin{aligned} & |\phi(n, \phi(t, x, \omega), \theta(t, \omega)) - Y(\theta(n, \theta(t, \omega)))| \\ &= |\phi(n + t, x, \omega) - Y(\theta(n + t, \omega))| \\ &\leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)(n+t)} \\ &\leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t} e^{(\lambda_{i_0} + \epsilon_1)n}. \end{aligned} \quad (28)$$

If  $\omega \in \Omega_5^*$ , then it follows from (26), (27), (28) and the definition of  $\tilde{\mathcal{S}}(\theta(t, \omega))$  that

there exists  $\tau_1(\omega) > 0$  such that  $\phi(t, x, \omega) \in \tilde{\mathcal{S}}(\theta(t, \omega))$  for all  $t \geq \tau_1(\omega)$ . This proves asymptotic invariance.

- Prove (d), the existence of the local unstable manifolds  $\tilde{\mathcal{U}}(\omega)$ , by running both the flow  $\phi$  and the shift  $\theta$  backward in time getting the cocycle  $(\tilde{Z}(t, \cdot, \omega), \tilde{\theta}(t, \omega), t \geq 0)$ :

$$\tilde{\phi}(t, x, \omega) := \phi(-t, x, \omega), \quad \tilde{Z}(t, x, \omega) := Z(-t, x, \omega),$$

$$\tilde{\theta}(t, \omega) := \theta(-t, \omega)$$

for all  $t \geq 0, \omega \in \Omega$ . The linearized flow  $(D_2\tilde{\phi}(t, Y(\omega), \omega), \tilde{\theta}(t, \omega), t \geq 0)$  is an  $L(\mathbf{R}^d)$ -valued perfect cocycle with a non-random finite Lyapunov spectrum  $\{-\lambda_1 < -\lambda_2 < \dots < -\lambda_i < -\lambda_{i+1} < \dots < -\lambda_m\}$  where  $\{\lambda_m <$

$\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$  is the Lyapunov spectrum of the forward linearized flow  $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ . Apply first part of the proof to get *stable manifolds* for the backward flow  $\tilde{\phi}$  satisfying assertions (a), (b), (c). This gives *unstable manifolds* for the original flow  $\phi$ , and (d), (e), (f) automatically hold.

- Measurability of the stable manifolds follows from the representations:

$$\tilde{\mathcal{S}}(\omega) = Y(\omega) + \tilde{\mathcal{S}}_d(\omega) \quad (29)$$

$$\tilde{\mathcal{S}}_d(\omega) = \lim_{n \rightarrow \infty} \bar{B}(0, \rho_1(\omega)) \cap \bigcap_{i=1}^n f_i(\cdot, \omega)^{-1}(\bar{B}(0, 1)) \quad (30)$$

$$f_i(x, \omega) := \beta_1(\omega)^{-1} e^{-(\lambda_{i_0} + \epsilon_1)i} Z(i, x, \omega), \quad x \in \mathbf{R}^d, \quad \omega \in \Omega_1^*,$$

for all integers  $i \geq 0$ . (Above limit is taken in the metric  $d^*$  on  $\mathcal{K}(\mathbf{R}^d)$ .) Use joint continuity of translation and measurability of  $Y$ ,  $f_i$ ,  $\rho_1$ , finite intersections and the continuity of the maps

$$\mathbf{R}^+ \ni r \mapsto \bar{B}(0, r) \in \mathcal{K}(\mathbf{R}^d).$$

$$\text{Hom}(\mathbf{R}^d) \ni f \mapsto f^{-1}(\bar{B}(0, 1)) \in \mathcal{K}(\mathbf{R}^d).$$

- For  $h, g_i$  in  $C_b^\infty$ , can adapt above argument to give a sure event in  $\mathcal{F}$ , also denoted by  $\Omega^*$  such that  $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$  are  $C^\infty$  for all  $\omega \in \Omega^*$ .  $\square$

# Some Technical Lemmas

$\|\cdot\|_{k,\epsilon} := C^{k,\epsilon}$ -norm on  $C^{k,\epsilon}$  mappings  $\bar{B}(0, \rho) \rightarrow \mathbf{R}^d$ .

## Lemma 2

*Assume that  $\log^+ |Y(\cdot)|$  is integrable. Then the cocycle  $\phi$  satisfies*

$$\int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \|\phi(t_2, Y(\theta(t_1, \omega)) + (\cdot), \theta(t_1, \omega))\|_{k,\epsilon} dP(\omega) < \infty \quad (10)$$

*for any fixed  $0 < T, \rho < \infty$  and any  $\epsilon \in (0, \delta)$ . Furthermore, the linearized flow  $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$ ,  $t \geq 0$ , is an  $L(\mathbf{R}^d)$ -valued perfect cocycle and*

$$\int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \|D_2\phi(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega))\|_{L(\mathbf{R}^d)} dP(\omega) < \infty \quad (11)$$

for any fixed  $0 < T < \infty$ . The forward cocycle  $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t > 0)$  has a non-random finite Lyapunov spectrum  $\{\lambda_m < \dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1\}$ . Each Lyapunov exponent  $\lambda_i$  has a non-random multiplicity  $q_i$ ,  $1 \leq i \leq m$ , and  $\sum_{i=1}^m q_i = d$ . The backward linearized cocycle  $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t < 0)$ , admits a “backward” non-random finite Lyapunov spectrum:

$$\lim_{t \rightarrow -\infty} \frac{1}{t} \log |D_2\phi(t, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbf{R}^d),$$

taking values in  $\{-\lambda_i\}_{i=1}^m$  with non-random multiplicities  $q_i$ ,  $1 \leq i \leq m$ , and  $\sum_{i=1}^m q_i = d$ .

## The Auxiliary Cocycle

To apply Ruelle's discrete non-linear ergodic theorem ([Ru.1], Theorem 5.1, p. 292), introduce the following auxiliary cocycle  $Z : \mathbf{R} \times \mathbf{R}^d \times \Omega \rightarrow \mathbf{R}^d$ . This a “centering” of the flow  $\phi$  about the stationary solution:

$$Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega)) \quad (16)$$

for  $t \in \mathbf{R}, x \in \mathbf{R}^d, \omega \in \Omega$ .

### Lemma 3

*$(Z, \theta)$  is a perfect cocycle on  $\mathbf{R}^d$  and  $Z(t, 0, \omega) = 0$  for all  $t \in \mathbf{R}$ , and all  $\omega \in \Omega$ .*

The proof of the local stable-manifold theorem (Theorem 1) uses a discretization argument that requires the following lemma.

**Lemma 4**

*Suppose that  $\log^+ |Y(\cdot)|$  is integrable. Then there is a sure event  $\Omega_3 \in \mathcal{F}$  with the following properties:*

- (i)  $\theta(t, \cdot)(\Omega_3) = \Omega_3$  for all  $t \in \mathbf{R}$ ,
- (ii) For every  $\omega \in \Omega_3$  and any  $x \in \mathbf{R}^d$ , the statement

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z(n, x, \omega)| < 0 \quad (17)$$

*implies*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |Z(t, x, \omega)| = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z(n, x, \omega)|. \quad (18)$$

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