6-2002

Mini-Course on Stochastic Systems with Memory (University of Campinas)

Salah-Eldin A. Mohammed
Southern Illinois University Carbondale, salah@sfde.math.siu.edu

Mathematics Institute; University of Campinas (UNICAMP); Campinas, Brazil; June 26, 27, 28, 2002

I. EXISTENCE

Campinas, Brazil

June 26, 2002

Salah-Eldin A. Mohammed

Southern Illinois University

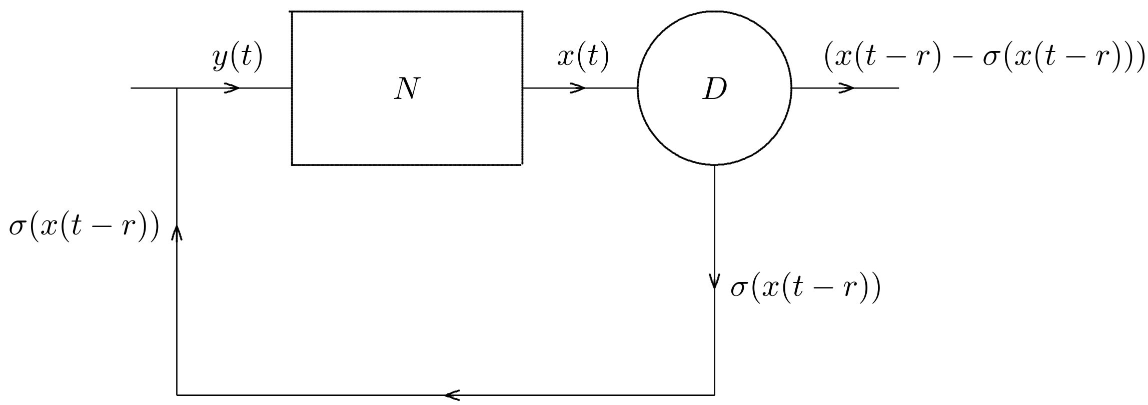
Carbondale, IL 62901–4408 USA

Web page: **http://sfde.math.siu.edu**

I. EXISTENCE

1. Examples

Example 1. (*Noisy Feedbacks*)



Box N: Input $= y(t)$, output $= x(t)$ at time $t > 0$ related by

$$x(t) = x(0) + \int_0^t y(u)\, dZ(u) \tag{1}$$

where $Z(u)$ is a semimartingale noise.

Box D: Delays signal $x(t)$ by $r\,(>0)$ units of time. A proportion σ $(0 \le \sigma \le 1)$ is transmitted through D and the rest $(1-\sigma)$ is used for other purposes.

Therefore

$$y(t) = \sigma x(t-r)$$

Take $\dot{Z}(u) :=$ white noise $= \dot{W}(u)$

Then substituting in (1) gives the Itô integral equation

$$x(t) = x(0) + \sigma \int_0^t x(u-r) dW(u)$$

or the stochastic differenial delay equation (sdde):

$$dx(t) = \sigma x(t-r) dW(t), \qquad t > 0 \tag{I}$$

To solve (I), need an *initial process* $\theta(t)$, $-r \le t \le 0$:

$$x(t) = \theta(t) \quad \text{a.s.,} \qquad -r \le t \le 0$$

$\mathbf{r = 0}$: (I) becomes a linear stochastic ode and has closed form solution

$$x(t) = x(0) e^{\sigma W(t) - \frac{\sigma^2 t}{2}}, \qquad t \ge 0.$$

$\mathbf{r > 0}$: Solve (I) by successive Itô integrations over steps of length r:

$$\begin{aligned} x(t) &= \theta(0) + \sigma \int_0^t \theta(u-r)\, dW(u), \quad 0 \le t \le r \\ x(t) &= x(r) + \sigma \int_r^t [\theta(0) + \sigma \int_0^{(v-r)} \theta(u-r)\, dW(u)]\, dW(v),\ r < t \le 2r, \\ \cdots &= \cdots \qquad\qquad 2r < t \le 3r, \end{aligned}$$

No closed form solution is known (even in deterministic case).

Curious Fact!

In the sdde (I) the Itô differential dW may be replaced by the Stratonovich differential $\circ dW$ *without changing the solution* x*.* Let x be the solution of (I) under an Itô differential dW. Then using finite partitions $\{u_k\}$ of the interval $[0,t]$:

$$\int_0^t x(u-r) \circ dW(t) = \lim \sum_k \frac{1}{2} [x(u_k - r) + x(u_{k+1} - r)][W(u_{k+1}) - W(u_k)]$$

where the limit in probability is taken as the mesh of the partition $\{u_k\}$ goes to zero. Compare the Stratonovich and Itô integrals using the corresponding partial sums:

$$\begin{aligned}
\lim E\bigg(\sum_k \frac{1}{2}[x(u_k - r) + x(u_{k+1} - r)][W(u_{k+1}) - W(u_k)] \\
- \sum_k [x(u_k - r)][W(u_{k+1}) - W(u_k)] \bigg)^2 \\
= \lim E\bigg(\sum_k \frac{1}{2}[x(u_{k+1} - r) - x(u_k - r)][W(u_{k+1}) - W(u_k)] \bigg)^2 \\
= \lim \sum_k \frac{1}{4} E[x(u_{k+1} - r) - x(u_k - r)]^2 \, E[W(u_{k+1}) - W(u_k)]^2 \\
= \lim \sum_k \frac{1}{4} E[x(u_{k+1} - r) - x(u_k - r)]^2 \, (u_{k+1} - u_k) \\
= 0
\end{aligned}$$

because W has independent increments, x is adapted to the Brownian filtration, $u \mapsto x(u) \in L^2(\Omega, \mathbf{R})$ is continuous, and the delay r is positive. Alternatively

$$\int_0^t x(u - r) \circ dW(u) = \int_0^t x(u - r) \, dW(u) + \frac{1}{2} < x(\cdot - r, W > (t)$$

and $< x(\cdot - r, W > (t) = 0$ for all $t > 0$.

Remark.

When $r > 0$, the solution process $\{x(t) : t \geq -r\}$ of (I) is a martingale but is *non-Markov*.

Example 2. (*Simple Population Growth*)

Consider a large population $x(t)$ at time t evolving with a constant birth rate $\beta > 0$ and a constant death rate α per capita. Assume immediate removal of the dead from the population. Let $r > 0$ (fixed, non-random$=9$, e.g.) be the development period of each individual and assume there is migration whose overall rate is distributed like white noise $\sigma\dot{W}$ (mean zero and variance $\sigma > 0$), where W is one-dimensional standard Brownian motion. The change in population $\Delta x(t)$ over a small time interval $(t, t+\Delta t)$ is

$$\Delta x(t) = -\alpha x(t)\Delta t + \beta x(t-r)\Delta t + \sigma\dot{W}\Delta t$$

Letting $\Delta t \to 0$ and using Itô stochastic differentials,

$$dx(t) = \{-\alpha x(t) + \beta x(t-r)\}\, dt + \sigma dW(t), \quad t > 0. \tag{II}$$

Associate with the above affine sdde the initial condition $(v,\eta) \in \mathbf{R} \times L^2([-r,0],\mathbf{R})$

$$x(0) = v, \quad x(s) = \eta(s), \quad -r \le s < 0.$$

Denote by $M_2 = \mathbf{R} \times L^2([-r,0],\mathbf{R})$ the Delfour-Mitter Hilbert space of all pairs (v,η), $v \in \mathbf{R}$, $\eta \in L^2([-r,0],\mathbf{R})$ with norm

$$\|(v,\eta)\|_{M_2} = \left(|v|^2 + \int_{-r}^0 |\eta(s)|^2\, ds\right)^{1/2}.$$

Let $W : \mathbf{R}^+ \times \Omega \to \mathbf{R}$ be defined on the canonical filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}^+}, P)$ where

$$\Omega = C(\mathbf{R}^+,\mathbf{R}),\ \mathcal{F} = \text{ Borel } \Omega,\ \mathcal{F}_t = \sigma\{\rho_u : u \le t\}$$

$\rho_u : \Omega \to \mathbf{R}, u \in \mathbf{R}^+$, are evaluation maps $\omega \mapsto \omega(u)$, and $P =$ Wiener measure on Ω.

Example 3. (*Logistic Population Growth*)

A single population $x(t)$ at time t evolving logistically with *development (incubation) period* $r > 0$ under Gaussian type noise (e.g. migration on a molecular level):

$$\dot{x}(t) = [\alpha - \beta x(t-r)]\, x(t) + \gamma x(t)\dot{W}(t), \quad t > 0$$

i.e.

$$dx(t) = [\alpha - \beta x(t-r)]\, x(t)\, dt + \gamma x(t) dW(t) \quad t > 0. \qquad (III)$$

with *initial condition*

$$x(t) = \theta(t) \quad -r \leq t \leq 0.$$

For positive delay r the above sdde can be solved *implicitly* using forward steps of length r, i.e. for $0 \leq t \leq r$, $x(t)$ satisfies the *linear* sode (without delay)

$$dx(t) = [\alpha - \beta\theta(t-r)]\, x(t)\, dt + \gamma x(t) dW(t) \quad 0 < t \leq r. \qquad (III')$$

$x(t)$ is a semimartingale and is non-Markov (Scheutzow [S], 1984).

Example 4. (*Heat bath*)

Model proposed by R. Kubo (1966) for physical Brownian motion. A molecule of mass m moving under random gas forces with position $\xi(t)$ and velocity $v(t)$ at time t; cf classical work by Einstein and Ornestein and Uhlenbeck. Kubo proposed the following modification of the Ornstein-Uhenbeck process

$$\left.\begin{aligned} d\xi(t) &= v(t)\, dt \\ mdv(t) &= -m[\int_{t_0}^{t} \beta(t-t')v(t')\, dt']\, dt + \gamma(\xi(t), v(t))\, dW(t),\ t > t_0. \end{aligned}\right\} \qquad (IV)$$

m =mass of molecule. No external forces.

$\beta =$ viscosity coefficient function with compact support.

γ a function $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ representing the random gas forces on the molecule.

$\xi(t) =$ position of molecule $\in \mathbf{R}^3$.

$v(t) =$ velocity of molecule $\in \mathbf{R}^3$.

$W = 3-$ dimensional Brownian motion.

([Mo], Pitman Books, RN # 99, 1984, pp. 223-226).

Further Examples

Delay equation with Poisson noise:

$$\left.\begin{aligned} dx(t) &= x((t-r)-)\, dN(t) \quad t > 0 \\ x_0 &= \eta \in D([-r,0],\mathbf{R}) \end{aligned}\right\} \qquad (V)$$

N := Poisson process with iid interarrival times ([S], Hab. 1988). $D([-r,0],\mathbf{R})$ = space of all cadlag paths $[-r,0] \to \mathbf{R}$, with sup norm.

Simple model of dye circulation in the blood (or pollution) (cf. Bailey and Williams [B-W], JMAA, 1966, Lenhart and Travis ([L-T], PAMS, 1986).

$$\left.\begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r))\}\, dt + \sigma x(t)\, dW(t) \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R} \times L^2([-r,0],\mathbf{R}), \end{aligned}\right\} \qquad (VI)$$

([Mo], Survey, 1992; [M-S], II, 1995.)

In above model:

$x(t)$:= dye concentration (gm/cc)

r = time taken by blood to traverse side tube (vessel)

Flow rate (cc/sec) is Gaussian with variance σ.

A fixed proportion of blood in main vessel is pumped into side vessel(s). Model will be analysed in Lecture V (Theorem V.5).

$$\left.\begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r))\}\,dt + \{\int_{-r}^{0} x(t+s)\sigma(s)\,ds\}\,dW(t), \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R} \times L^2([-r, 0], \mathbf{R}),\ t > 0. \end{aligned}\right\} \qquad (VII)$$

([Mo], Survey, 1992; [M-S], II, 1995.)

Linear d-dimensional systems driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$ with constant coefficients.

$$\left.\begin{aligned} dx(t) &= H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\ &\quad + \sum_{i=1}^{m} g_i x(t)\,dW_i(t), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (VIII)$$

$H := (\mathbf{R}^d)^N \times M_2 \to \mathbf{R}^d$ linear functional on $(\mathbf{R}^d)^N \times M_2$; g_i $d \times d$-matrices ([Mo], Stochastics, 1990).

Linear systems driven by (helix) semimartingale noise (N, L), and memory driven by a (stationary) measure-valued process ν and a (stationary) process K ([M-S], I, AIHP, 1996):

$$\left.\begin{aligned} dx(t) &= \Big\{\int_{[-r,0]} \nu(t)(ds)\,x(t+s)\Big\}\,dt \\ &\quad + dN(t)\int_{-r}^{0} K(t)(s)\,x(t+s)\,ds + dL(t)\,x(t-), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (IX)$$

Multidimensional affine systems driven by (helix) noise Q ([M-S], Stochastics, 1990):

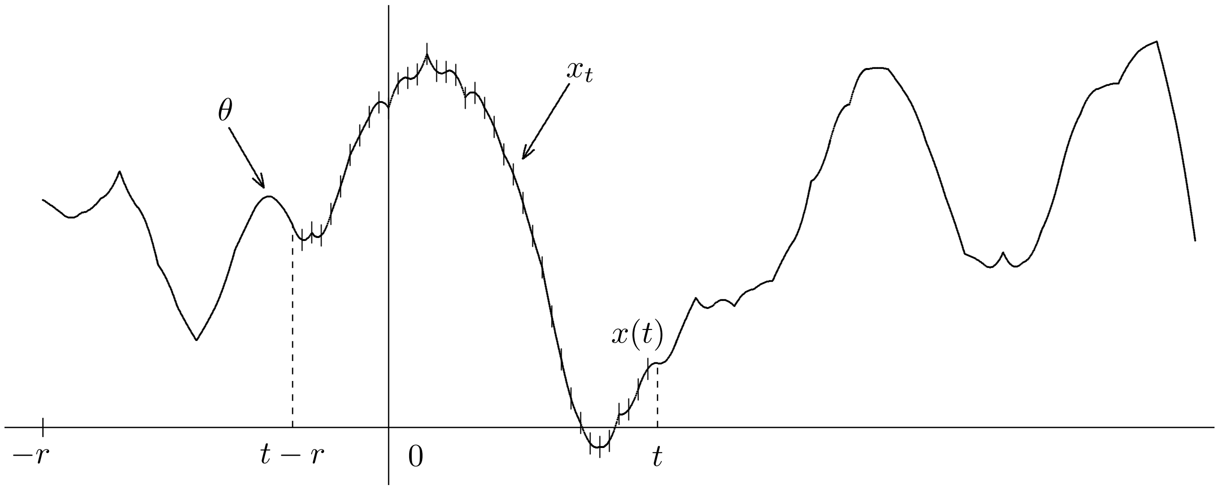
$$\left.\begin{aligned} dx(t) &= \Big\{\int_{[-r,0]} \nu(t)(ds)\,x(t+s)\Big\}\,dt + dQ(t), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (X)$$

Memory driven by white noise:

$$\left.\begin{aligned} dx(t) &= \left\{ \int_{[-r,0]} x(t+s)\, dW(s) \right\} dW(t) \quad t > 0 \\ x(0) &= v \in \mathbf{R}, \quad x(s) = \eta(s), \quad -r < s < 0, \quad r \geq 0 \end{aligned}\right\} \qquad (XI)$$

([Mo], Survey, 1992).

Formulation



Slice each solution path x over the interval $[t-r, t]$ to get *segment* x_t as a process on $[-r, 0]$:

$$x_t(s) := x(t+s) \qquad \text{a.s.},\ t \geq 0, s \in J := [-r, 0].$$

Therefore sdde's (I), (II), (III) and (XI) become

$$\left.\begin{aligned} dx(t) &= \sigma x_t(-r) dW(t), \quad t > 0 \\ x_0 &= \theta \in C([-r,0], \mathbf{R}) \end{aligned}\right\} \qquad (I)$$

$$\left.\begin{aligned} dx(t) &= \{-\alpha x(t) + \beta x_t(-r)\}\, dt + \sigma dW(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in \mathbf{R} \times L^2([-r,0], \mathbf{R}) \end{aligned}\right\} \qquad (II)$$

$$\left.\begin{aligned} dx(t) &= [\alpha - \beta x_t(-r)]x_t(0)\,dt + \gamma x_t(0)\,dW(t) \\ x_0 &= \theta \in C([-r,0],\mathbf{R}) \end{aligned}\right\} \qquad (III)$$

$$\left.\begin{aligned} dx(t) &= \left\{\int_{[-r,0]} x_t(s)\,dW(s)\right\} dW(t) \quad t>0 \\ (x(0),x_0) &= (v,\eta) \in \mathbf{R} \times L^2([-r,0],\mathbf{R}), \quad r \geq 0 \end{aligned}\right\} \qquad (XI)$$

Think of R.H.S.'s of the above equations as functionals of x_t (and $x(t)$) and generalize to *stochastic functional differential equation* (sfde)

$$\left.\begin{aligned} dx(t) &= h(t,x_t)dt + g(t,x_t)dW(t) \quad t>0 \\ x_0 &= \theta \end{aligned}\right\} \qquad (XII)$$

on filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions:

$(\mathcal{F}_t)_{t\geq 0}$ right-continuous and each $\mathcal{F}_t$ contains all P-null sets in $\mathcal{F}$.

$C := C([-r,0],\mathbf{R}^d)$ Banach space, sup norm.

$W(t) = m$–dimensional Brownian motion.

$L^2(\Omega, C) :=$ Banach space of all $(\mathcal{F}, Borel C)$-measurable L^2 (Bochner sense) maps $\Omega \to C$ with the L^2-norm

$$\|\theta\|_{L^2(\Omega,C)} := \left[\int_\Omega \|\theta(\omega)\|_C^2 \, dP(\omega)\right]^{1/2}$$

Coefficients:

$$h : [0, T] \times L^2(\Omega, C) \to L^2(\Omega, \mathbf{R}^d) \qquad \text{(Drift)}$$

$$g : [0, T] \times L^2(\Omega, C) \to L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d) \qquad \text{(Diffusion).}$$

Initial data:

$$\theta \in L^2(\Omega, C, \mathcal{F}_0).$$

Solution:

$x : [-r, T] \times \Omega \to \mathbf{R}^d$ measurable and sample-continuous, $x|[0, T]$ $(\mathcal{F}_t)_{0 \leq t \leq T}$-adapted and $x(s)$ is $\mathcal{F}_0$-measurable for all $s \in [-r, 0]$.

Exercise: $[0, T] \ni t \mapsto x_t \in C([-r, 0], \mathbf{R}^d)$ is $(\mathcal{F}_t)_{0 \leq t \leq T}$-adapted.

(*Hint:* $Borel\, C$ is generated by all evaluations.)

Hypotheses (E_1)**.**

(i) h, g are jointly continuous and uniformly Lipschitz in the second variable with respect to the first:

$$\|h(t,\psi_1) - h(t,\psi_2)\|_{L^2(\Omega,\mathbf{R}^d)} \leq L\|\psi_1 - \psi_2\|_{L^2(\Omega,C)}$$

for all $t \in [0,T]$ and $\psi_1, \psi_2 \in L^2(\Omega, C)$. Similarly for the diffusion coefficent g.

(ii) For each $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted process $y : [0,T] \to L^2(\Omega, C)$, the processes $h(\cdot, y(\cdot)), g(\cdot, y(\cdot))$ are also $(\mathcal{F}_t)_{0\leq t\leq T}$- adapted.

Theorem I.1. ([Mo], 1984) (Existence and Uniqueness).

Suppose h and g satisfy Hypotheses (E_1). Let $\theta \in L^2(\Omega, C; \mathcal{F}_0)$.

Then the sfde (XII) has a unique solution ${}^\theta x : [-r,\infty) \times \Omega \to \mathbf{R}^d$ starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ with $t \longmapsto {}^\theta x_t$ continuous and ${}^\theta x \in L^2(\Omega, C([-r,T]\mathbf{R}^d))$ for all $T > 0$. For a given θ, uniqueness holds up to equivalence among all $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted processes in $L^2(\Omega, C([-r,T], \mathbf{R}^d))$.

Proof.

[Mo], Pitman Books, 1984, Theorem 2.1, pp. 36-39. □

Theorem I.1 covers equations (I), (II), (IV), (VI), (VII), (VIII), (XI) and a large class of sfde's driven by white noise. Note that (XI) *does not satisfy the hypotheses underlying the classical results* of Doleans-Dade [Dol], 1976, Metivier and Pellaumail [Met-P], 1980, Protter, Ann. Prob. 1987, Lipster and Shiryayev [Lip-Sh], [Met], 1982. This is because the coefficient

$$\eta \to \int_{-r}^{0} \eta(s)\, dW(s)$$

on the RHS of (XI) *does not admit almost surely Lipschitz (or even linear) versions* $C \to \mathbf{R}$! This will be shown later.

When the coeffcients h, g factor through functionals

$$H : [0,T] \times C \to \mathbf{R}^d, \qquad G : [0,T] \times C \to \mathbf{R}^{d \times m}$$

we can impose the following local Lipschitz and global linear growth conditions on the sfde

$$\left.\begin{aligned} dx(t) &= H(t, x_t)\, dt + G(t, x_t)\, dW(t) \quad t > 0 \\ x_0 &= \theta \end{aligned}\right\} \qquad (XIII)$$

with W m-dimensional Brownian motion:

Hypotheses (E_2)

(i) H, G are Lipschitz on bounded sets in C: For each integer $n \geq 1$ there exists $L_n > 0$ such that

$$|H(t,\eta_1) - H(t,\eta_2)| \leq L_n \|\eta_1 - \eta_2\|_C$$

for all $t \in [0,T]$ and $\eta_1, \eta_2 \in C$ with $\|\eta_1\|_C \leq n$, $\|\eta_2\|_C \leq n$. Similarly for the diffusion coefficent G.

(ii) There is a constant $K > 0$ such that

$$|H(t,\eta)| + \|G(t,\eta)\| \leq K(1 + \|\eta\|_C)$$

for all $t \in [0,T]$ and $\eta \in C$.

Note that the adaptability condition is not needed (explicitly) because H, G are deterministic and because the sample-continuity and adaptability of x imply that the segment $[0,T] \ni t \mapsto x_t \in C$ is also adapted.

Exercise: Formulate the heat-bath model (IV) as a sfde of the form (XIII).(β has compact support in $\mathbf{R}^+$.)

Theorem I.2. ([Mo], 1984) (Existence and Uniqueness).

Suppose H and G satisfy Hypotheses (E_2) and let $\theta \in L^2(\Omega, C; \mathcal{F}_0)$.

Then the sfde (XIII) has a unique $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted solution ${}^\theta x : [-r, T] \times \Omega \to \mathbf{R}^d$ starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ with $t \longmapsto {}^\theta x_t$ continuous and ${}^\theta x \in L^2(\Omega, C([-r, T], \mathbf{R}^d))$ for all $T > 0$. For a given θ, uniqueness holds up to equivalence among all $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted processes in $L^2(\Omega, C([-r, T], \mathbf{R}^d))$.

Furthermore if $\theta \in L^{2k}(\Omega, C; \mathcal{F}_0)$, then ${}^\theta x_t \in L^{2k}(\Omega, C; \mathcal{F}_t)$ and

$$E\|{}^\theta x_t\|_C^{2k} \leq C_k[1 + \|\theta\|_{L^{2k}(\Omega,C)}^{2k}]$$

for all $t \in [0, T]$ and some positive constants C_k.

Proofs of Theorems I.1, I.2.(Outline)

[Mo], pp. 150-152. Generalize sode proofs in Gihman and Skorohod ([G-S], 1973) or Friedman ([Fr], 1975):

(1) Truncate coefficients outside bounded sets in C. Reduce to globally Lipschitz case.

(2) Successive approx. in globally Lipschitz situation.

(3) Use local uniqueness ([Mo], Theorem 4.2, p. 151) to "patch up" solutions of the truncated sfde's.

For (2) consider globally Lipschitz case and $h \equiv 0$.

We look for solutions of (XII) by successive approximation in $L^2(\Omega, C([-r, a], \mathbf{R}^d))$. Let $J := [-r, 0]$.

Suppose $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$ is $\mathcal{F}_0$-measurable. Note that this is equivalent to saying that $\theta(\cdot)(s)$ is $\mathcal{F}_0$-measurable for all $s \in J$, because θ has a.a. sample paths continuous.

We prove by induction that there is a sequence of processes ${}^k x : [-r, a] \times \Omega \to \mathbf{R}^d$, $k = 1, 2, \cdots$ having the

Properties $P(k)$:

(i) ${}^k x \in L^2(\Omega, C([-r,a], \mathbf{R}^d))$ and is adapted to $(\mathcal{F}_t)_{t\in[0,a]}$.

(ii) For each $t \in [0,a]$, ${}^k x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_t$-measur-able.

(iii)

$$\left.\begin{aligned} \|{}^{k+1}x - {}^k x\|_{L^2(\Omega,C)} &\le (ML^2)^{k-1}\frac{a^{k-1}}{(k-1)!}\|{}^2x - {}^1x\|_{L^2(\Omega,C)} \\ \|{}^{k+1}x_t - {}^k x_t\|_{L^2(\Omega,C)} &\le (ML^2)^{k-1}\frac{t^{k-1}}{(k-1)!}\|{}^2x - {}^1x\|_{L^2(\Omega,C)} \end{aligned}\right\} \tag{1}$$

where M is a "martingale" constant and L is the Lipschitz constant of g.

Take ${}^1x : [-r,a] \times \Omega \to \mathbf{R}^d$ to be

$${}^1x(t,\omega) = \begin{cases} \theta(\omega)(0) & t \in [0,a] \\ \theta(\omega)(t) & t \in J \end{cases}$$

a.s., and

$${}^{k+1}x(t,\omega) = \begin{cases} \theta(\omega)(0) + (\omega)\displaystyle\int_0^t g(u, {}^k x_u)dW(\cdot)(u) & t \in [0,a] \\ \theta(\omega)(t) \qquad t \in J \end{cases} \tag{2}$$

a.s.

Since $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_0$-measurable, then ${}^1x \in L^2(\Omega, C([-r,a], \mathbf{R}^d))$ and is trivially adapted to $(\mathcal{F}_t)_{t\in[0,a]}$. Hence ${}^1x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_t$-measurable for all $t \in [0,a]$. $P(1)$ (iii) holds trivially.

Now suppose $P(k)$ is satisfied for some $k > 1$. Then by Hypothesis $(E_1)(i), (ii)$ and the continuity of the slicing map (*stochastic memory*), it follows from $P(k)(ii)$ that the process

$$[0, a] \ni u \longmapsto g(u, {}^k x_u) \in L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d))$$

is continuous and adapted to $(\mathcal{F}_t)_{t \in [0,a]}$. $P(k+1)(i)$ and $P(k+1)(ii)$ follow from the continuity and adaptability of the stochastic integral. Check $P(k+1)(iii)$, by using Doob's inequality.

For each $k > 1$, write

$${}^k x = {}^1 x + \sum_{i=1}^{k-1} ({}^{i+1} x - {}^i x).$$

Now $L^2_A(\Omega, C([-r, a], \mathbf{R}^d))$ is closed in $L^2(\Omega, C([-r, a], \mathbf{R}^d))$; so the series

$$\sum_{i=1}^{\infty} ({}^{i+1} x - {}^i x)$$

converges in $L^2_A(\Omega, C([-r, a], \mathbf{R}^d))$ because of (1) and the convergence of

$$\sum_{i=1}^{\infty} \left[(ML^2)^{i-1} \frac{a^{i-1}}{(i-1)!} \right]^{1/2}.$$

Hence $\{{}^k x\}_{k=1}^{\infty}$ converges to some $x \in L^2_A(\Omega, C([-r, a], \mathbf{R}^d))$.

Clearly $x|J = \theta$ and is $\mathcal{F}_0$-measurable, so applying Doob's inequality to the Itô integral of the difference

$$u \longmapsto g(u, {}^k x_u) - g(u, x_u)$$

gives

$$\begin{aligned} E\bigg(\sup_{t\in[0,a]} \bigg| \int_0^t g(u, {}^k x_u)\, dW(\cdot)(u) - \int_0^t g(u, x_u)\, dW(\cdot)(u)\bigg|^2\bigg) \\ < ML^2 a \|{}^k x - x\|^2_{L^2(\Omega,C)} \\ \longrightarrow 0 \text{ as } k \to \infty. \end{aligned}$$

Thus viewing the right-hand side of (2) as a process in $L^2(\Omega, C\,([-r,a], \mathbf{R}^d))$ and letting $k \to \infty$, it follows from the above that x must satisfy the sfde (XII) a.s. for all $t \in [-r, a]$.

For uniqueness, let $\tilde{x} \in L^2_A(\Omega, ([-r,a], \mathbf{R}^d))$ be also a solution of (XII) with initial process θ. Then by the Lipschitz condition:

$$\|x_t - \tilde{x}_t\|^2_{L^2(\Omega,C)} < ML^2 \int_0^t \|x_u - \tilde{x}_u\|^2_{L^2(\Omega,C)}\, du$$

for all $t \in [0,a]$. Therefore we must have $x_t - \tilde{x}_t = 0$ for all $t \in [0,a]$; so $x = \tilde{x}$ in $L^2(\Omega, C([-r,a], \mathbf{R}^d))$ a.s. □

Remarks and Generalizations.

(i) In Theorem I.2 replace the process $(t, W(t))$ by a (square integrable) semimartingale $Z(t)$ satisfying appropriate conditions.([Mo], 1984, Chapter II).

(ii) Results on existence of solutions of sfde's driven by white noise were first obtained by Itô and Nisio ([I-N], J. Math. Kyoto University, 1968) and then Kushner (JDE, 197).

(iii) Extensions to sfde's with *infinite* memory. Fading memory case: work by Mizel and Trützer [M-T],JIE, 1984, Marcus and Mizel [M-M], Stochastics, 1988; general infinite memory: Itô and Nisio [I-N], J. Math. Kyoto University, 1968.

(iii) Pathwise local uniqueness holds for sfde's of type (XIII) under a global Lipschitz condition: If coeffcients of two sfde's agree on an open set in C, then the corresponding trajectories leave the open set at the same time and agree almost surely up to the time they leave the open set ([Mo], Pitman Books, 1984, Theorem 4.2, pp. 150-151.)

(iv) Replace the state space C by the Delfour-Mitter Hilbert space $M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d)$ with the Hilbert norm

$$\|(v,\eta)\|_{M_2} = \left(|v|^2 + \int_{-r}^0 |\eta(s)|^2\, ds\right)^{1/2}$$

for $(v,\eta) \in M_2$ (T. Ahmed, S. Elsanousi and S. Mohammed, 1983).

(v) Have Lipschitz and smooth dependence of ${}^\theta x_t$ on the initial process $\theta \in L^2(\Omega, C)$ ([Mo], 1984, Theorems 3.1, 3.2, pp. 41-45).

II. MARKOV BEHAVIOR
AND THE WEAK GENERATOR

Campinas, Brazil

June 27, 2002

Salah-Eldin A. Mohammed

Southern Illinois University

Carbondale, IL 62901–4408 USA

Web page: **http://sfde.math.siu.edu**

II. MARKOV BEHAVIOR AND THE GENERATOR

Consider the sfde

$$\left.\begin{aligned} dx(t) &= H(t, x_t)\,dt + G(t, x_t)\,dW(t), \qquad t > 0 \\ x_0 &= \eta \in C := C([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (XIII)$$

with coefficients $H : [0,T]\times C \to \mathbf{R}^d$, $G : [0,T]\times C \to \mathbf{R}^{d\times m}$, m-dimensional Brownian motion W and trajectory field $\{{}^\eta x_t : t \geq 0, \eta \in C\}$.

1. Questions

(i) For the sfde (XIII) does the trajectory field x_t give a diffusion in C (or M_2)?

(ii) How does the trajectory x_t transform under smooth non-linear functionals $\phi : C \to \mathbf{R}$?

(iii) What "diffusions" on C (or M_2) correspond to sfde's on $\mathbf{R}^d$?

We will only answer the first two questions. More details in [Mo], Pitman Books, 1984, Chapter III, pp. 46-112. Third question is OPEN.

Difficulties

(i) Although the current state $x(t)$ is a semimartingale, the trajectory x_t does *not* seem to possess any martingale properties when viewed as C-(or M_2)-valued process: e.g. for Brownian motion W $\big(H \equiv 0, G \equiv 1\big)$:

$$[E(W_t|\mathcal{F}_{t_1})](s) = W(t_1) = W_{t_1}(0), \qquad s \in [-r, 0]$$

whenever $t_1 \leq t - r$.

(ii) Lack of strong continuity leads to the use of weak limits in C which tend to live outside C.

(iii) We will show that x_t is a Markov process in C. However almost all tame functions lie *outside* the domain of the (weak) generator.

(iv) Lack of an Itô formula makes the computation of the generator hard.

Hypotheses (M)

(i) $\mathcal{F}_t :=$ completion of $\sigma\{W(u) : 0 \leq u \leq t\}$, $\quad t \geq 0$.

(ii) H, G are jointly continuous and globally Lipschitz in second variable uniformly wrt the first:

$$|H(t,\eta_1) - H(t,\eta_2)| + \|G(t,\eta_1) - G(t,\eta_2)\| \leq L\|\eta_1 - \eta_2\|_C$$

for all $t \in [0,T]$ and $\eta_1, \eta_2 \in C$.

2. The Markov Property

${}^{\eta}x^{t_1} :=$ solution starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_{t_1})$ at $t = t_1$ for the sfde:

$${}^{\eta}x^{t_1}(t) = \begin{cases} \eta(0) + \int_{t_1}^{t} H(u, x_u^{t_1})\, du + \int_{t_1}^{t} G(u, x_u^{t_1})\, dW(u), & t > t_1 \\ \eta(t - t_1), & t_1 - r \leq t \leq t_1. \end{cases}$$

This gives a two-parameter family of mappings

$$T_{t_2}^{t_1} : L^2(\Omega, C; \mathcal{F}_{t_1}) \to L^2(\Omega, C; \mathcal{F}_{t_2}),\ t_1 \leq t_2,$$

$$T_{t_2}^{t_1}(\theta) := {}^{\theta}x_{t_2}^{t_1}, \qquad \theta \in L^2(\Omega, C; \mathcal{F}_{t_1}). \tag{1}$$

Uniqueness of solutions gives the *two-parameter* semigroup property:

$$T_{t_2}^{t_1} \circ T_{t_1}^{0} = T_{t_2}^{0}, \quad t_1 \leq t_2. \tag{2}$$

([Mo], Pitman Books, 1984, Theorem II (2.2), p. 40.)

Theorem II.1 (Markov Property)([Mo], 1984).

In (XIII) suppose Hypotheses (M) hold. Then the trajectory field $\{{}^{\eta}x_t : t \geq 0, \eta \in C\}$ *is a Feller process on* C *with transition probabilities*

$$p(t_1, \eta, t_2, B) := P\big({}^{\eta}x_{t_2}^{t_1} \in B\big) \quad t_1 \leq t_2, \quad B \in \text{ Borel } C, \quad \eta \in C.$$

i.e.

$$P\big(x_{t_2} \in B \big| \mathcal{F}_{t_1}\big) = p(t_1, x_{t_1}(\cdot), t_2, B) = P\big(x_{t_2} \in B \big| x_{t_1}\big) \text{ a.s.}$$

Further, if H *and* G *do not depend on* t*, then the trajectory is time-homogeneous:*

$$p(t_1, \eta, t_2, \cdot) = p(0, \eta, t_2 - t_1, \cdot), \quad 0 \leq t_1 \leq t_2, \quad \eta \in C.$$

Proof.

[Mo], 1984, Theorem III.1.1, pp. 51-58. [Mo], 1984, Theorem III.2.1, pp. 64-65. □

3. The Semigroup

In the autonomous sfde

$$\left.\begin{aligned} dx(t) &= H(x_t)\,dt + G(x_t)\,dW(t) \quad t>0 \\ x_0 &= \eta \in C \end{aligned}\right\} \qquad (XIV)$$

suppose the coefficients $H : C \to \mathbf{R}^d$, $G : C \to \mathbf{R}^{d\times m}$ are *globally bounded* and globally Lipschitz.

$C_b :=$ Banach space of all bounded uniformly continuous functions $\phi : C \to \mathbf{R}$, with the sup norm

$$\|\phi\|_{C_b} := \sup_{\eta\in C} |\phi(\eta)|, \quad \phi \in C_b.$$

Define the operators $P_t : C_b \hookrightarrow C_b, t \geq 0$, on C_b by

$$P_t(\phi)(\eta) := E\phi({}^\eta x_t) \quad t \geq 0,\ \eta \in C.$$

A family $\phi_t, t > 0$, *converges weakly* to $\phi \in C_b$ as $t \to 0+$ if $\lim_{t\to 0+} < \phi_t, \mu > = < \phi, \mu >$ for all finite regular Borel measures μ on C. Write $\phi := w - \lim_{t\to 0+} \phi_t$. This is equivalent to

$$\begin{cases} \phi_t(\eta) \to \phi(\eta) \text{ as } t \to 0+, \text{ for all } \eta \in C \\ \\ \{\|\phi_t\|_{C_b} : t \geq 0\} \text{ is bounded .} \end{cases}$$

(Dynkin, [Dy], Vol. 1, p. 50). Proof uses uniform boundedness principle and dominated convergence theorem.

Theorem II.2([Mo], Pitman Books, 1984)

(i) $\{P_t\}_{t\geq 0}$ *is a one-parameter contraction semigroup on* C_b.

(ii) $\{P_t\}_{t\geq 0}$ *is weakly continuous at* $t=0$*:*

$$\begin{cases} P_t(\phi)(\eta) \to \phi(\eta) \text{ as } t \to 0+ \\ \\ \{|P_t(\phi)(\eta)| : t \geq 0, \eta \in C\} \text{is bounded by } \|\phi\|_{C_b}. \end{cases}$$

(iii) If $r > 0$, $\{P_t\}_{t\geq 0}$ *is never strongly continuous on* C_b *under the sup norm.*

Proof.

(i) One parameter semigroup property

$$P_{t_2} \circ P_{t_1} = P_{t_1+t_2}, \quad t_1, t_2 \geq 0$$

follows from the continuation property (2) and time-homogeneity of the Feller process x_t (Theorem II.1).

(ii) Definition of P_t, continuity and boundedness of ϕ and sample-continuity of trajectory ${}^\eta x_t$ give weak continuity of $\{P_t(\phi) : t > 0\}$ at $t = 0$ in C_b.

(iii) Lack of strong continuity of semigroup:

Define the canonical shift (static) semigroup

$$S_t : C_b \to C_b,\ t \geq 0,$$

by

$$S_t(\phi)(\eta) := \phi(\tilde{\eta}_t), \quad \phi \in C_b, \quad \eta \in C,$$

where $\tilde{\eta} : [-r, \infty) \to \mathbf{R}^d$ is defined by

$$\tilde{\eta}(t) = \begin{cases} \eta(0) & t \geq 0 \\ \eta(t) & t \in [-r, 0). \end{cases}$$

Then P_t is strongly continuous iff S_t is strongly continuous. P_t and S_t have the same "domain of strong continuity" independently of H, G, and W. This follows from the global boundedness of H and G. ([Mo], Theorem IV.2.1, pp. 72-73). Key relation is

$$\lim_{t\to 0+} E\|{}^\eta x_t - \tilde{\eta}_t\|_C^2 = 0$$

uniformly in $\eta \in C$. But $\{S_t\}$ is strongly continuous on C_b iff C is locally compact iff $r = 0$ (no memory) ! ([Mo], Theorems IV.2.1 and IV.2.2, pp.72-73). Main idea is to pick any $s_0 \in [-r, 0)$ and consider the function $\phi_0 : C \to \mathbf{R}$ defined by

$$\phi_0(\eta) := \begin{cases} \eta(s_0) & \|\eta\|_C \leq 1 \\ \dfrac{\eta(s_0)}{\|\eta\|_C} & \|\eta\|_C > 1 \end{cases}$$

Let C_b^0 be the domain of strong continuity of P_t, viz.

$$C_b^0 := \{\phi \in C_b : P_t(\phi) \to \phi \text{ as } t \to 0+ \text{ in } C_b\}.$$

Then $\phi_0 \in C_b$, but $\phi_0 \notin C_b^0$ because $r > 0$. $\square$

4. The Generator

Define the *weak generator* $A : D(A) \subset C_b \to C_b$ by the weak limit

$$A(\phi)(\eta) := w - \lim_{t \to 0+} \frac{P_t(\phi)(\eta) - \phi(\eta)}{t}$$

where $\phi \in D(A)$ iff the above weak limit exists. Hence $D(A) \subset C_0^b$ (Dynkin [Dy], Vol. 1, Chapter I, pp. 36-43). Also $D(A)$ is weakly dense in C_b and A is weakly closed. Further

$$\frac{d}{dt} P_t(\phi) = A(P_t(\phi)) = P_t(A(\phi)), \quad t > 0$$

for all $\phi \in D(A)$ ([Dy], pp. 36-43).

Next objective is to derive a formula for the weak generator A. *We need to augment C by adjoining a canonical d-dimensional direction. The generator A will be equal to the weak generator of the shift semigroup $\{S_t\}$ plus a second order linear partial differential operator along this new direction.* Computation requires the following lemmas.

Let

$$F_d = \{v\chi_{\{0\}} : v \in \mathbf{R}^d\}$$
$$C \oplus F_d = \{\eta + v\chi_{\{0\}} : \eta \in C, v \in \mathbf{R}^d\}, \quad \|\eta + v\chi_{\{0\}}\| = \|\eta\|_C + |v|$$

Lemma II.1.([Mo], Pitman Books, 1984)

Suppose $\phi : C \to \mathbf{R}$ is C^2 and $\eta \in C$. Then $D\phi(\eta)$ and $D^2\phi(\eta)$ have unique weakly continuous linear and bilinear extensions

$$\overline{D\phi(\eta)} : C \oplus F_d \to \mathbf{R}, \quad \overline{D^2\phi(\eta)} : (C \oplus F_d) \times (C \oplus F_d) \to \mathbf{R}$$

respectively.

Proof.

First reduce to the one-dimensional case $d = 1$ by using coordinates.

Let $\alpha \in C^* = [C([-r, 0], \mathbf{R})]^*$. We will show that there is a weakly continuous linear extension $\overline{\alpha} : C \oplus F_1 \to \mathbf{R}$ of α; viz. If $\{\xi^k\}$ is a bounded sequence in C such that $\xi^k(s) \to \xi(s)$ as $k \to \infty$ for all $s \in [-r, 0]$, where $\xi \in C \oplus F_1$, then $\alpha(\xi^k) \to \overline{\alpha}(\xi)$ as $k \to \infty$. By the Riesz representation theorem there is a unique finite regular Borel measure μ on $[-r, 0]$ such that

$$\alpha(\eta) = \int_{-r}^{0} \eta(s)\, d\mu(s)$$

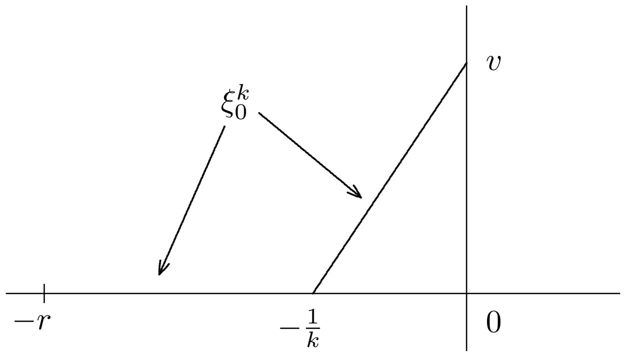
for all $\eta \in C$. Define $\overline{\alpha} \in [C \oplus F_1]^*$ by

$$\overline{\alpha}(\eta + v\chi_{\{0\}}) = \alpha(\eta) + v\mu(\{0\}), \quad \eta \in C, \quad v \in \mathbf{R}.$$

Easy to check that $\overline{\alpha}$ is weakly continuous. (*Exercise:* Use Lebesgue dominated convergence theorem.)

Weak extension $\overline{\alpha}$ is unique because each function $v\chi_{\{0\}}$ can be approximated weakly by a sequence of continuous functions $\{\xi_0^k\}$:

$$\xi_0^k(s) := \begin{cases} (ks+1)v, & -\frac{1}{k} \le s \le 0 \\ 0 & -r \le s < -\frac{1}{k}. \end{cases}$$



Put $\alpha = D\phi(\eta)$ to get first assertion of lemma.

To construct a weakly continuous bilinear extension $\overline{\beta} : (C \oplus F_1) \times (C \oplus F_1) \to \mathbf{R}$ for any continuous bilinear form $\beta : C \times C \to \mathbf{R}$, use classical theory of vector measures (Dunford and Schwartz, [D-S], Vol. I, Section 6.3). Think of β as a continuos *linear* map $C \to C^*$. Since C^* is weakly complete ([D-S], I.13.22, p. 341), then β is a weakly compact linear operator ([D-S], Theorem I.7.6, p. 494): i.e. it maps norm-bounded sets in C into weakly sequentially compact sets in C^*. By the Riesz representation theorem (for vector measures), there is a unique C^*-valued Borel measure λ on $[-r, 0]$ (of finite semi-variation) such that

$$\beta(\xi) = \int_{-r}^{0} \xi(s)\, d\lambda(s)$$

for all $\xi \in C$. ([D-S], Vol. I, Theorem VI.7.3, p. 493). By the dominated convergence theorem for vector measures ([D-S], Theorem IV.10.10, p. 328), one could reach elements in F_1 using weakly convergent sequences of type $\{\xi_0^k\}$. This gives a unique weakly continuous extension $\hat{\beta} : C \oplus F_1 \to C^*$. Next for each $\eta \in C$, $v \in \mathbf{R}$, extend $\hat{\beta}(\eta + v\chi_{\{0\}}) : C \to \mathbf{R}$ to a weakly continuous linear map $\overline{\hat{\beta}(\eta + v\chi_{\{0\}})} : C \oplus F_1 \to \mathbf{R}$. Thus $\overline{\beta}$ corresponds to the weakly continuous bilinear extension $\overline{\hat{\beta}(\cdot)}(\cdot) : [C \oplus F_1] \times [C \oplus F_1] \to \mathbf{R}$ of β. (Check this as exercise).

Finally use $\beta = D^2\phi(\eta)$ for each fixed $\eta \in C$ to get the required bilinear extension $\overline{D^2\phi(\eta)}$. □

Lemma II.2. ([Mo], Pitman Books, 1984)

For $t > 0$ define $W_t^* \in C$ by

$$W_t^*(s) := \begin{cases} \frac{1}{\sqrt{t}}[W(t+s) - W(0)], & -t \le s < 0, \\ 0 & -r \le s \le -t. \end{cases}$$

Let β be a continuous bilinear form on C. Then

$$\lim_{t\to 0+}\left[\frac{1}{t}E\beta({}^{\eta}x_t - \tilde{\eta}_t, {}^{\eta}x_t - \tilde{\eta}_t) - E\beta(G(\eta)\circ W_t^*, G(\eta)\circ W_t^*)\right] = 0$$

Proof.

Use

$$\lim_{t\to 0+} E\|\frac{1}{\sqrt{t}}({}^{\eta}x_t - \tilde{\eta}_t - G(\eta)\circ W_t^*\|_C^2 = 0.$$

The above limit follows from the Lipschitz continuity of H and G and the martingale properties of the Itô integral. Conclusion of lemma is obtained by a computation using the bilinearity of β, Hölder's inequality and the above limit.([Mo], Pitman Books, 1984, pp. 86-87.)

□

Lemma II.3. ([Mo], Pitman Books, 1984)

Let β be a continuous bilinear form on C and $\{e_i\}_{i=1}^m$ be any basis for $\mathbf{R}^m$. Then

$$\lim_{t\to 0+}\frac{1}{t}E\beta({}^{\eta}x_t - \tilde{\eta}_t, {}^{\eta}x_t - \tilde{\eta}_t) = \sum_{i=1}^{m}\overline{\beta}\big(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}\big)$$

for each $\eta \in C$.

Proof.

By taking coordinates reduce to the one-dimensional case $d = m = 1$:

$$\lim_{t\to 0+} E\beta(W_t^*, W_t^*) = \overline{\beta}(\chi_{\{0\}}, \chi_{\{0\}})$$

with W one-dimensional Brownian motion. The proof of the above relation is lengthy and difficult. A key idea is the use of the projective tensor product $C \otimes_\pi C$ in order to view the continuous *bilinear* form β as a continuous *linear* functional on $C \otimes_\pi C$. At this level β commutes with the (Bochner) expectation. Rest of computation is effected using Mercer's theorem and some Fourier analysis. See [Mo], 1984, pp. 88-94. □

Theorem II.3.([Mo], Pitman Books, 1984)

In (XIV) suppose H and G are globally bounded and Lipschitz. Let $S : D(S) \subset C_b \to C_b$ be the weak generator of $\{S_t\}$. Suppose $\phi \in D(S)$ is sufficiently smooth (e.g. ϕ is C^2, $D\phi$, $D^2\phi$ globally bounded and Lipschitz). Then $\phi \in D(A)$ and

$$A(\phi)(\eta) = S(\phi)(\eta) + \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big) + \frac{1}{2}\sum_{i=1}^{m} \overline{D^2\phi(\eta)}\big(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}\big).$$

where $\{e_i\}_{i=1}^m$ is any basis for $\mathbf{R}^m$.

Proof.

Step 1.

For fixed $\eta \in C$, use Taylor's theorem:

$$\phi({}^\eta x_t) - \phi(\eta) = \phi(\tilde{\eta}_t) - \phi(\eta) + D\phi(\tilde{\eta}_t)({}^\eta x_t - \tilde{\eta}_t) + R(t)$$

a.s. for $t > 0$; where

$$R(t) := \int_0^1 (1-u) D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)]({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t)\, du.$$

Take expectations and divide by $t > 0$:

$$\left.\begin{aligned}\frac{1}{t}E[\phi({}^\eta x_t) - \phi(\eta)] = \frac{1}{t}[S_t(\phi(\eta) - \phi(\eta)] + D\phi(\tilde{\eta}_t)\Big\{E[\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)]\Big\} \\ + \frac{1}{t}ER(t)\end{aligned}\right\} \tag{3}$$

for $t > 0$.

As $t \to 0+$, the first term on the RHS converges to $S(\phi)(\eta)$, because $\phi \in D(S)$.

Step 2.

Consider second term on the RHS of (3). Then

$$\begin{aligned}\lim_{t\to 0+}\left[E\left\{\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right\}\right](s) &= \begin{cases}\lim\limits_{t\to 0+}\dfrac{1}{t}\displaystyle\int_0^t E[H({}^\eta x_u)]\,du, & s = 0\\ 0 & -r \le s < 0.\end{cases}\\ &= [H(\eta)\chi_{\{0\}}](s), \qquad\qquad -r \le s \le 0.\end{aligned}$$

Since H is bounded, then $\|E\{\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\}\|_C$ is bounded in $t > 0$ and $\eta \in C$ (*Exercise*). Hence

$$w - \lim_{t\to 0+}\left[E\left\{\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right\}\right] = H(\eta)\chi_{\{0\}} \quad (\notin C).$$

Therefore by Lemma II.1 and the continuity of $D\phi$ at η:

$$\begin{aligned}\lim_{t\to 0+} D\phi(\tilde{\eta}_t)\left\{E\left[\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right]\right\} &= \lim_{t\to 0+} D\phi(\eta)\left\{E\left[\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right]\right\}\\ &= \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big)\end{aligned}$$

Step 3.

To compute limit of third term in RHS of (3), consider

$$\begin{aligned}
&\Big| \frac{1}{t} E D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)]({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t) \\
&\quad - \frac{1}{t} E D^2\phi(\eta)({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t)\Big| \\
&\quad \le (E\|D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)] - D^2\phi(\eta)\|^2)^{1/2} \left[\frac{1}{t^2} E\|{}^\eta x_t - \tilde{\eta}_t\|^4\right]^{1/2} \\
&\quad \le K(t^2+1)^{1/2}[E\|D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)] - D^2\phi(\eta)\|^2]^{1/2} \\
&\quad \to 0
\end{aligned}$$

as $t \to 0+$, uniformly for $u \in [0,1]$, by martingale properties of the Itô integral and the Lipschitz continuity of $D^2\phi$. Therefore by Lemma II.3

$$\begin{aligned}
\lim_{t\to 0+} \frac{1}{t} ER(t) &= \int_0^1 (1-u) \lim_{t\to 0+} \frac{1}{t} E D^2\phi(\eta)({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t)\, du \\
&= \frac{1}{2} \sum_{i=1}^m \overline{D^2\phi(\eta)}\big(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}\big).
\end{aligned}$$

The above is a weak limit since $\phi \in D(S)$ and has first and second derivatives globally bounded on C. □

5. Quasitame Functions

Recall that a function $\phi : C \to \mathbf{R}$ is *tame* (or a *cylinder function*) if there is a finite set $\{s_1 < s_2 < \cdots < s_k\}$ in $[-r, 0]$ and a C^∞-bounded function $f : (\mathbf{R}^d)^k \to \mathbf{R}$ such that

$$\phi(\eta) = f(\eta(s_1), \cdots, \eta(s_k)), \qquad \eta \in C.$$

The set of all tame functions is a weakly dense subalgebra of C_b, invariant under the static shift S_t and generates $Borel\, C$. For $k \ge 2$ the tame function ϕ *lies outside* the domain of strong continuity C_b^0 of P_t, and hence *outside* $D(A)$ ([Mo], Pitman Books, 1984, pp.98-103; see also proof of Theorem IV .2.2, pp. 73-76). To overcome this difficulty we introduce

Definition.

Say $\phi : C \to \mathbf{R}$ is *quasitame* if there are C^∞-bounded maps $h : (\mathbf{R}^d)^k \to \mathbf{R},\ f_j : \mathbf{R}^d \to \mathbf{R}^d$, and piecewise C^1 functions $g_j : [-r, 0] \to \mathbf{R}, 1 \leq j \leq k-1$, such that

$$\phi(\eta) = h\left(\int_{-r}^0 f_1(\eta(s))g_1(s)\,ds, \cdots, \int_{-r}^0 f_{k-1}(\eta(s))g_{k-1}(s)\,ds, \eta(0)\right) \tag{4}$$

for all $\eta \in C$.

Theorem II.4. ([Mo], Pitman Books, 1984)

The set of all quasitame functions is a weakly dense subalgebra of C_b^0, invariant under S_t, generates Borel C and belongs to $D(A)$. In particular, if ϕ is the quasitame function given by (4), then

$$\begin{aligned} A(\phi)(\eta) = \sum_{j=1}^{k-1} D_j h(m(\eta))\{&f_j(\eta(0))g_j(0) - f_j(\eta(-r))g_j(-r) \\ &- \int_{-r}^0 f_j(\eta(s))g_j'(s)\,ds\} \\ &+ D_k h(m(\eta))(H(\eta)) + \frac{1}{2}\text{trace}[D_k^2 h(m(\eta)) \circ (G(\eta) \times G(\eta))]. \end{aligned}$$

for all $\eta \in C$, where

$$m(\eta) := \left(\int_{-r}^0 f_1(\eta(s))g_1(s)\,ds, \cdots, \int_{-r}^0 f_{k-1}(\eta(s))g_{k-1}(s)\,ds, \eta(0)\right).$$

Remarks.

(i) Replace C by the Hilbert space M_2. No need for the weak extensions because M_2 is weakly complete. Extensions of $D\phi(v, \eta)$ and $D^2\phi(v, \eta)$ correspond to partial derivatives in the $\mathbf{R}^d$-variable. *Tame functions do not exist on M_2* but quasitame functions do! (with $\eta(0)$ replaced by $v \in \mathbf{R}^d$).

Analysis of supermartingale behavior and stability of $\phi({}^\eta x_t)$ given in Kushner ([Ku], JDE, 1968). Infinite fading memory setting by Mizel and Trützer ([M-T], JIE, 1984) in the weighted state space $\mathbf{R}^d \times L^2((-\infty, 0], \mathbf{R}^d; \rho)$.

(ii) For each quasitame ϕ on C, $\phi({}^\eta x_t)$ *is* a semimartingale, and the Itô formula holds:

$$d[\phi({}^\eta x_t)] = A(\phi)({}^\eta x_t)\, dt + \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big)\, dW(t).$$

III. THE STABLE MANIFOLD THEOREM

FOR

STOCHASTIC SYSTEMS WITH MEMORY

Campinas, Brazil

June 28, 2002

Salah-Eldin A. Mohammed

Southern Illinois University

Carbondale, IL 62901–4408 USA

Web site: **http://sfde.math.siu.edu**

Outline

- Smooth cocycles in Hilbert space. Stationary trajectories.
- Linearization of a cocycle along a stationary trajectory.
- Ergodic theory of cocycles in Hilbert space.
- Hyperbolicity of stationary trajectories. Lyapunov exponents.
- Cocycles generated by stochastic systems with memory. Via random diffeomorphism groups.
- The *Local Stable Manifold Theorem* for stochastic differential equations with memory (SFDE's): Existence of smooth stable and unstable manifolds in a neighborhood of a hyperbolic stationary trajectory.
- Proofs based on Ruelle-Oseledec (non-linear) multiplicative ergodic theory and perfection techniques.

The Cocycle

$(\Omega, \mathcal{F}, P) :=$ complete probability space.

$\theta : \mathbf{R}^+ \times \Omega \to \Omega$ a P-preserving (ergodic) semigroup on $(\Omega, \mathcal{F}, P)$.

$E :=$ real (separable) Hilbert space, norm $\|\cdot\|$, Borel σ-algebra.

Definition.

Let k be a non-negative integer and $\epsilon \in (0, 1]$. A $C^{k,\epsilon}$ *perfect cocycle* (X, θ) on E is a measurable random field $X : \mathbf{R}^+ \times E \times \Omega \to E$ such that:

(i) For each $\omega \in \Omega$, the map $\mathbf{R}^+ \times E \ni (t, x) \mapsto X(t, x, \omega) \in E$ is continuous; for fixed $(t, \omega) \in \mathbf{R}^+ \times \Omega$, the map $E \ni x \mapsto X(t, x, \omega) \in E$ is $C^{k,\epsilon}$.

(ii) $X(t+s, \cdot, \omega) = X(t, \cdot, \theta(s, \omega)) \circ X(s, \cdot, \omega)$ for all $s, t \in \mathbf{R}^+$ and all $\omega \in \Omega$.

(iii) $X(0, x, \omega) = x$ for all $x \in E, \omega \in \Omega$.

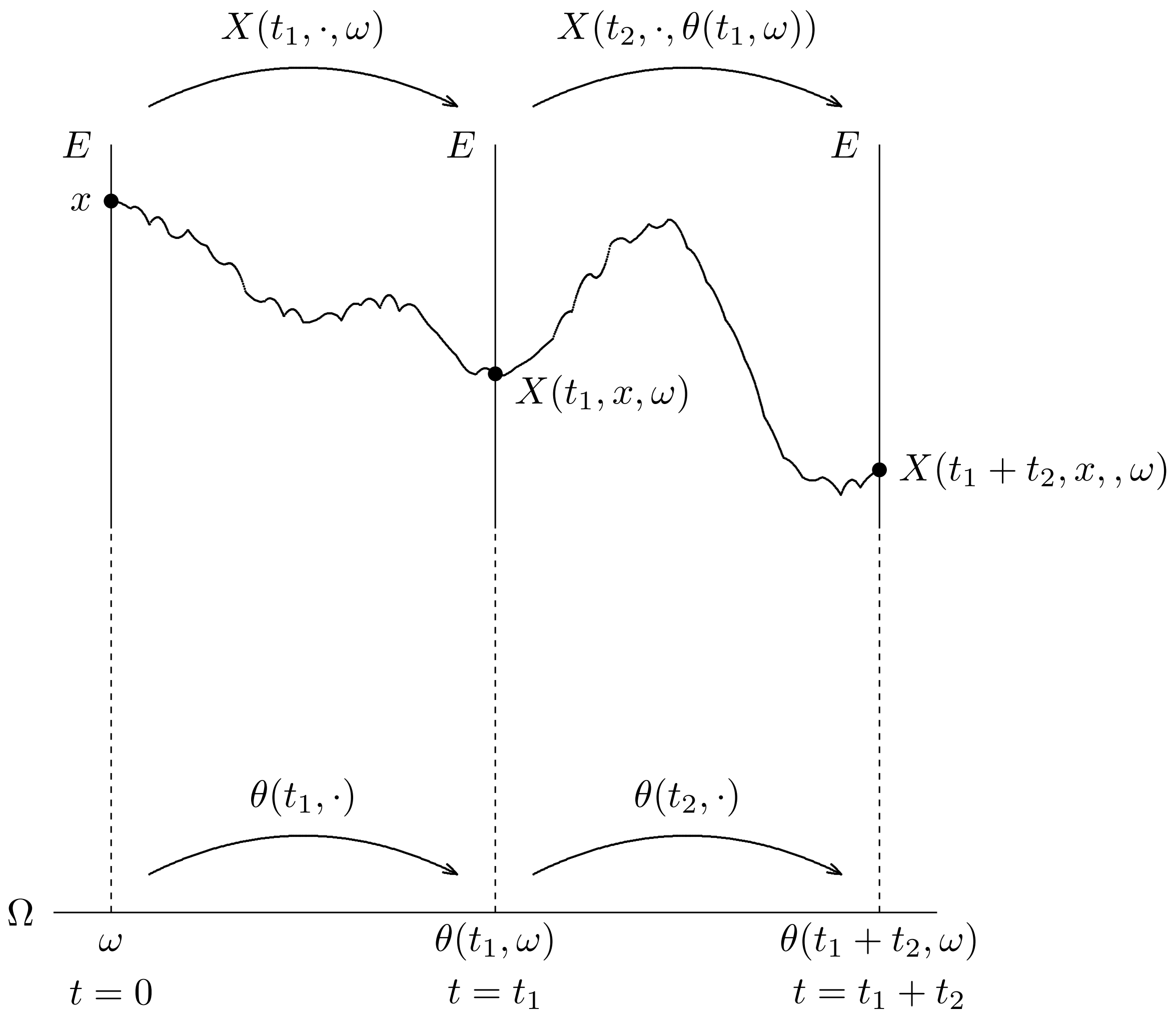


Figure illustrates the cocycle property. Vertical solid lines represent random fibers consisting of copies of E. (X, θ) is a "vector-bundle morphism".

Definition

The cocycle (X, θ) has a *stationary point* if there exists a random variable $Y : \Omega \to E$ such that

$$X(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{1}$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $X(t, Y) = Y(\theta(t))$.

Linearization. Hyperbolicity.

Linearize a $C^{k,\epsilon}$ cocycle (X,θ) along a stationary random point Y: Get an $L(E)$-valued cocycle $(DX(t,Y(\omega),\omega),\theta(t,\omega))$. (Follows from cocycle property of X and chain rule.)

Theorem. *(Oseledec-Ruelle)*

Let $T:\mathbf{R}^+\times\Omega\to L(E)$ be strongly measurable, such that (T,θ) is an $L(E)$-valued cocycle, with each $T(t,\omega)$ compact. Suppose that

$$E\sup_{0\le t\le 1}\log^+\|T(t,\cdot)\|_{L(E)}<\infty,\quad E\sup_{0\le t\le 1}\log^+\|T(1-t,\theta(t,\cdot))\|_{L(E)}<\infty.$$

Then there is a sure event $\Omega_0\in\mathcal{F}$ such that $\theta(t,\cdot)(\Omega_0)\subseteq\Omega_0$ for all $t\in\mathbf{R}^+$, and for each $\omega\in\Omega_0$, the limit

$$\lim_{n\to\infty}[T(t,\omega)^*\circ T(t,\omega)]^{1/(2t)}:=\Lambda(\omega)$$

exists in the uniform operator norm. $\Lambda(\omega)$ is self-adjoint with a non-random spectrum

$$e^{\lambda_1}>e^{\lambda_2}>e^{\lambda_3}>\cdots$$

where the λ_i's are distinct. Each e^{λ_i} has a fixed finite non-random multiplicity m_i and eigen-space $F_i(\omega)$, with $m_i := dimF_i(\omega)$. Define

$$E_1(\omega) := E, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \ i > 1.$$

Then

$$\cdots \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$\lim_{t\to\infty} \frac{1}{t} \log \|T(t,\omega)x\| = \lambda_i(\omega) \quad \textit{if} \quad x \in E_i(\omega)\backslash E_{i+1}(\omega),$$

and

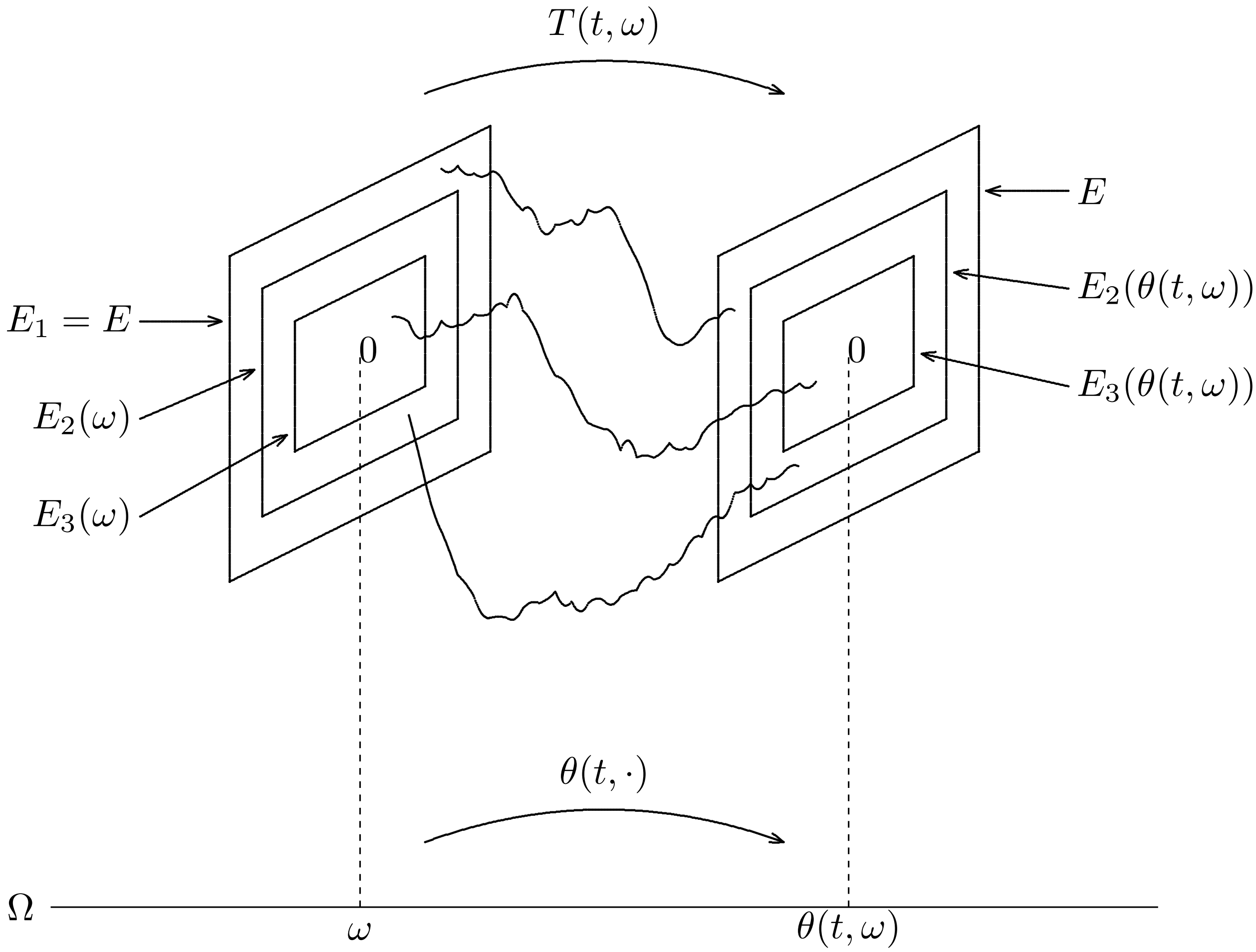
$$T(t,\omega)(E_i(\omega)) \subseteq E_i(\theta(t,\omega))$$

for all $t \geq 0$, $i \geq 1$.

Proof.

Based on the discrete version of Oseledec's multiplicative ergodic theorem and the perfect ergodic theorem. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]). □

Spectral Theorem



Definition

A stationary point $Y(\omega)$ of (I) is said to be *hyperbolic* if the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t,\omega))$ has a nonvanishing Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, viz. $\lambda_i \neq 0$ for all $i \geq 1$.

Let $i_0 > 1$ be such that $\lambda_{i_0} < 0 < \lambda_{i_0 - 1}$.

Suppose

$$E \log^+ \sup_{0 \leq t_1, t_2 \leq r} \|D_2 X(t_2, Y(\theta(t_1)), \theta(t_1))\|_{L(M_2)}) < \infty.$$

By Oseledec-Ruelle Theorem, there is a sequence of closed finite-codimen-sional (Oseledec) spaces

$$\cdots E_{i-1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$E_i(\omega) = \{(v, \eta) \in M_2 : \lim_{t \to \infty} \frac{1}{t} \log \|DX(t, Y(\omega), \omega)(v, \eta)\| \leq \lambda_i\}, \quad i \geq 1,$$

for all $\omega \in \Omega^*$, a sure event in $\mathcal{F}$ satisfying $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$.

Denote by $\{U(\omega), S(\omega) : \omega \in \Omega^*\}$ the unstable and stable subspaces associated with the linearized cocycle (DX, θ) as given by ([Mo.1], Theorem 4, Corollary 2) and ([M-S.1], Theorem 5.3). Then get a measurable invariant splitting

$$E = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \Omega^*,$$

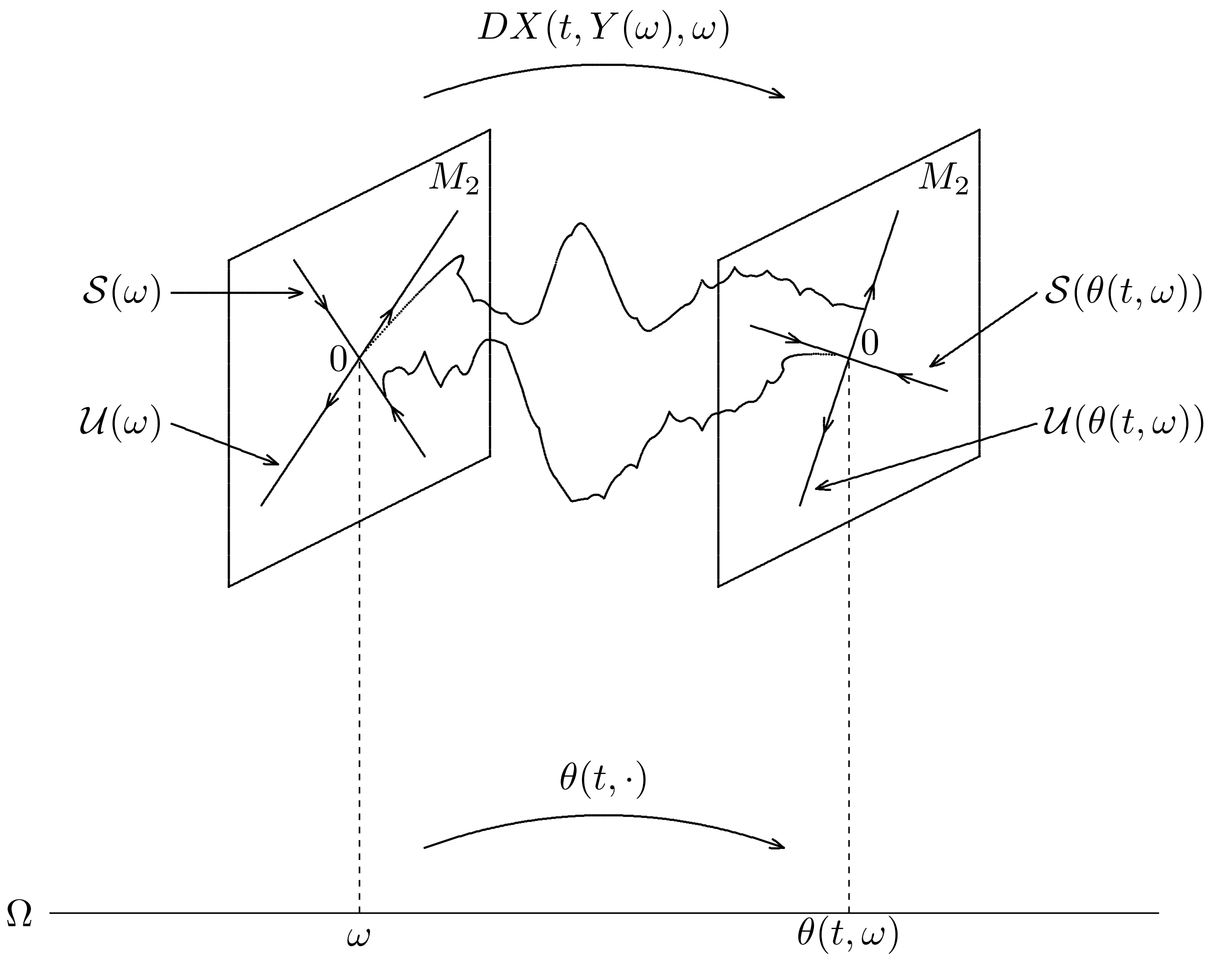
$$DX(t,Y(\omega),\omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t,\omega)),\ DX(t,Y(\omega),\omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)),$$

for all $t \geq 0$, together with the exponential dichotomies

$$\|DX(t,Y(\omega),\omega)(x)\| \geq \|x\| e^{\delta_1 t} \quad \text{for all} \quad t \geq \tau_1^*, x \in \mathcal{U}(\omega),$$

$$\|DX(t,Y(\omega),\omega)(x)\| \leq \|x\| e^{-\delta_2 t} \quad \text{for all} \quad t \geq \tau_2^*, x \in \mathcal{S}(\omega),$$

where $\tau_i^* = \tau_i^*(x,\omega) > 0, i = 1,2,$ are random times and $\delta_i > 0, i = 1,2,$ are fixed.



$DX(t, Y(\omega), \omega)$
M_2
M_2
$\mathcal{S}(\omega)$
$\mathcal{S}(\theta(t,\omega))$
0
0
$\mathcal{U}(\omega)$
$\mathcal{U}(\theta(t,\omega))$
$\theta(t,\cdot)$
Ω
ω
$\theta(t,\omega)$

Stochastic Systems with Memory

"Regular" Itô SFDE with finite memory:

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t)\,dt + \sum_{i=1}^{m} G_i(x(t))\,dW_i(t), \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (I)$$

Solution segment $x_t(s) := x(t+s),\ t \geq 0, s \in [-r, 0]$.

m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$, $W(0) = 0$.

Ergodic Brownian shift θ on Wiener space $(\Omega, \mathcal{F}, P)$. $\bar{\mathcal{F}} := P-$completion of $\mathcal{F}$.

State space M_2, Hilbert with usual norm $\|\cdot\|$.

Can allow for "smooth memory" in diffusion coefficient.

$H : M_2 \to \mathbf{R}^d$ of class $C^{k,\delta}$, globally bounded.

$G : \mathbf{R}^d \to L(\mathbf{R}^p, \mathbf{R}^d)$ is of class $C_b^{k+1,\delta}$.

$B((v, \eta), \rho)$ open ball of radius ρ and center $(v, \eta) \in M_2$;

$\bar{B}((v,\eta),\rho)$ corresponding closed ball.

Then (I) has a stochastic semiflow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ with $X(t,(v,\eta),\cdot) = (x(t),x_t)$. X is of class $C^{k,\epsilon}$ for any $\epsilon \in (0,\delta)$, takes bounded sets into relatively compact sets in M_2. (X,θ) is a perfect cocycle on M_2 ([M-S.4]).

Theorem. *([M-S], 1999) (Local Stable and Unstable Manifolds)*

Assume smoothness hypotheses on H and G. Let $Y : \Omega \to M_2$ be a hyperbolic stationary point of the SFDE (I) such that $E(\|Y(\cdot)\|^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$

Suppose the linearized cocycle $(DX(t,Y(\omega),\omega),\theta(t,\omega),t \geq 0)$ of (I) has a Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. Define $\lambda_{i_0} := max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all finite λ_i are positive, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized semiflow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Fix $\epsilon_1 \in (0,-\lambda_{i_0})$ and $\epsilon_2 \in (0,\lambda_{i_0-1})$. Then there exist

(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,

(ii) $\mathcal{F}$*-measurable random variables* $\rho_i, \beta_i : \Omega^* \to (0,1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, *such that for each* $\omega \in \Omega^*$, *the following is true:*

There are $C^{k,\epsilon}$ *(*$\epsilon \in (0, \delta)$*) submanifolds* $\tilde{\mathcal{S}}(\omega)$, $\tilde{\mathcal{U}}(\omega)$ *of* $\bar{B}(Y(\omega), \rho_1(\omega))$ *and* $\bar{B}(Y(\omega), \rho_2(\omega))$ *(resp.) with the following properties:*

(a) $\tilde{\mathcal{S}}(\omega)$ *is the set of all* $(v, \eta) \in \bar{B}(Y(\omega), \rho_1(\omega))$ *such that*

$$\|X(n, (v, \eta), \omega) - Y(\theta(n, \omega))\| \leq \beta_1(\omega)\, e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers $n \geq 0$. *Furthermore,*

$$\limsup_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega) - Y(\theta(t, \omega))\| \leq \lambda_{i_0}$$

for all $(v, \eta) \in \tilde{\mathcal{S}}(\omega)$. *Each stable subspace* $\mathcal{S}(\omega)$ *of the linearized semiflow* DX *is tangent at* $Y(\omega)$ *to the submanifold* $\tilde{\mathcal{S}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. *In particular, codim* $\tilde{\mathcal{S}}(\omega) =$ *codim* $\mathcal{S}(\omega)$, *is fixed and finite.*

(b) $\displaystyle \limsup_{t \to \infty} \frac{1}{t} \log \left[\sup \left\{ \frac{\|X(t, (v_1, \eta_1), \omega) - X(t, (v_2, \eta_2), \omega)\|}{\|(v_1, \eta_1) - (v_2, \eta_2)\|} : (v_1, \eta_1) \neq (v_2, \eta_2),\ (v_1, \eta_1),\ (v_2, \eta_2) \in \tilde{\mathcal{S}}(\omega) \right\} \right] \leq \lambda_{i_0}.$

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ *such that*

$$X(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega))$$

for all $t \geq \tau_1(\omega)$. *Also*

$$DX(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)), \quad t \geq 0.$$

(d) $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $(v, \eta) \in \bar{B}(Y(\omega), \rho_2(\omega))$ *with the property that there is a unique "history" process* $y(\cdot, \omega) : \{-nr : n \geq 0\} \to M_2$ *such that* $y(0, \omega) = (v, \eta)$ *and for each integer* $n \geq 1$, *one has* $X(r, y(-nr, \omega), \theta(-nr, \omega)) = y(-(n-1)r, \omega)$ *and*

$$\|y(-nr, \omega) - Y(\theta(-nr, \omega))\|_{M_2} \leq \beta_2(\omega) e^{-(\lambda_{i_0 - 1} - \epsilon_2)nr}.$$

Furthermore, for each $(v, \eta) \in \tilde{\mathcal{U}}(\omega)$, *there is a unique continuous-time "history" process also denoted by* $y(\cdot, \omega) : (-\infty, 0] \to M_2$ *such that* $y(0, \omega) = (v, \eta)$, $X(t, y(s, \omega), \theta(s, \omega)) = y(t + s, \omega)$ *for all* $s \leq 0, 0 \leq t \leq -s$, *and*

$$\limsup_{t \to \infty} \frac{1}{t} \log \|y(-t, \omega) - Y(\theta(-t, \omega))\| \leq -\lambda_{i_0 - 1}.$$

Each unstable subspace $\mathcal{U}(\omega)$ *of the linearized semiflow* DX *is tangent at* $Y(\omega)$ *to* $\tilde{\mathcal{U}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. *In particular,* $\dim \tilde{\mathcal{U}}(\omega)$ *is finite and non-random.*

(e) *Let* $y(\cdot,(v_i,\eta_i),\omega), i = 1,2$, *be the history processes associated with* $(v_i,\eta_i) = y(0,(v_i,\eta_i),\omega) \in \tilde{\mathcal{U}}(\omega)$, $i = 1,2$. *Then*

$$\limsup_{t\to\infty} \frac{1}{t}\log\Bigg[\sup\Bigg\{\frac{\|y(-t,(v_1,\eta_1),\omega) - y(-t,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1)-(v_2,\eta_2)\|} : (v_1,\eta_1) \neq (v_2,\eta_2),\ (v_i,\eta_i) \in \tilde{\mathcal{U}}(\omega), i = 1,2\Bigg\}\Bigg] \leq -\lambda_{i_0-1}.$$

(f) *(Cocycle-invariance of the unstable manifolds):*

There exists $\tau_2(\omega) \geq 0$ *such that*

$$\tilde{\mathcal{U}}(\omega) \subseteq X(t,\cdot,\theta(-t,\omega))(\tilde{\mathcal{U}}(\theta(-t,\omega)))$$

for all $t \geq \tau_2(\omega)$. *Also*

$$DX(t,\cdot,\theta(-t,\omega))(\mathcal{U}(\theta(-t,\omega))) = \mathcal{U}(\omega), \quad t \geq 0;$$

and the restriction

$$DX(t,\cdot,\theta(-t,\omega))|\mathcal{U}(\theta(-t,\omega)) : \mathcal{U}(\theta(-t,\omega)) \to \mathcal{U}(\omega), \quad t \geq 0,$$

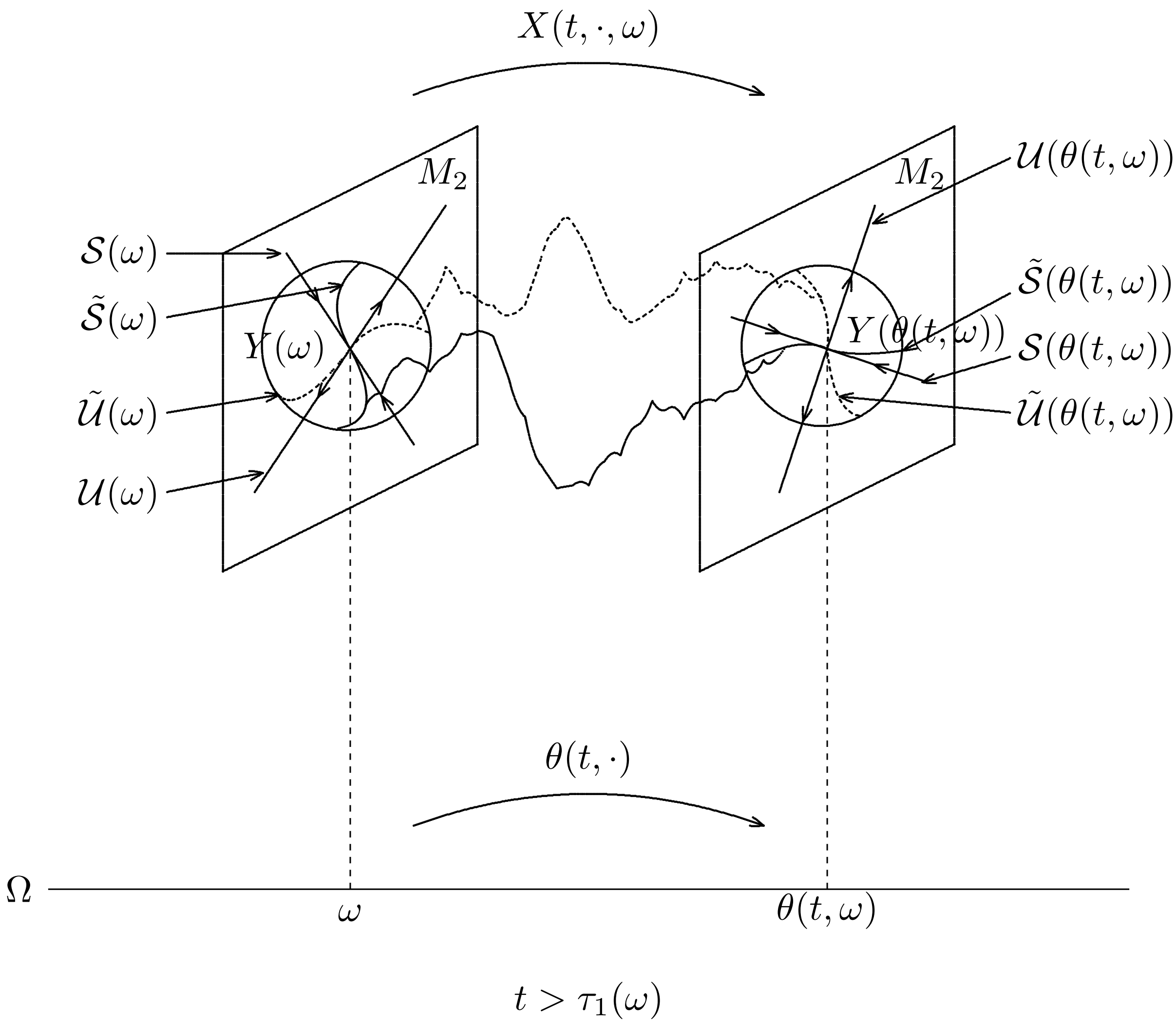
is a linear homeomorphism onto.

(g) *The submanifolds $\tilde{\mathcal{U}}(\omega)$ and $\tilde{\mathcal{S}}(\omega)$ are transversal, viz.*

$$M_2 = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

Assume, in addition, that H, G are C_b^∞. Then the local stable and unstable manifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞.

Figure summarizes essential features of Stable Manifold Theorem:



A picture is worth a 1000 words!

Example

Consider the affine linear sfde

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t)\, dt + G\, dW(t), \quad t > 0 \\ x(0) &= v \in \mathbf{R}^d, \quad x_0 = \eta \in L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (I'')$$

where $H : M_2 \to \mathbf{R}^d$ is a continuous linear map, G is a fixed $(d \times p)$-matrix, and W is p-dimensional Brownian motion. Assume that the linear deterministic $(d \times d)$-matrix-valued FDE

$$dy(t) = H \circ (y(t), y_t)\, dt$$

has a semiflow

$$T_t : L(\mathbf{R}^d) \times L^2([-r,0], L(\mathbf{R}^d)) \to L(\mathbf{R}^d) \times L^2([-r,0], L(\mathbf{R}^d)), t \geq 0,$$

which is uniformly asymptotically stable. Set

$$Y := \int_{-\infty}^{0} T_{-u}(I, 0) G\, dW(u) \qquad (2)$$

where I is the identity $(d \times d)$-matrix. Integration by parts and

$$W(t, \theta(t_1, \omega)) = W(t + t_1, \omega) - W(t_1, \omega), \quad t, t_1 \in \mathbf{R}, \qquad (3)$$

imply that Y has a measurable version satisfying (1). Y is Gaussian and thus has finite moments of all orders. See

([Mo.1], Theorem 4.2, Corollary 4.2.1, pp. 208-217.) More generally, when H is hyperbolic, one can show that a stationary point of (I'') exists ([Mo.1]).

In the general white-noise case an invariant measure on M_2 for the one-point motion gives rise to a stationary point provided we suitably enlarge the underlying probability space. Conversely, let $Y : \Omega \to M_2$ be a stationary random point independent of the Brownian motion $W(t), t \geq 0$. Let $\rho := P \circ Y^{-1}$ be the distribution of Y. By independence of Y and W, ρ is an invariant measure for the one-point motion

Outline of Proof

- By definition, a *stationary* random point $Y(\omega) \in M_2$ is invariant under the semiflow X; viz $X(t, Y) = Y(\theta(t, \cdot))$ for all times t.
- We linearize the semiflow X along the stationary point $Y(\omega)$ in M_2. In view of the stationarity of Y and the cocycle property of X, this gives a linear perfect cocycle $(DX(t, Y), \theta(t, \cdot))$ in $L(M_2)$, where D denotes spatial (Fréchet) derivatives.
- Ergodicity of θ allows for the notion of *hyperbolicity* of a stationary solution of (I) via Oseledec-Ruelle theorem: Use local compactness of the semiflow for times greater than the delay r ([M-S.4]), and apply multiplicative ergodic theorem in order to yield a discrete non-random Lyapunov spectrum $\{\lambda_i : i \geq 1\}$ for the linearized cocycle. Y is *hyperbolic* if $\lambda_i \neq 0$ for every i.
- Assuming that $\|Y\|^{\epsilon_0}$ is integrable (for small ϵ_0) and using the variational method of construction of the semiflow, we show that the linearized cocycle satisfies the hypotheses for "perfect versions" of ergodic theorem and Kingman's subadditive ergodic theorem. These refined versions yield invariance of the Oseledec

spaces under the continuous-time linearized cocycle. In particular, the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear semiflow X.

- We establish continuous-time integrability estimates on the spatial derivatives of the non-linear cocycle X in a neighborhood of the stationary point Y. These esitmates follow from the variational construction of the stochastic semiflow coupled with known global spatial estimates for finite-dimensional stochastic flows.
- We introduce the auxiliary perfect cocycle

$$Z(t,\cdot,\omega) := X(t,(\cdot)+Y(\omega),\omega) - Y(\theta(t,\omega)), \quad t \in \mathbf{R}^+, \omega \in \Omega.$$

By refining the arguments in ([Ru.2], Theorems 5.1 and 6.1), we construct local stable/unstable manifolds for the discrete cocycle $(Z(nr,\cdot,\omega), \theta(nr,\omega))$ near 0 and hence (by translation) for $X(nr,\cdot,\omega)$ near $Y(\omega)$ for all ω sampled from a $\theta(t,\cdot)$-invariant sure event in Ω. This is possible because of the continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between delay periods of length r and further refining the arguments in [Ru.2], we then show that the above manifolds also serve as local stable/unstable manifolds for the *continuous-time* semiflow X near Y.

- The final key step is to establish the asymptotic invariance of the local stable manifolds under the stochastic semiflow X. This is achieved by appealing to the arguments underlying the proofs of Theorems 4.1 and 5.1 in [Ru.2] and some difficult estimates using the continuous-time integrability properties, and the perfect subadditive ergodic theorem. The asymptotic invariance of the local unstable manifolds follows by employing the concept of a *stochastic history process* for X coupled with similar arguments to the above. The existence of the history process compensates for the lack of invertibility of the semiflow.