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The Stable Manifold Theorem for Stochastic Differential Equations (Dynamical Systems and Probability Seminar, Loughborough University)

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THE STABLE MANIFOLD THEOREM FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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SDE's: Stable Manifolds

- Formulate a *Local Stable Manifold Theorem* for SDE's driven by Brownian motion (or general noise with stationary ergodic increments): Stratonovich or Itô type.
- Start with the existence of a stochastic flow for SDE.
- Concept of a hyperbolic stationary trajectory. The stationary trajectory is a solution of the forward/backward anticipating SDE for all time (Stratonovich case).

- Existence of a stationary random family of asymptotically invariant stable and unstable manifolds within a stationary neighborhood of the hyperbolic stationary solution.
- Stable and unstable manifolds dynamically characterized using forward and backward solutions of anticipating versions of the (Stratonovich) SDE.
- Proof based on Ruelle-Oseledec (nonlinear) multiplicative ergodic theory and anticipating stochastic calculus.

Formulation of the Theorem

Stratonovich SDE on $\mathbf{R}^d$

$$dx(t) = h(x(t))\,dt + \sum_{i=1}^{m} g_i(x(t)) \circ dW_i(t), \qquad (I)$$

driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$.

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}}, P) :=$ canonical filtered Wiener space.

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R} \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F} :=$ Borel σ-field of Ω;

$\mathcal{F}_t :=$ sub-σ-field of $\mathcal{F}$ generated by the evaluations $\omega \to \omega(u), \quad u \le t, \qquad t \in \mathbf{R}$.

$P :=$ Wiener measure on Ω.

$h : \mathbf{R}^d \to \mathbf{R}^d, 1 \le i \le m,$ $C_b^{k,\delta}$ vector fields on $\mathbf{R}^d$; viz. h has all derivatives $D^j h, 1 \le j \le k$, continuous and globally bounded, $D^k h$ Hölder continuous with exponent $\delta \in (0,1)$.

$g_i,\ 1 \le i \le m$, globally bounded and $C_b^{k+1,\delta}$.

$\theta : \mathbf{R} \times \Omega \to \Omega$ is the (ergodic) Brownian shift

$$\theta(t,\omega)(s) := \omega(t+s) - \omega(t), \qquad t, s \in \mathbf{R},\ \omega \in \Omega.$$

Let $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ be the stochastic flow generated by (I) $\big(\phi(t,\cdot,\omega) = [\phi(-t,\cdot,\theta(t,\omega))]^{-1}$,

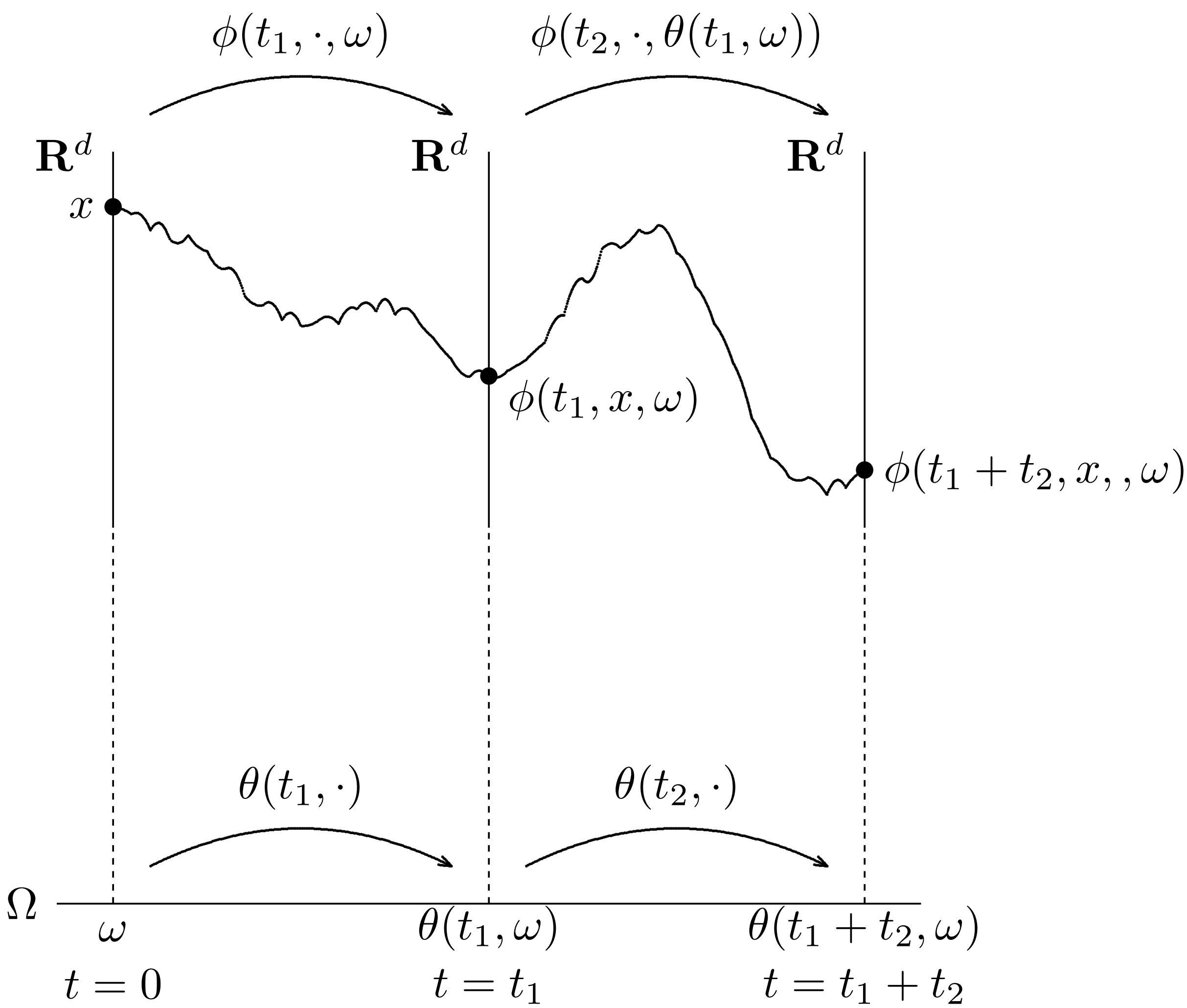
$t < 0\big)$. Then ϕ is a perfect *cocycle:*

$$\phi(t_1 + t_2, \cdot, \omega) = \phi(t_2, \cdot, \theta(t_1,\omega)) \circ \phi(t_1, \cdot, \omega),$$

for all $t_1, t_2 \in \mathbf{R}$ and all $\omega \in \Omega$ ([I-W], [A-S], [A]).

Figure illustrates the cocycle property. Vertical solid lines represent random fibers consisting of copies of $\mathbf{R}^d$. (ϕ, θ) is a "random vector-bundle morphism" over the "base" probability space Ω.

The Cocycle


$\phi(t_1, \cdot, \omega)$
$\phi(t_2, \cdot, \theta(t_1, \omega))$
$\mathbf{R}^d$
$\mathbf{R}^d$
$\mathbf{R}^d$
x
$\phi(t_1, x, \omega)$
$\phi(t_1 + t_2, x, , \omega)$
$\theta(t_1, \cdot)$
$\theta(t_2, \cdot)$
Ω
ω
$\theta(t_1, \omega)$
$\theta(t_1 + t_2, \omega)$
$t = 0$
$t = t_1$
$t = t_1 + t_2$

Definition

The SDE (I) has a *stationary trajectory* if there exists an $\mathcal{F}$-measurable random variable $Y : \Omega \to \mathbf{R}^d$ such that

$$\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{1}$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $\phi(t, Y) = Y(\theta(t))$.

Examples of Stationary Solutions

1. Fixed points:

$$d\phi(t) = h(\phi(t))\,dt + \sum_{i=1}^{m} g_i(\phi(t)) \circ dW_i(t)$$

$$h(x_0) = g_i(x_0) = 0, \qquad 1 \le i \le m$$

Take $Y(\omega) = x_0$ for all $\omega \in \Omega$.

2. Linear affine case $d = 1$:

$$d\phi(t) = \lambda\phi(t)\,dt + dW(t)$$

$\lambda > 0$ fixed, $W(t) \in \mathbf{R}$. Take

$$\phi(t, x, \omega) = e^{\lambda t}\left[x + \int_0^t e^{-\lambda u}\,dW(u)\right],$$

$$Y(\omega) := -\int_0^{\infty} e^{-\lambda u}\,dW(u),$$

$$\theta(t, \omega)(s) = \omega(t + s) - \omega(t).$$

Check that $\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega))$, using integration by parts and variation of parameters.

3. Affine linear SDE in $d = 2$:

$$d\phi(t) = A\phi(t)\,dt + GdW(t)$$

with A a fixed hyperbolic 2×2-diagonal matrix; G a constant 2×2-matrix, and W 2-dimensional Brownian motion.

4. Non-linear transforms of (3) under a global diffeomorphism.

5. Invariant measure for SDE: Enlarge probability space ([M-S.3]).

Let $\phi(t, Y)$ be a stationary solution of (I). Cocycle property of ϕ implies that the linearization

$$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$$

along the stationary solution is also a $d \times d$-matrix-valued cocycle. Using Kolmogorov's theorem, the random variables

$$\sup_{x \in \mathbf{R}^d} \frac{\|D_2\phi(t, x)\|}{(1 + |x|^\gamma)}, \ \gamma > 0,$$

have moments of all orders. If $E \log^+ |Y| < \infty$, then $E \log^+ \|D_2\phi(1, Y)\| < \infty$. Apply Oseledec's Theorem to get a *non-random* finite Lyapunov spectrum:

$$\lim_{n \to \infty} \frac{1}{n} \log |D_2\phi(n, Y(\omega), \omega)(v(\omega))|, \qquad v \in L^0(\Omega, \mathbf{R}^d).$$

Spectrum takes finitely many *fixed* values $\{\lambda_i\}_{i=1}^p$ with non-random multiplicities q_i, $1 \leq i \leq p$, and $\sum_{i=1}^{p} q_i = d$ ([Ru.1], Theorem I.6).

Definition

Stationary trajectory $\phi(t, Y)$ of (I) is *hyperbolic* if $E \log^+ |Y(\cdot)| < \infty$, and if the linearized cocycle $(D_2\phi(n, Y(\omega), \omega), \theta(n, \omega))$ has a non-vanishing Lyapunov spectrum

$$\{\lambda_p < \cdots < \lambda_{i_0+1} < \lambda_{i_0} < 0 < \lambda_{i_0-1} < \cdots < \lambda_2 < \lambda_1\}$$

i.e. $\lambda_i \neq 0$ for all $1 \leq i \leq p$.

Define $\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all $\lambda_i > 0$, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized flow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Let $\rho \in \mathbf{R}^+$, $x \in \mathbf{R}^d$.

$B(x, \rho) :=$ open ball in $\mathbf{R}^d$, center x and radius ρ;

$\bar{B}(x, \rho) :=$ corresponding closed ball;

$\mathcal{K}(\mathbf{R}^d) :=$ the class of all non-empty compact subsets of $\mathbf{R}^d$ with Hausdorff metric d^*:

$d^*(A_1, A_2) := \sup\{d(x, A_1) : x \in A_2\} \vee \sup\{d(y, A_2) : y \in A_1\}$ where $A_1, A_2 \in \mathcal{K}(\mathbf{R}^d)$;

$d(x, A_i) := \inf\{|x - y| : y \in A_i\}$, $x \in \mathbf{R}^d$, $i = 1, 2$;

$\mathcal{B}(\mathcal{K}(\mathbf{R}^d)) :=$ Borel σ-algebra on $\mathcal{K}(\mathbf{R}^d)$ with respect to the metric d^*.

Theorem 1 (The Stable Manifold Theorem) (M.+ Scheutzow, AOP '99)

Assume that the coefficients of SDE (I) satisfy the given hypotheses. Suppose $\phi(t, Y)$ is a hyperbolic stationary trajectory of (I) with $E \log^+ |Y| < \infty$.

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0 - 1})$. Then there exist

(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,

(ii) $\mathcal{F}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k,\epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) $\tilde{\mathcal{S}}(\omega)$ *is the set of all* $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ *such that*

$$|\phi(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega)\, e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers $n \geq 0$. *Furthermore,*

$$\limsup_{t\to\infty} \frac{1}{t} \log |\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0} \qquad (2)$$

for all $x \in \tilde{\mathcal{S}}(\omega)$. *Each stable subspace* $\mathcal{S}(\omega)$ *of the linearized flow* $D_2\phi$ *is tangent at* $Y(\omega)$ *to the submanifold* $\tilde{\mathcal{S}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. *In particular, dim* $\tilde{\mathcal{S}}(\omega) =$ *dim* $\mathcal{S}(\omega)$ *and is non-random.*

(b) $\limsup_{t\to\infty} \frac{1}{t} \log \left[\sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{\mathcal{S}}(\omega)}} \left\{ \frac{|\phi(t, x_1, \omega) - \phi(t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq \lambda_{i_0}$.

(c) *(Cocycle-invariance of the stable manifolds):*

There exists $\tau_1(\omega) \geq 0$ *such that*

$$\phi(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega)), \quad t \geq \tau_1(\omega). \qquad (3)$$

Also

$$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t, \omega)), \quad t \geq 0. \tag{4}$$

(d) $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $x \in \bar{B}(Y(\omega), \rho_2(\omega))$ *with the property that*

$$|\phi(-n, x, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega)\, e^{(-\lambda_{i_0-1}+\epsilon_2)n} \tag{5}$$

for all integers $n \geq 0$*. Also*

$$\limsup_{t\to\infty} \frac{1}{t} \log |\phi(-t, x, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1}. \tag{6}$$

for all $x \in \tilde{\mathcal{U}}(\omega)$*. Furthermore, the unstable subspace* $\mathcal{U}(\omega)$ *of* $D_2\phi$ *is the tangent space to* $\tilde{\mathcal{U}}(\omega)$ *at* $Y(\omega)$*, viz.* $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$*. In particular,* $\dim \tilde{\mathcal{U}}(\omega) = \dim \mathcal{U}(\omega)$ *and is non-random.*

(e) $\limsup_{t\to\infty} \frac{1}{t} \log\left[\sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{\mathcal{U}}(\omega)}} \left\{ \frac{|\phi(-t, x_1, \omega) - \phi(-t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq$

$-\lambda_{i_0-1}$.

(f) *(Cocycle-invariance of the unstable manifolds):*

There exists $\tau_2(\omega) \geq 0$ *such that*

$$\phi(-t, \cdot, \omega)(\tilde{\mathcal{U}}(\omega)) \subseteq \tilde{\mathcal{U}}(\theta(-t, \omega)), \quad t \geq \tau_2(\omega). \tag{7}$$

Also

$$D_2\phi(-t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(-t, \omega)), \quad t \geq 0. \tag{8}$$

(g) *The submanifolds* $\tilde{\mathcal{U}}(\omega)$ *and* $\tilde{\mathcal{S}}(\omega)$ *are transversal, viz.*

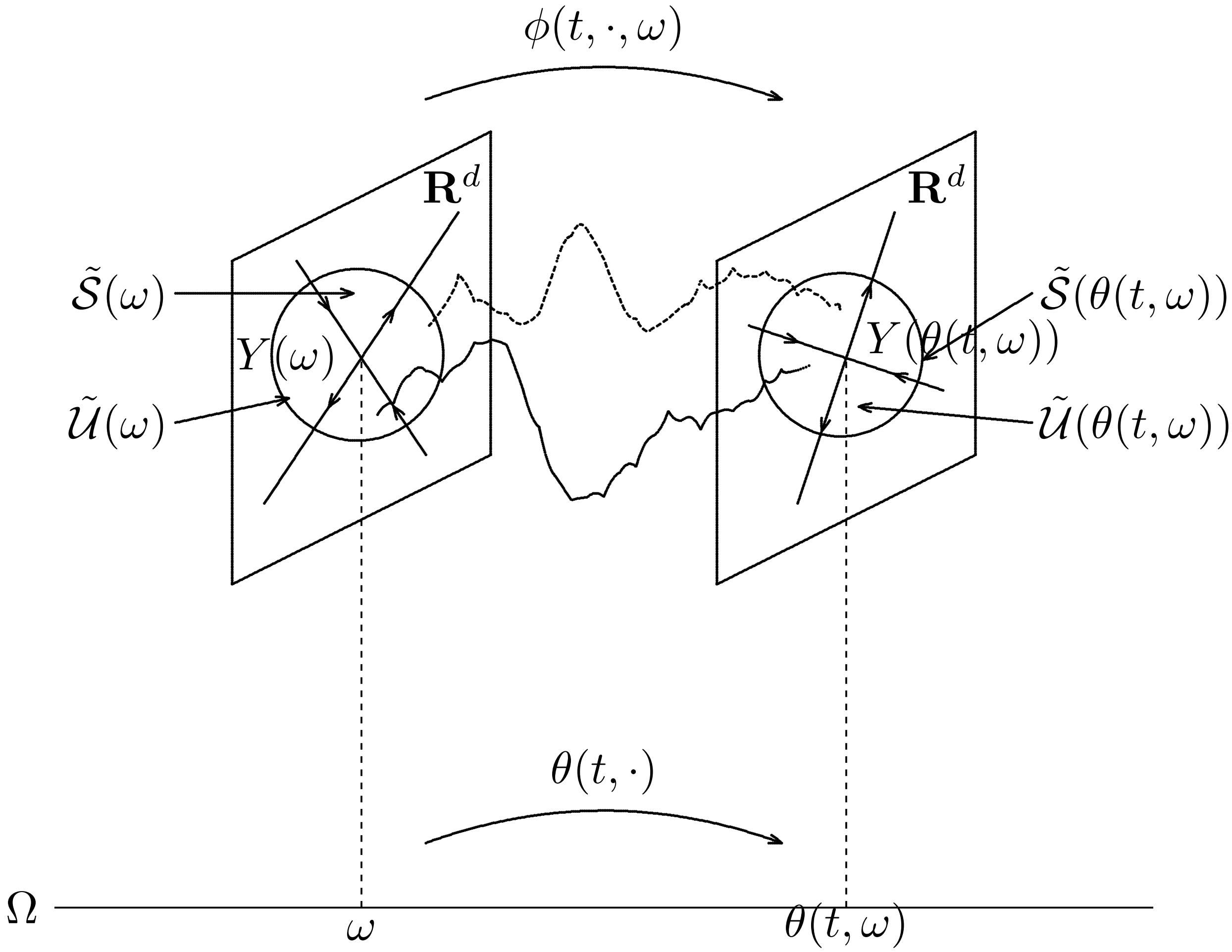
$$\mathbf{R}^d = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega). \tag{9}$$

(h) *The mappings*

$$\begin{array}{ll} \Omega \to \mathcal{K}(\mathbf{R}^d), & \Omega \to \mathcal{K}(\mathbf{R}^d), \\ \omega \mapsto \tilde{\mathcal{S}}(\omega) & \omega \mapsto \tilde{\mathcal{U}}(\omega) \end{array}$$

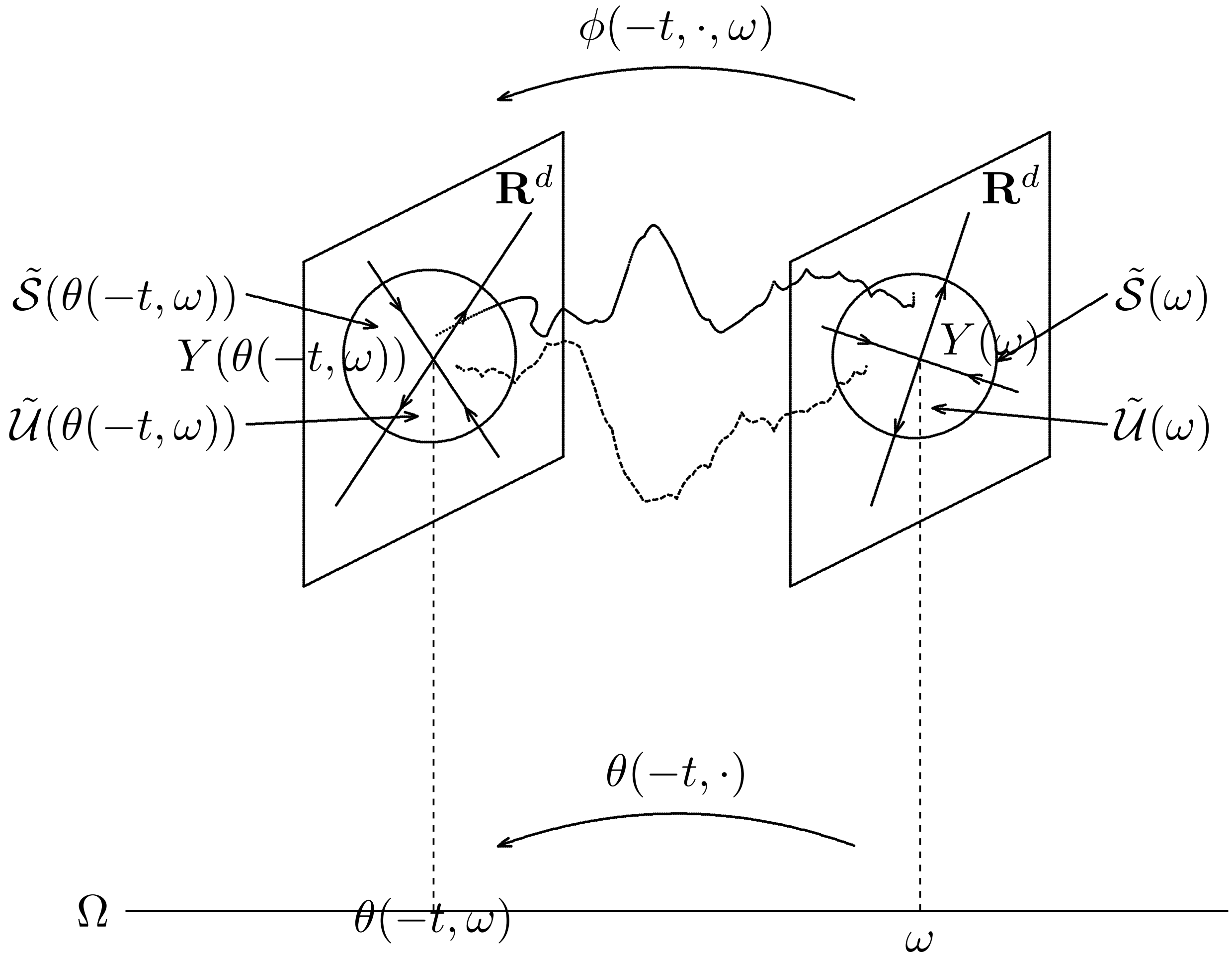
are $(\mathcal{F}, \mathcal{B}(\mathcal{K}(\mathbf{R}^d)))$*-measurable.*

Assume, further, that $h, g_i, 1 \leq i \leq m$*, are* C_b^∞*. Then the local stable and unstable manifolds* $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ *are* C^∞*.*



$$t > \tau_1(\omega)$$

A picture is worth a 1000 words!


$\phi(-t,\cdot,\omega)$
$\mathbf{R}^d$
$\mathbf{R}^d$
$\tilde{\mathcal{S}}(\theta(-t,\omega))$
$Y(\theta(-t,\omega))$
$\tilde{\mathcal{U}}(\theta(-t,\omega))$
$\tilde{\mathcal{S}}(\omega)$
$Y(\omega)$
$\tilde{\mathcal{U}}(\omega)$
$\theta(-t,\cdot)$
Ω
$\theta(-t,\omega)$
ω

$t > \tau_2(\omega)$

Sketch of Proof

Linearization and Substitution

Assume regularity conditions on the coefficients h, g_i. By the *Substitution Rule*, $\phi(t, Y(\omega), \omega)$ is a stationary *solution* of the *anticipating* Stratonovich SDE

$$\left.\begin{aligned} d\phi(t,Y) &= h(\phi(t,Y))\,dt + \sum_{i=1}^{m} g_i(\phi(t,Y)) \circ dW_i(t), \qquad t > 0 \\ \phi(0,Y) &= Y. \end{aligned}\right\} \quad (II)$$

([N-P]).

Linearize the SDE (I) along the stationary trajectory. By substitution, match the solution of the linearized equation with the linearized cocycle $D_2\phi(t, Y(\omega), \omega)$. Hence $D_2\phi(t, Y(\omega), \omega)$, $t \geq 0$, solves the SDE:

$$\left.\begin{aligned} dD_2\phi(t,Y) &= Dh(\phi(t,Y))D_2\phi(t,Y)\,dt \\ &\quad + \sum_{i=1}^{m} Dg_i(\phi(t,Y))D_2\phi(t,Y)\circ dW_i(t), \qquad t>0 \\ D_2\phi(0,Y) &= I. \end{aligned}\right\} \tag{III}$$

D_2, D denotes spatial (Fréchet) derivatives.

Similarly, the backward trajectories

$$\phi(t,Y),\ D_2\phi(t,Y),\ \ t<0,$$

solve the corresponding backward Stratonovich SDE's:

$$\left.\begin{aligned} d\phi(t,Y) &= -h(\phi(t,Y))\,dt - \sum_{i=1}^{m} g_i(\phi(t,Y))\circ \hat{d}W_i(t), \qquad t<0 \\ \phi(0,Y) &= Y. \end{aligned}\right\} \tag{II^-}$$

$$\left.\begin{aligned} dD_2\phi(t,Y) &= -Dh(\phi(t,Y))D_2\phi(t,Y)\,dt \\ &\qquad - \sum_{i=1}^{m} Dg_i(\phi(t,Y))D_2\phi(t,Y)\circ \hat{d}W_i(t), \qquad t<0 \\ D_2\phi(0,Y) &= I. \end{aligned}\right\} \qquad (III^-)$$

Above SDE's (II)-(III)⁻ give dynamic characterizations of the stable and unstable manifolds.

The following lemma is used to construct the shift-invariant sure event appearing in the statement of the local stable manifold theorem. Gives "perfect versions" of the ergodic theorem and Kingman's subadditive ergodic theorem.

Lemma 1

(i) Let $h : \Omega \to \mathbf{R}^+$ *be* $\mathcal{F}$*-measurable and such that*

$$\int_\Omega \sup_{0 \le u \le 1} h(\theta(u,\omega))\, dP(\omega) < \infty.$$

Then there is a sure event $\Omega_1 \in \mathcal{F}$ *such that* $\theta(t,\cdot)(\Omega_1) = \Omega_1$ *for all* $t \in \mathbf{R}$*, and*

$$\lim_{t \to \infty} \frac{1}{t} h(\theta(t,\omega)) = 0$$

for **all** $\omega \in \Omega_1$.

(ii) Suppose $f : \mathbf{R}^+ \times \Omega \to \mathbf{R} \cup \{-\infty\}$ *is a measurable process on* $(\Omega, \mathcal{F}, P)$ *satisfying the following conditions*

(a) $E \sup_{0 \le u \le 1} f^+(u) < \infty, \quad E \sup_{0 \le u \le 1} f^+(1-u, \theta(u)) < \infty$

(b) $f(t_1+t_2,\omega) \le f(t_1,\omega) + f(t_2, \theta(t_1,\omega))$ *for all* $t_1, t_2 \ge 0$ *and* **all** $\omega \in \Omega$.

Then there is sure event $\Omega_2 \in \mathcal{F}$ *such that* $\theta(t,\cdot)(\Omega_2) = \Omega_2$ *for all* $t \in \mathbf{R}$, *and a fixed number* $f^* \in \mathbf{R} \cup \{-\infty\}$ *such that*

$$\lim_{t\to\infty} \frac{1}{t} f(t,\omega) = f^*$$

for **all** $\omega \in \Omega_2$.

Proof

[Mo.1], Lemma 7. □

Theorem 2 ([O], 1968)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\theta : \mathbf{R}^+ \times \Omega \to \Omega$ a measurable family of ergodic P-preserving transformations. Let $T : \mathbf{R}^+ \times \Omega \to L(\mathbf{R}^d)$ be measurable, such that (T, θ) is an $L(\mathbf{R}^d)$-valued cocycle. Suppose that

$$E \sup_{0 \le t \le 1} \log^+ \|T(t, \cdot)\| < \infty, \quad E \sup_{0 \le t \le 1} \log^+ \|T(1-t, \theta(t, \cdot))\| < \infty.$$

Then there is a set $\Omega_0 \in \mathcal{F}$ of full P-measure such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \in \mathbf{R}^+$, and for each $\omega \in \Omega_0$, the limit

$$\lim_{t \to \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. Each $\Lambda(\omega)$ has a discrete non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_p}$$

where the λ_i's are distinct. Each e^{λ_i} has an eigen-space $F_i(\omega)$ and a fixed non-random multiplicity $m_i := dim F_i(\omega)$. Define

$$E_1(\omega) := \mathbf{R}^d, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \ 1 < i \le p.$$

Then

$$E_p(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = \mathbf{R}^d$$

$$\lim_{t\to\infty} \frac{1}{t} \log \|T(t,\omega)x\| = \lambda_i(\omega), \quad \textit{if} \quad x \in E_i(\omega) \backslash E_{i+1}(\omega),$$

and

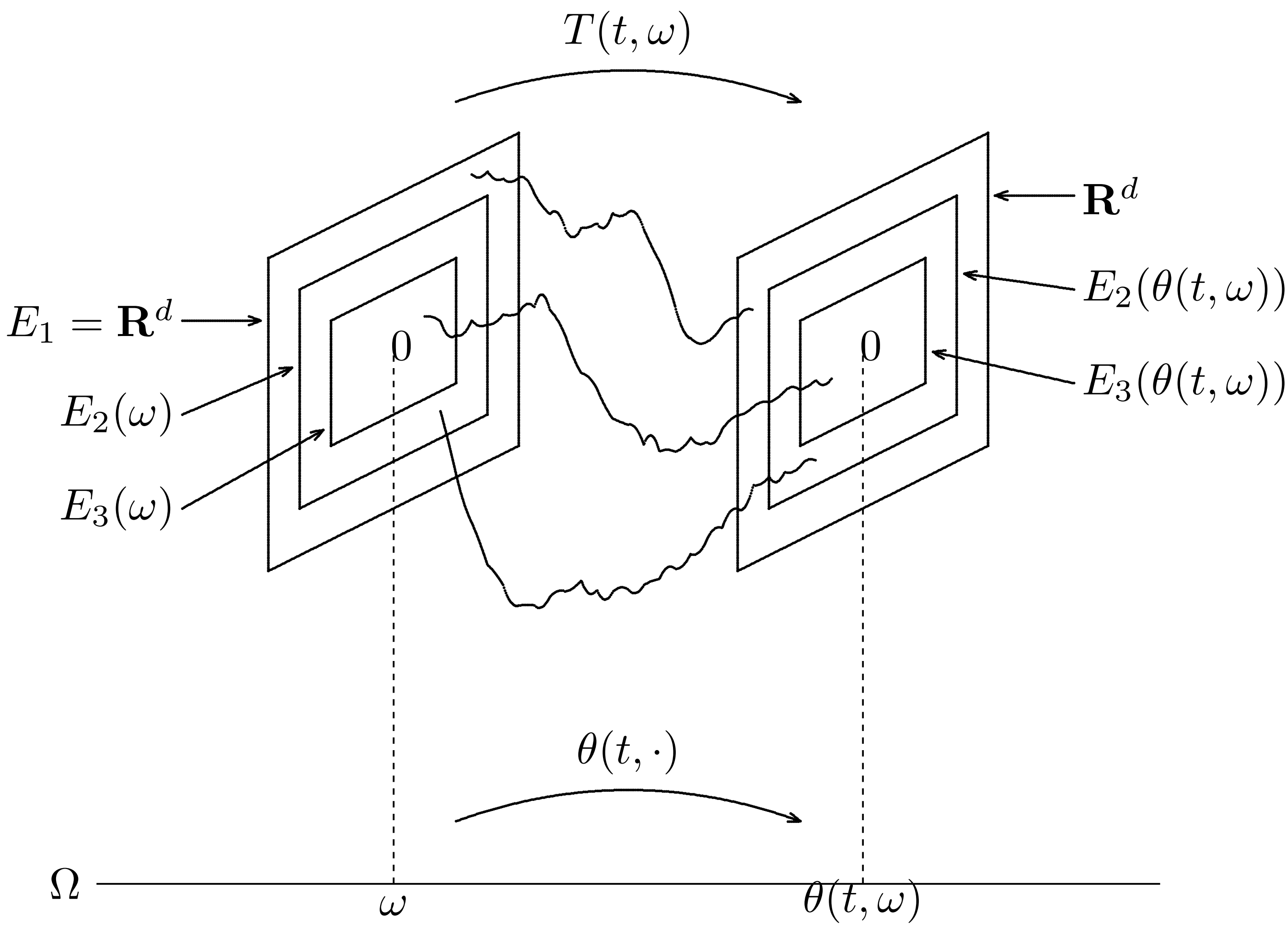
$$T(t,\omega)(E_i(\omega)) \subseteq E_i(\theta(t,\omega))$$

for all $t \ge 0$, $1 \le i \le p$.

Proof.

Based on the discrete version of Oseledec's multiplicative ergodic theorem and Lemma 1. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]), "perfect" infinite-dimensional version and application to SFDE's. $\square$

Spectral Theorem



Apply Theorem 2 with

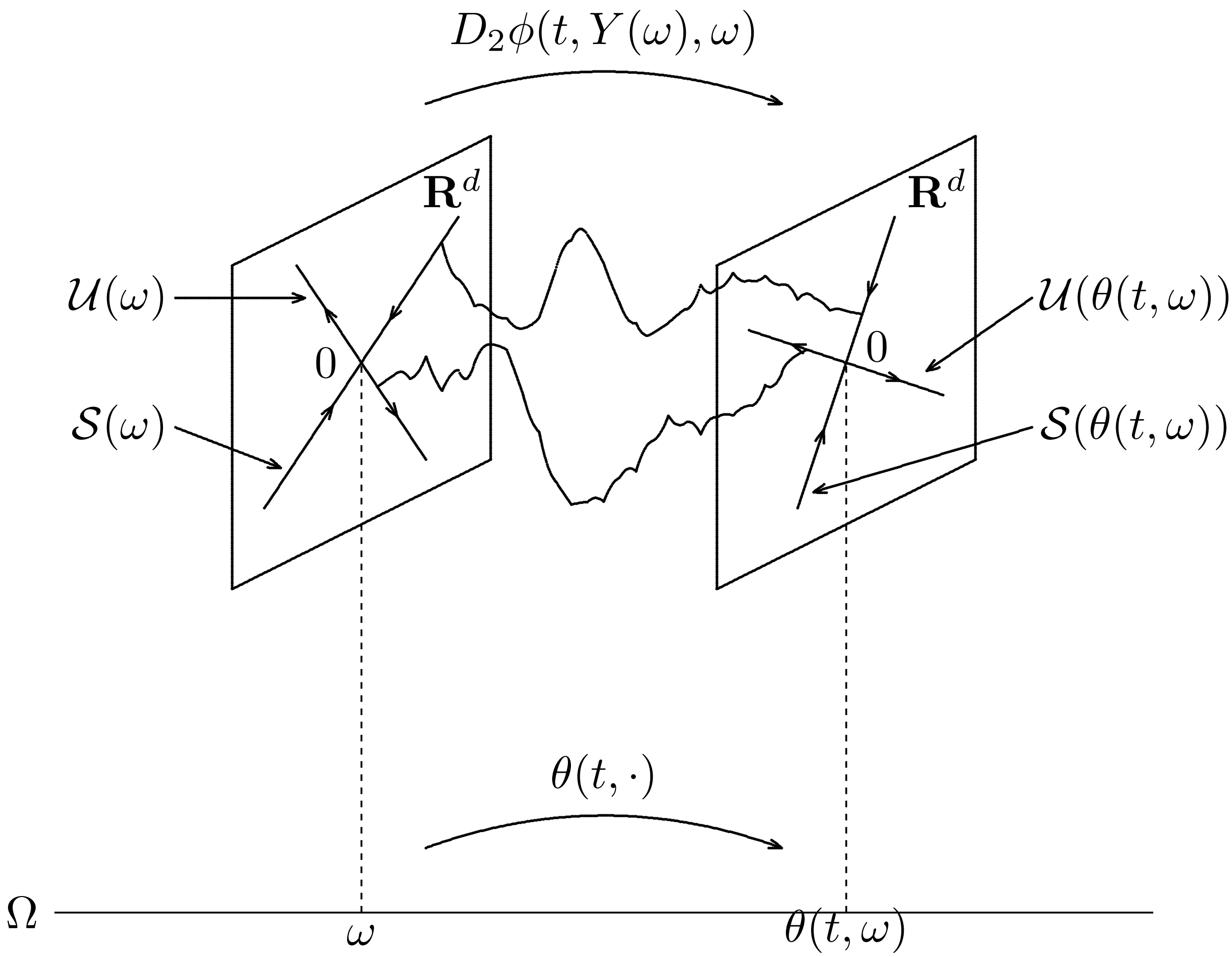
$$T(t,\omega) := D_2\phi(t, Y(\omega), \omega)$$

Then linearized cocycle has random invariant stable and unstable subspaces $\{S(\omega), U(\omega) : \omega \in \Omega\}$:

$$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t,\omega)),$$

$$D_2\phi(-t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(-t,\omega)), \qquad t \geq 0.$$

[Mo.1].


$D_2\phi(t, Y(\omega), \omega)$
$\mathbf{R}^d$
$\mathbf{R}^d$
$\mathcal{U}(\omega)$
$\mathcal{U}(\theta(t,\omega))$
0
0
$\mathcal{S}(\omega)$
$\mathcal{S}(\theta(t,\omega))$
$\theta(t, \cdot)$
Ω
ω
$\theta(t,\omega)$

Estimates on the non-linear cocycle

Theorem 3 **(M. + Scheutzow [M-S.2])**

There exists a jointly measurable modification of the trajectory random field of (I) (with initial condition x *at* $t = s$*), denoted by* $\{\phi_{s,t}(x) : -\infty < s, t < \infty,\ x \in \mathbf{R}^d\}$*, having the following properties:*

The cocycle $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ *is given by*

$$\phi(t, x, \omega) := \phi_{0,t}(x, \omega), \quad x \in \mathbf{R}^d, \omega \in \Omega, t \in \mathbf{R}.$$

Then for all $\omega \in \Omega$, $\epsilon \in (0, \delta), \gamma, \rho, T > 0, 1 \leq |\alpha| \leq k$, $\phi(t, \cdot, \omega)$ *is* $C^{k,\epsilon}, 0 < \epsilon < \delta$*, and the quantities*

$$\sup_{\substack{0 \leq s,t \leq T, \\ x \in \mathbf{R}^d}} \frac{|\phi_{s,t}(x, \omega)|}{[1 + |x|(\log^+ |x|)^\gamma]}, \qquad \sup_{\substack{0 \leq s,t \leq T, \\ x \in \mathbf{R}^d}} \frac{\|D_x^\alpha \phi_{s,t}(x, \omega)\|}{(1 + |x|^\gamma)},$$

$$\sup_{x \in \mathbf{R}^d} \sup_{\substack{0 \leq s,t \leq T, \\ 0 < |x' - x| \leq \rho}} \frac{\|D_x^\alpha \phi_{s,t}(x, \omega) - D_x^\alpha \phi_{s,t}(x', \omega)\|}{|x - x'|^\epsilon (1 + |x|)^\gamma},$$

are finite. The random variables defined by the above expressions have p-th moments for all $p \geq 1$.

Ruelle's Non-linear Ergodic Theorem

Theorem 4 ([Ru.1], 1979)

Let $\Omega \ni \mapsto F_\omega \in C^{k,\epsilon}(\mathbf{R}^d, 0; \mathbf{R}^d, 0)$ *be measurable such that* $E \log^+ \|F_\cdot|\bar{B}(0,1)\|_{k,\epsilon} < \infty$. *Set* $F^n(\omega) := F_{\theta(n-1,\omega)} \circ \cdots \circ F_{\theta(1,\omega)} \circ F_\omega$. *Suppose* $\lambda < 0$ *is not in the spectrum of the cocycle* $(DF_\omega^n(0), \theta(n,\omega))$. *Then there is a sure event* $\Omega_0 \in \mathcal{F}$ *such that* $\theta(1,\cdot)(\Omega_0) \subseteq \Omega_0$, *and measurable functions* $0 < \alpha(\omega) < \beta(\omega) < 1, \gamma(\omega) > 1$ *with the following properties:*

(a) *If* $\omega \in \Omega_0$, *the set*

$$V_\omega^\lambda := \{x \in \bar{B}(0, \alpha(\omega)) : |F_\omega^n(x)| \leq \beta(\omega) e^{n\lambda} \text{ for all } n \geq 0\}$$

is a $C^{k,\epsilon}$ *submanifold of* $\bar{B}(0, \alpha(\omega))$.

(b) *If* $x_1, x_2 \in V_\omega^\lambda$, *then*

$$|F_\omega^n(x_1) - F_\omega^n(x_2)| \leq \gamma(\omega)|x_1 - x_2| e^{n\lambda}$$

for all integers $n \geq 0$. *If* $\lambda' < \lambda$ *and* $[\lambda', \lambda]$ *is disjoint from the spectrum of* $(DF^n_\omega(0), \theta(n, \omega))$, *then there exists a measurable* $\gamma'(\omega) > 1$ *such that*

$$|F^n_\omega(x_1) - F^n_\omega(x_2)| \leq \gamma'(\omega)|x_1 - x_2|e^{n\lambda'}$$

for all $x_1, x_2 \in V^\lambda_\omega$ *and all integers* $n \geq 0$.

Proof

[Ru.1], Theorem 5.1, p. 292.

Construction of the Stable/Unstable Manifolds

- Use auxiliary cocycle (Z, θ):

$$Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega)) \quad (16)$$

for $t \in \mathbf{R}, x \in \mathbf{R}^d, \omega \in \Omega$. Set $\tau := \theta(1, \cdot) : \Omega \to \Omega$. Define maps $F_\omega, F_\omega^n : \mathbf{R}^d \to \mathbf{R}^d$:

$$F_\omega(x) := Z(1, x, \omega) \qquad x \in \mathbf{R}^d$$

$$F_\omega^n := F_{\tau^{n-1}(\omega)} \circ \cdots \circ F_{\tau(\omega)} \circ F_\omega$$

for all $\omega \in \Omega$. Then cocycle property for Z gives $F_\omega^n = Z(n, \cdot, \omega)$ for each $n \geq 1$. F_ω is $C^{k,\epsilon}$ $(\epsilon \in (0, \delta))$ and $(DF_\omega)(0) = D_2\phi(1, Y(\omega), \omega)$.

- Integrability of the map

$$\omega \mapsto \log^+ \|D_2\phi(1, Y(\omega), \omega)\|_{L(\mathbf{R}^d)}$$

(Lemma 2) implies discrete cocycle $((DF^n_\omega)(0), \theta(n,\omega), n \geq 0)$ has same non-random Lyapunov spectrum as that of linearized continuous cocycle

$$(D_2\phi(t, Y(\omega), \omega), \theta(t,\omega), t \geq 0),$$

viz. $\{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, where each λ_i has fixed multiplicity q_i, $1 \leq i \leq m$ (Lemma 2).

- If $\lambda_i > 0$ for all $1 \leq i \leq m$, then take $\tilde{S}(\omega) := \{Y(\omega)\}$ for all $\omega \in \Omega$. Theorem is trivial in this case. Hence assume there is at least one $\lambda_i < 0$.
- Use discrete non-linear ergodic theorem of Ruelle (Theorem 4) and its proof to obtain a sure event $\Omega_1^* \in \mathcal{F}$

such that $\theta(t,\cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$, $\mathcal{F}$-measurable positive random variables $\rho_1, \beta_1 : \Omega_1^* \to (0,\infty)$, $\rho_1 < \beta_1$, and a random family of $C^{k,\epsilon}$ $\big(\epsilon \in (0,\delta)\big)$ submanifolds of $\bar{B}(0,\rho_1(\omega))$ denoted by $\tilde{\mathcal{S}}_d(\omega)$, $\omega \in \Omega_1^*$, and satisfying the following properties for each $\omega \in \Omega_1^*$: $\tilde{\mathcal{S}}_d(\omega)$ is the set of all $x \in \bar{B}(0,\rho_1(\omega))$ such that

$$|Z(n,x,\omega)| \leq \beta_1(\omega) e^{(\lambda_{i_0}+\epsilon_1)n}, \qquad n \in \mathbf{Z}^+ \tag{21}$$

$\tilde{\mathcal{S}}_d(\omega)$ is tangent at 0 to the stable subspace $\mathcal{S}(\omega)$ of the linearized flow $D_2\phi$, viz. $T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$. Therefore $\dim \tilde{\mathcal{S}}_d(\omega)$ is non-random by ergodicity of θ. Also

$$\limsup_{n\to\infty} \frac{1}{n} \log \left[\sup_{\substack{x_1 \neq x_2, \\ x_1, x_2 \in \tilde{\mathcal{S}}_d(\omega)}} \frac{|Z(n,x_1,\omega) - Z(n,x_2,\omega)|}{|x_1 - x_2|} \right] \leq \lambda_{i_0}. \tag{22}$$

The $\theta(t,\cdot)$-invariant sure event $\Omega_1^* \in \mathcal{F}$ is constructed using the ideas in Ruelle's proof (of Theorem 5.1 in [Ru.1], p. 293), combined with the estimate (10) of Lemma 2 and the subadditive ergodic theorem (Lemma 1 (ii)).

- For each $\omega \in \Omega_1^*$, let $\tilde{\mathcal{S}}(\omega)$ be as defined in part (a) of the theorem. Then by definition of $\tilde{\mathcal{S}}_d(\omega)$ and Z:

$$\tilde{\mathcal{S}}(\omega) = \tilde{\mathcal{S}}_d(\omega) + Y(\omega). \tag{23}$$

Since $\tilde{\mathcal{S}}_d(\omega)$ is a $C^{k,\epsilon}$ $\big(\epsilon \in (0,\delta)\big)$ submanifold of $\bar{B}(0,\rho_1(\omega))$, then $\tilde{\mathcal{S}}(\omega)$ is a $C^{k,\epsilon}$ $\big(\epsilon \in (0,\delta)\big)$ submanifold of $\bar{B}(Y(\omega),\rho_1(\omega))$. Furthermore, $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$.

Hence $\dim \tilde{\mathcal{S}}(\omega) = \dim \mathcal{S}(\omega) = \sum_{i=i_0}^{m} q_i$, and is non-random.

- (22) implies that

$$\limsup_{n\to\infty} \frac{1}{n} \log |Z(n, x, \omega)| \leq \lambda_{i_0} \tag{24}$$

for all ω in Ω_1^* and all $x \in \tilde{\mathcal{S}}_d(\omega)$. Lemma 4 implies there is a sure event $\Omega_2^* \subseteq \Omega_1^*$ such that $\theta(t, \cdot)(\Omega_2^*) = \Omega_2^*$ for all $t \in \mathbf{R}$, and

$$\limsup_{t\to\infty} \frac{1}{t} \log |Z(t, x, \omega)| \leq \lambda_{i_0} \tag{25}$$

for all $\omega \in \Omega_2^*$ and all $x \in \tilde{\mathcal{S}}_d(\omega)$. Therefore (2) holds.

- To prove (b), let $\omega \in \Omega_1^*$. By (22), there is a positive integer $N_0 := N_0(\omega)$ (independent of $x \in \tilde{\mathcal{S}}_d(\omega)$) such that

$Z(n,x,\omega) \in \bar{B}(0,1)$ for all $n \geq N_0$. Let $\Omega_4^* := \Omega_2^* \cap \Omega_3$, where Ω_3 is the shift-invariant sure event defined in the proof of Lemma 4. Then Ω_4^* is a sure event and $\theta(t,\cdot)(\Omega_4^*) = \Omega_4^*$ for all $t \in \mathbf{R}$. By cocycle property, Mean-Value theorem and the ergodic theorem (Lemma 1(i)), we get (b).

- To prove the invariance property (4), apply the Oseledec theorem to $(D_2\phi(t,Y(\omega),\omega),\theta(t,\omega))$. Get a sure $\theta(t,\cdot)$-invariant event, also denoted by Ω_1^*, such that $D_2\phi(t,Y(\omega),\omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$. Equality holds because $D_2\phi(t,Y(\omega),\omega)$ is injective and $\dim \mathcal{S}(\omega) = \dim \mathcal{S}(\theta(t,\omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$.

- To prove the asymptotic invariance property (3), use ideas from Ruelle's Theorems 5.1 and 4.1 in [Ru.1], to pick random variables ρ_1, β_1 and a sure event (also denoted by) Ω_1^* such that $\theta(t,\cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$, and for any $\epsilon \in (0, \epsilon_1)$ and every $\omega \in \Omega_1^*$, there exists a positive $K_1^\epsilon(\omega)$ for which the inequalities

$$\begin{aligned} \rho_1(\theta(t,\omega)) &\geq K_1^\epsilon(\omega)\rho_1(\omega)e^{(\lambda_{i_0}+\epsilon)t}, \\ \beta_1(\theta(t,\omega)) &\geq K_1^\epsilon(\omega)\beta_1(\omega)e^{(\lambda_{i_0}+\epsilon)t} \end{aligned} \tag{26}$$

hold for all $t \geq 0$. Use (b) to obtain a sure event $\Omega_5^* \subseteq \Omega_4^*$ such that $\theta(t,\cdot)(\Omega_5^*) = \Omega_5^*$ for all $t \in \mathbf{R}$, and for any $0 < \epsilon < \epsilon_1$

and $\omega \in \Omega_4^*$, there exists $\beta^\epsilon(\omega) > 0$ (independent of x) with

$$|\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t} \qquad (27)$$

for all $x \in \tilde{\mathcal{S}}(\omega)$, $t \geq 0$. Fix $t \geq 0$, $\omega \in \Omega_5^*$ and $x \in \tilde{\mathcal{S}}(\omega)$. Let n be a non-negative integer. Then the cocycle property and (27) imply that

$$\begin{aligned}
|\phi(n, \phi(t, x, \omega), \theta(t, \omega)) - Y(\theta(n, \theta(t, \omega)))| \\
&= |\phi(n + t, x, \omega) - Y(\theta(n + t, \omega))| \\
&\leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)(n+t)} \\
&\leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t} e^{(\lambda_{i_0} + \epsilon_1)n}.
\end{aligned} \qquad (28)$$

If $\omega \in \Omega_5^*$, then it follows from (26),(27), (28) and the definition of $\tilde{\mathcal{S}}(\theta(t, \omega))$ that there exists $\tau_1(\omega) > 0$ such that $\phi(t, x, \omega) \in \tilde{\mathcal{S}}(\theta(t, \omega))$ for all $t \geq \tau_1(\omega)$. This proves asymptotic invariance.

- Prove (d), the existence of the local unstable manifolds $\tilde{\mathcal{U}}(\omega)$, by running both the flow ϕ and the shift θ backward in time getting the cocycle $(\tilde{Z}(t, \cdot, \omega), \tilde{\theta}(t, \omega), t \geq 0)$:

$$\tilde{\phi}(t, x, \omega) := \phi(-t, x, \omega),\ \tilde{Z}(t, x, \omega) := Z(-t, x, \omega),$$
$$\tilde{\theta}(t, \omega) := \theta(-t, \omega)$$

for all $t \geq 0, \omega \in \Omega$. The linearized flow $(D_2\tilde{\phi}(t, Y(\omega), \omega), \tilde{\theta}(t, \omega), t \geq 0)$ is an $L(\mathbf{R}^d)$-valued perfect cocycle with a non-random finite Lyapunov spectrum $\{-\lambda_1 < -\lambda_2 < \cdots < -\lambda_i < -\lambda_{i+1} < \cdots < -\lambda_m\}$ where $\{\lambda_m <$

$\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ is the Lyapunov spectrum of the forward linearized flow $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega),\ t \geq 0)$. Apply first part of the proof to get *stable manifolds* for the backward flow $\tilde{\phi}$ satisfying assertions (a), (b), (c). This gives *unstable manifolds* for the original flow ϕ, and (d), (e), (f) automatically hold.

- Measurability of the stable manifolds follows from the representations:

$$\tilde{\mathcal{S}}(\omega) = Y(\omega) + \tilde{\mathcal{S}}_d(\omega) \tag{29}$$

$$\tilde{\mathcal{S}}_d(\omega) = \lim_{n\to\infty} \bar{B}(0, \rho_1(\omega)) \cap \bigcap_{i=1}^{n} f_i(\cdot, \omega)^{-1}(\bar{B}(0,1)) \tag{30}$$

$f_i(x, \omega) := \beta_1(\omega)^{-1} e^{-(\lambda_{i_0}+\epsilon_1)i} Z(i, x, \omega)$, $x \in \mathbf{R}^d$, $\omega \in \Omega_1^*$, for all integers $i \geq 0$. (Above limit is taken in the metric d^* on $\mathcal{K}(\mathbf{R}^d)$.) Use joint continuity of translation and measurability of Y, f_i, ρ_1, finite intersections and the continuity of the maps

$$\mathbf{R}^+ \ni r \mapsto \bar{B}(0, r) \in \mathcal{K}(\mathbf{R}^d).$$

$$\mathrm{Hom}(\mathbf{R}^d) \ni f \mapsto f^{-1}(\bar{B}(0, 1)) \in \mathcal{K}(\mathbf{R}^d).$$

- For h, g_i in C_b^∞, can adapt above argument to give a sure event in $\mathcal{F}$, also denoted by Ω^* such that $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞ for all $\omega \in \Omega^*$. $\square$

Some Technical Lemmas

$\|\cdot\|_{k,\epsilon} := C^{k,\epsilon}$-norm on $C^{k,\epsilon}$ mappings $\bar{B}(0,\rho) \to \mathbf{R}^d$.

Lemma 2

Assume that $\log^+ |Y(\cdot)|$ *is integrable. Then the cocycle* ϕ *satisfies*

$$\int_\Omega \log^+ \sup_{-T \le t_1, t_2 \le T} \|\phi(t_2, Y(\theta(t_1,\omega)) + (\cdot), \theta(t_1,\omega))\|_{k,\epsilon} \, dP(\omega) < \infty \tag{10}$$

for any fixed $0 < T, \rho < \infty$ *and any* $\epsilon \in (0,\delta)$*. Furthermore, the linearized flow* $(D_2\phi(t, Y(\omega), \omega), \theta(t,\omega))$*,* $t \ge 0$*, is an* $L(\mathbf{R}^d)$*-valued perfect cocycle and*

$$\int_\Omega \log^+ \sup_{-T \le t_1, t_2 \le T} \|D_2\phi(t_2, Y(\theta(t_1,\omega)), \theta(t_1,\omega))\|_{L(\mathbf{R}^d)} \, dP(\omega) < \infty \tag{11}$$

for any fixed $0 < T < \infty$. The forward cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t > 0)$ has a non-random finite Lyapunov spectrum $\{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. Each Lyapunov exponent λ_i has a non-random multiplicity q_i, $1 \le i \le m$, and $\sum_{i=1}^{m} q_i = d$. The backward linearized cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t < 0)$, admits a "backward" non-random finite Lyapunov spectrum:

$$\lim_{t \to -\infty} \frac{1}{t} \log |D_2\phi(t, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbf{R}^d),$$

taking values in $\{-\lambda_i\}_{i=1}^{m}$ with non-random multiplicities q_i, $1 \le i \le m$, and $\sum_{i=1}^{m} q_i = d$.

The Auxiliary Cocycle

To apply Ruelle's discrete non-linear ergodic theorem ([Ru.1], Theorem 5.1, p. 292), introduce the following auxiliary cocycle $Z : \mathbf{R} \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$. This a "centering" of the flow ϕ about the stationary solution:

$$Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega)) \qquad (16)$$

for $t \in \mathbf{R}, x \in \mathbf{R}^d, \omega \in \Omega$.

Lemma 3

(Z, θ) is a perfect cocycle on $\mathbf{R}^d$ and $Z(t, 0, \omega) = 0$ for all $t \in \mathbf{R}$, and all $\omega \in \Omega$.

The proof of the local stable-manifold theorem (Theorem 1) uses a discretization argument that requires the following lemma.

Lemma 4

Suppose that $\log^+ |Y(\cdot)|$ *is integrable. Then there is a sure event* $\Omega_3 \in \mathcal{F}$ *with the following properties:*

(i) $\theta(t, \cdot)(\Omega_3) = \Omega_3$ *for all* $t \in \mathbf{R}$,

(ii) For every $\omega \in \Omega_3$ *and any* $x \in \mathbf{R}^d$, *the statement*

$$\limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)| < 0 \tag{17}$$

implies

$$\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| = \limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)|. \tag{18}$$

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