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Stochastic Differential Systems with Memory (Spring School on Stochastic Delay Differential Equations)

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STOCHASTIC DIFFERENTIAL SYSTEMS

WITH MEMORY*

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Abstract

This series of talks is intended as an introduction to certain aspects of stochastic differential systems, whose evolution depends on the history of the state. We shall frequently refer to such systems as *stochastic functional differential equations (sfde's)*. In the deterministic case, sfde's reduce to *retarded functional differential equations (rfde's)*. Such equations have received a great deal of attention by analysts during the last few decades. The reader may refer to fundamental works by J. Hale, Mallet-Paret, Mizel, etc.. The lectures will cover some of the following topics as much as time permits.

Part I: Existence.

Simple motivating examples: *the noisy feedback loop*, *the logistic time-lag model with Gaussian noise*, and *the classical "heat-bath" model of R. Kubo*, modeling the motion of a "large" molecule in a viscous fluid. These examples are embedded in a general class of stochastic functional differential equations (sfde's). Pathwise existence and uniqueness of solutions to these classes of sfde's under local Lipschitz and linear growth hypotheses on the coefficients. Existence of solutions under smooth constraints.

Part II: Markov Behavior.

The Markov (Feller) property holds for the trajectory random field of a sfde. The trajectory Markov semigroup is *not* strongly continuous for positive delays, and its domain of strong continuity does not contain tame (or cylinder) functions with evaluations away from 0. Quasitame functions. The weak infinitesimal generator.

Part III: Classification of SFDE's.

Non-existence of stochastic semiflows for SDDE's. Classification of sfde's into *regular* and *singular* types. Sufficient conditions for regularity of linear systems driven by white noise or semimartingales.

Part IV: Dynamics of Regular SFDE's.

Linear sfde's. Existence of a compacting stochastic semiflow. The multiplicative ergodic theory for regular linear sfde's. The saddle point property. Examples of one-dimensional linear sfde's: Estimates for the top Lyapunov exponent.

Nonlinear sfde's. Existence of semiflows. The local stable manifold theorem.

Part V: Miscellaneous Topics.

Existence of smooth densities for solutions of sfde's using the Malliavin calculus. Numerical solution. Small delays. Existence of stationary solutions. Applications to finance: The delayed Black-Scholes formula.

I. EXISTENCE

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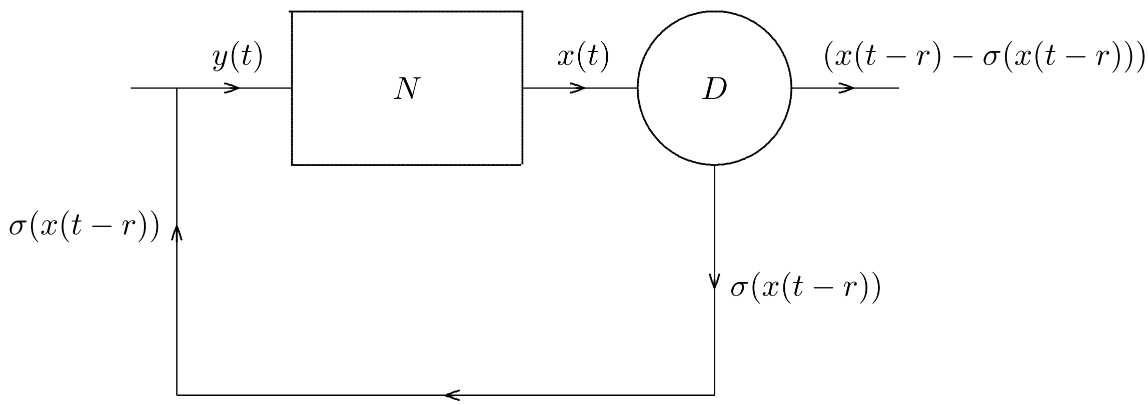
I. EXISTENCE

1. Examples

Example 1. (*Noisy Feedbacks*)



Box N: Input $= y(t)$, output $= x(t)$ at time $t > 0$ related by

$$x(t) = x(0) + \int_0^t y(u)\, dZ(u) \tag{1}$$

where $Z(u)$ is a semimartingale noise.

Box D: Delays signal $x(t)$ by $r\,(>0)$ units of time. A proportion σ $(0 \le \sigma \le 1)$ is transmitted through D and the rest $(1-\sigma)$ is used for other purposes.

Therefore

$$y(t) = \sigma x(t-r)$$

Take $\dot{Z}(u) :=$ white noise $= \dot{W}(u)$

Then substituting in (1) gives the Itô integral equation

$$x(t) = x(0) + \sigma \int_0^t x(u-r) dW(u)$$

or the stochastic differential delay equation (sdde):

$$dx(t) = \sigma x(t-r)dW(t), \qquad t > 0 \tag{I}$$

To solve (I), need an *initial process* $\theta(t)$, $-r \le t \le 0$:

$$x(t) = \theta(t) \quad \text{a.s.}, \qquad -r \le t \le 0$$

$\mathbf{r = 0}$: (I) becomes a linear stochastic ode and has closed form solution

$$x(t) = x(0)e^{\sigma W(t) - \frac{\sigma^2 t}{2}}, \qquad t \ge 0.$$

$\mathbf{r>0}$: Solve (I) by successive Itô integrations over steps of length r:

$$\begin{aligned}
x(t) &= \theta(0) + \sigma \int_0^t \theta(u-r)\, dW(u), \quad 0 \le t \le r \\
x(t) &= x(r) + \sigma \int_r^t [\theta(0) + \sigma \int_0^{(v-r)} \theta(u-r)\, dW(u)]\, dW(v),\ r < t \le 2r, \\
\cdots &= \cdots \qquad\qquad 2r < t \le 3r,
\end{aligned}$$

No closed form solution is known (even in deterministic case).

Curious Fact!

In the sdde (I) the Itô differential dW may be replaced by the Stratonovich differential $\circ dW$ *without changing the solution* x. Let x be the solution of (I) under an Itô differential dW. Then using finite partitions $\{u_k\}$ of the interval $[0, t]$:

$$\int_0^t x(u-r) \circ dW(t) = \lim \sum_k \frac{1}{2}[x(u_k - r) + x(u_{k+1} - r)][W(u_{k+1}) - W(u_k)]$$

where the limit in probability is taken as the mesh of the partition $\{u_k\}$ goes to zero. Compare the Stratonovich and Itô integrals using the corresponding partial sums:

$$\begin{aligned}
\lim E\bigg(&\sum_k \frac{1}{2}[x(u_k - r) + x(u_{k+1} - r)][W(u_{k+1}) - W(u_k)] \\
&- \sum_k [x(u_k - r)][W(u_{k+1}) - W(u_k)]\bigg)^2 \\
&= \lim E\bigg(\sum_k \frac{1}{2}[x(u_{k+1} - r) - x(u_k - r)][W(u_{k+1}) - W(u_k)]\bigg)^2 \\
&= \lim \sum_k \frac{1}{4} E[x(u_{k+1} - r) - x(u_k - r)]^2 \, E[W(u_{k+1}) - W(u_k)]^2 \\
&= \lim \sum_k \frac{1}{4} E[x(u_{k+1} - r) - x(u_k - r)]^2 \, (u_{k+1} - u_k) \\
&= 0
\end{aligned}$$

because W has independent increments, x is adapted to the Brownian filtration, $u \mapsto x(u) \in L^2(\Omega, \mathbf{R})$ is continuous, and the delay r is positive. Alternatively

$$\int_0^t x(u - r) \circ dW(u) = \int_0^t x(u - r)\, dW(u) + \frac{1}{2} < x(\cdot - r, W > (t)$$

and $< x(\cdot - r, W > (t) = 0$ for all $t > 0$.

Remark.

When $r > 0$, the solution process $\{x(t) : t \geq -r\}$ of (I) is a martingale but is *non-Markov*.

Example 2. (*Simple Population Growth*)

Consider a large population $x(t)$ at time t evolving with a constant birth rate $\beta > 0$ and a constant death rate α per capita. Assume immediate removal of the dead from the population. Let $r > 0$ (fixed, non-random= 9, e.g.) be the development period of each individual and assume there is migration whose overall rate is distributed like white noise $\sigma\dot{W}$ (mean zero and variance $\sigma > 0$), where W is one-dimensional standard Brownian motion. The change in population $\Delta x(t)$ over a small time interval $(t, t+\Delta t)$ is

$$\Delta x(t) = -\alpha x(t)\Delta t + \beta x(t-r)\Delta t + \sigma\dot{W}\Delta t$$

Letting $\Delta t \to 0$ and using Itô stochastic differentials,

$$dx(t) = \{-\alpha x(t) + \beta x(t-r)\}\, dt + \sigma dW(t), \quad t > 0. \tag{II}$$

Associate with the above affine sdde the initial condition $(v, \eta) \in \mathbf{R} \times L^2([-r, 0], \mathbf{R})$

$$x(0) = v, \quad x(s) = \eta(s), \quad -r \le s < 0.$$

Denote by $M_2 = \mathbf{R} \times L^2([-r, 0], \mathbf{R})$ the Delfour-Mitter Hilbert space of all pairs (v, η), $v \in \mathbf{R}$, $\eta \in L^2([-r, 0], \mathbf{R})$ with norm

$$\|(v, \eta)\|_{M_2} = \left(|v|^2 + \int_{-r}^{0} |\eta(s)|^2\, ds \right)^{1/2}.$$

Let $W : \mathbf{R}^+ \times \Omega \to \mathbf{R}$ be defined on the canonical filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}^+}, P)$ where

$$\Omega = C(\mathbf{R}^+, \mathbf{R}),\ \mathcal{F} = \text{ Borel } \Omega,\ \mathcal{F}_t = \sigma\{\rho_u : u \le t\}$$

$\rho_u : \Omega \to \mathbf{R}, u \in \mathbf{R}^+$, are evaluation maps $\omega \mapsto \omega(u)$, and $P =$ Wiener measure on Ω.

Example 3. (*Logistic Population Growth*)

A single population $x(t)$ at time t evolving logistically with *development (incubation) period* $r > 0$ under Gaussian type noise (e.g. migration on a molecular level):

$$\dot{x}(t) = [\alpha - \beta x(t-r)]\, x(t) + \gamma x(t)\dot{W}(t), \quad t > 0$$

i.e.

$$dx(t) = [\alpha - \beta x(t-r)]\, x(t)\, dt + \gamma x(t) dW(t) \quad t > 0. \tag{III}$$

with *initial condition*

$$x(t) = \theta(t) \quad -r \le t \le 0.$$

For positive delay r the above sdde can be solved *implicitly* using forward steps of length r, i.e. for $0 \le t \le r$, $x(t)$ satisfies the *linear* sode (without delay)

$$dx(t) = [\alpha - \beta\theta(t-r)]\, x(t)\, dt + \gamma x(t) dW(t) \quad 0 < t \le r. \tag{III'}$$

$x(t)$ is a semimartingale and is non-Markov (Scheutzow [S], 1984).

Example 4. (*Heat bath*)

Model proposed by R. Kubo (1966) for physical Brownian motion. A molecule of mass m moving under random gas forces with position $\xi(t)$ and velocity $v(t)$ at time t; cf classical work by Einstein and Ornestein and Uhlenbeck. Kubo proposed the following modification of the Ornstein-Uhenbeck process

$$\left.\begin{aligned} d\xi(t) &= v(t)\, dt \\ mdv(t) &= -m\Big[\int_{t_0}^{t} \beta(t-t')v(t')\, dt'\Big]\, dt + \gamma(\xi(t), v(t))\, dW(t),\ t > t_0. \end{aligned}\right\} \tag{IV}$$

m =mass of molecule. No external forces.

$\beta =$ viscosity coefficient function with compact support.

γ a function $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ representing the random gas forces on the molecule.

$\xi(t) =$ position of molecule $\in \mathbf{R}^3$.

$v(t) =$ velocity of molecule $\in \mathbf{R}^3$.

$W = 3-$ dimensional Brownian motion.

([Mo], Pitman Books, RN # 99, 1984, pp. 223-226).

Further Examples

Delay equation with Poisson noise:

$$\left.\begin{aligned} dx(t) &= x((t-r)-)\,dN(t) \quad t>0 \\ x_0 &= \eta \in D([-r,0],\mathbf{R}) \end{aligned}\right\} \qquad (V)$$

$N :=$ Poisson process with iid interarrival times ([S], Hab. 1988). $D([-r,0],\mathbf{R}) =$ space of all cadlag paths $[-r,0] \to \mathbf{R}$, with sup norm.

Simple model of dye circulation in the blood (or pollution) (cf. Bailey and Williams [B-W], JMAA, 1966, Lenhart and Travis ([L-T], PAMS, 1986).

$$\left.\begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r))\}\,dt + \sigma x(t)\,dW(t) \quad t>0 \\ (x(0),x_0) &= (v,\eta) \in M_2 = \mathbf{R}\times L^2([-r,0],\mathbf{R}), \end{aligned}\right\} \qquad (VI)$$

([Mo], Survey, 1992; [M-S], II, 1995.)

In above model:

$x(t) :=$ dye concentration (gm/cc)

$r =$ time taken by blood to traverse side tube (vessel)

Flow rate (cc/sec) is Gaussian with variance σ.

A fixed proportion of blood in main vessel is pumped into side vessel(s). Model will be analysed in Lecture V (Theorem V.5).

$$\left.\begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r))\}\, dt + \{\int_{-r}^{0} x(t+s)\sigma(s)\, ds\}\, dW(t), \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R} \times L^2([-r,0], \mathbf{R}),\ t > 0. \end{aligned}\right\} \qquad (VII)$$

([Mo], Survey, 1992; [M-S], II, 1995.)

Linear d-dimensional systems driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$ with constant coefficients.

$$\left.\begin{aligned} dx(t) &= H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\ &\qquad + \sum_{i=1}^{m} g_i x(t)\, dW_i(t), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (VIII)$$

$H := (\mathbf{R}^d)^N \times M_2 \to \mathbf{R}^d$ linear functional on $(\mathbf{R}^d)^N \times M_2$; g_i $d \times d$-matrices ([Mo], Stochastics, 1990).

Linear systems driven by (helix) semimartingale noise (N, L), and memory driven by a (stationary) measure-valued process ν and a (stationary) process K ([M-S], I, AIHP, 1996):

$$\left.\begin{aligned} dx(t) &= \Big\{\int_{[-r,0]} \nu(t)(ds)\, x(t+s)\Big\}\, dt \\ &\quad + dN(t) \int_{-r}^{0} K(t)(s)\, x(t+s)\, ds + dL(t)\, x(t-), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (IX)$$

Multidimensional affine systems driven by (helix) noise Q ([M-S], Stochastics, 1990):

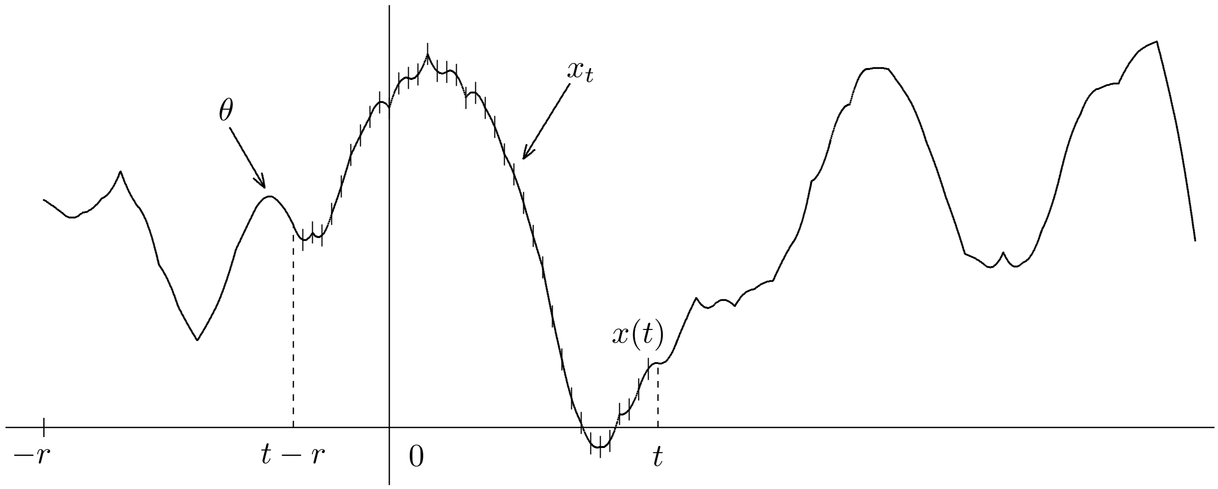
$$\left.\begin{aligned} dx(t) &= \Big\{\int_{[-r,0]} \nu(t)(ds)\, x(t+s)\Big\}\, dt + dQ(t), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (X)$$

Memory driven by white noise:

$$\left.\begin{array}{l} dx(t) = \left\{ \displaystyle\int_{[-r,0]} x(t+s)\, dW(s) \right\} dW(t) \qquad t > 0 \\ x(0) = v \in \mathbf{R}, \quad x(s) = \eta(s), \quad -r < s < 0, \quad r \geq 0 \end{array}\right\} \qquad (XI)$$

([Mo], Survey, 1992).

Formulation



Slice each solution path x over the interval $[t-r, t]$ to get *segment* x_t as a process on $[-r, 0]$:

$$x_t(s) := x(t+s) \qquad \text{a.s.}, \; t \geq 0, s \in J := [-r, 0].$$

Therefore sdde's (I), (II), (III) and (XI) become

$$\left.\begin{array}{r} dx(t) = \sigma x_t(-r) dW(t), \quad t > 0 \\ \\ x_0 = \theta \in C([-r,0], \mathbf{R}) \end{array}\right\} \qquad (I)$$

$$\left.\begin{array}{r} dx(t) = \{-\alpha x(t) + \beta x_t(-r)\}\, dt + \sigma dW(t), \quad t > 0 \\ \\ (x(0), x_0) = (v, \eta) \in \mathbf{R} \times L^2([-r,0], \mathbf{R}) \end{array}\right\} \qquad (II)$$

$$\left.\begin{aligned} dx(t) &= [\alpha - \beta x_t(-r)]x_t(0)\,dt + \gamma x_t(0)\,dW(t) \\ x_0 &= \theta \in C([-r,0],\mathbf{R}) \end{aligned}\right\} \qquad (III)$$

$$\left.\begin{aligned} dx(t) &= \left\{\int_{[-r,0]} x_t(s)\,dW(s)\right\} dW(t) \qquad t > 0 \\ (x(0), x_0) &= (v,\eta) \in \mathbf{R} \times L^2([-r,0],\mathbf{R}), \qquad r \geq 0 \end{aligned}\right\} \qquad (XI)$$

Think of R.H.S.'s of the above equations as functionals of x_t (and $x(t)$) and generalize to *stochastic functional differential equation* (sfde)

$$\left.\begin{aligned} dx(t) &= h(t,x_t)dt + g(t,x_t)dW(t) \qquad t > 0 \\ x_0 &= \theta \end{aligned}\right\} \qquad (XII)$$

on filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions:

$(\mathcal{F}_t)_{t\geq 0}$ right-continuous and each $\mathcal{F}_t$ contains all P-null sets in $\mathcal{F}$.

$C := C([-r,0],\mathbf{R}^d)$ Banach space, sup norm.

$W(t) = m$–dimensional Brownian motion.

$L^2(\Omega, C) :=$ Banach space of all $(\mathcal{F}, Borel C)$-measurable L^2 (Bochner sense) maps $\Omega \to C$ with the L^2-norm

$$\|\theta\|_{L^2(\Omega,C)} := \left[\int_\Omega \|\theta(\omega)\|_C^2 \, dP(\omega)\right]^{1/2}$$

Coefficients:

$$h : [0,T] \times L^2(\Omega, C) \to L^2(\Omega, \mathbf{R}^d) \qquad \text{(Drift)}$$

$$g : [0,T] \times L^2(\Omega, C) \to L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d) \qquad \text{(Diffusion)}.$$

Initial data:

$$\theta \in L^2(\Omega, C, \mathcal{F}_0).$$

Solution:

$x : [-r,T] \times \Omega \to \mathbf{R}^d$ measurable and sample-continuous, $x|[0,T]$ $(\mathcal{F}_t)_{0 \leq t \leq T}$-adapted and $x(s)$ is $\mathcal{F}_0$-measurable for all $s \in [-r, 0]$.

Exercise: $[0,T] \ni t \mapsto x_t \in C([-r,0], \mathbf{R}^d)$ is $(\mathcal{F}_t)_{0 \leq t \leq T}$-adapted.

(*Hint:* $Borel\, C$ is generated by all evaluations.)

Hypotheses (E_1)**.**

(i) h, g are jointly continuous and uniformly Lipschitz in the second variable with respect to the first:

$$\|h(t,\psi_1) - h(t,\psi_2)\|_{L^2(\Omega,\mathbf{R}^d)} \le L\|\psi_1 - \psi_2\|_{L^2(\Omega,C)}$$

for all $t \in [0,T]$ and $\psi_1, \psi_2 \in L^2(\Omega, C)$. Similarly for the diffusion coefficent g.

(ii) For each $(\mathcal{F}_t)_{0\le t\le T}$-adapted process $y : [0,T] \to L^2(\Omega, C)$, the processes $h(\cdot, y(\cdot)), g(\cdot, y(\cdot))$ are also $(\mathcal{F}_t)_{0\le t\le T}$- adapted.

Theorem I.1. ([Mo], 1984) (Existence and Uniqueness).

Suppose h and g satisfy Hypotheses (E_1). Let $\theta \in L^2(\Omega, C; \mathcal{F}_0)$.

Then the sfde (XII) has a unique solution ${}^\theta x : [-r,\infty) \times \Omega \to \mathbf{R}^d$ starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ with $t \longmapsto {}^\theta x_t$ continuous and ${}^\theta x \in L^2(\Omega, C([-r,T]\mathbf{R}^d))$ for all $T > 0$. For a given θ, uniqueness holds up to equivalence among all $(\mathcal{F}_t)_{0\le t\le T}$-adapted processes in $L^2(\Omega, C([-r,T], \mathbf{R}^d))$.

Proof.

[Mo], Pitman Books, 1984, Theorem 2.1, pp. 36-39. $\square$

Theorem I.1 covers equations (I), (II), (IV), (VI), (VII), (VIII), (XI) and a large class of sfde's driven by white noise. Note that (XI) *does not satisfy the hypotheses underlying the classical results* of Doleans-Dade [Dol], 1976, Metivier and Pellaumail [Met-P], 1980, Protter, Ann. Prob. 1987, Lipster and Shiryayev [Lip-Sh], [Met], 1982. This is because the coefficient

$$\eta \to \int_{-r}^{0} \eta(s)\, dW(s)$$

on the RHS of (XI) *does not admit almost surely Lipschitz (or even linear) versions* $C \to \mathbf{R}$! This will be shown later.

When the coeffcients h, g factor through functionals

$$H : [0,T] \times C \to \mathbf{R}^d, \qquad G : [0,T] \times C \to \mathbf{R}^{d \times m}$$

we can impose the following local Lipschitz and global linear growth conditions on the sfde

$$\left. \begin{aligned} dx(t) &= H(t, x_t)\, dt + G(t, x_t)\, dW(t) \quad t > 0 \\ x_0 &= \theta \end{aligned} \right\} \qquad (XIII)$$

with W m-dimensional Brownian motion:

Hypotheses (E_2)

(i) H, G are Lipschitz on bounded sets in C: For each integer $n \geq 1$ there exists $L_n > 0$ such that

$$|H(t, \eta_1) - H(t, \eta_2)| \leq L_n \|\eta_1 - \eta_2\|_C$$

for all $t \in [0, T]$ and $\eta_1, \eta_2 \in C$ with $\|\eta_1\|_C \leq n$, $\|\eta_2\|_C \leq n$. Similarly for the diffusion coefficent G.

(ii) There is a constant $K > 0$ such that

$$|H(t, \eta)| + \|G(t, \eta)\| \leq K(1 + \|\eta\|_C)$$

for all $t \in [0, T]$ and $\eta \in C$.

Note that the adaptability condition is not needed (explicitly) because H, G are deterministic and because the sample-continuity and adaptability of x imply that the segment $[0, T] \ni t \mapsto x_t \in C$ is also adapted.

Exercise: Formulate the heat-bath model (IV) as a sfde of the form (XIII).(β has compact support in $\mathbf{R}^+$.)

Theorem I.2. ([Mo], 1984) (Existence and Uniqueness).

Suppose H and G satisfy Hypotheses (E_2) and let $\theta \in L^2(\Omega, C; \mathcal{F}_0)$.

Then the sfde (XIII) has a unique $(\mathcal{F}_t)_{0\le t\le T}$-adapted solution ${}^\theta x : [-r, T] \times \Omega \to \mathbf{R}^d$ starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ with $t \longmapsto {}^\theta x_t$ continuous and ${}^\theta x \in L^2(\Omega, C([-r, T], \mathbf{R}^d))$ for all $T > 0$. For a given θ, uniqueness holds up to equivalence among all $(\mathcal{F}_t)_{0\le t\le T}$-adapted processes in $L^2(\Omega, C([-r, T], \mathbf{R}^d))$.

Furthermore if $\theta \in L^{2k}(\Omega, C; \mathcal{F}_0)$, then ${}^\theta x_t \in L^{2k}(\Omega, C; \mathcal{F}_t)$ and

$$E\|{}^\theta x_t\|_C^{2k} \le C_k[1 + \|\theta\|_{L^{2k}(\Omega,C)}^{2k}]$$

for all $t \in [0, T]$ and some positive constants C_k.

Proofs of Theorems I.1, I.2.(Outline)

[Mo], pp. 150-152. Generalize sode proofs in Gihman and Skorohod ([G-S], 1973) or Friedman ([Fr], 1975):

(1) Truncate coefficients outside bounded sets in C. Reduce to globally Lipschitz case.

(2) Successive approx. in globally Lipschitz situation.

(3) Use local uniqueness ([Mo], Theorem 4.2, p. 151) to "patch up" solutions of the truncated sfde's.

For (2) consider globally Lipschitz case and $h \equiv 0$.

We look for solutions of (XII) by successive approximation in $L^2(\Omega, C([-r,a], \mathbf{R}^d))$. Let $J := [-r, 0]$.

Suppose $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$ is $\mathcal{F}_0$-measurable. Note that this is equivalent to saying that $\theta(\cdot)(s)$ is $\mathcal{F}_0$-measurable for all $s \in J$, because θ has a.a. sample paths continuous.

We prove by induction that there is a sequence of processes ${}^k x : [-r, a] \times \Omega \to \mathbf{R}^d$, $k = 1, 2, \cdots$ having the

Properties $P(k)$*:*

(i) ${}^k x \in L^2(\Omega, C([-r,a], \mathbf{R}^d))$ and is adapted to $(\mathcal{F}_t)_{t \in [0,a]}$.

(ii) For each $t \in [0,a]$, ${}^k x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_t$-measur-able.

(iii)

$$\left.\begin{aligned} \|{}^{k+1}x - {}^k x\|_{L^2(\Omega,C)} &\le (ML^2)^{k-1} \frac{a^{k-1}}{(k-1)!} \|{}^2 x - {}^1 x\|_{L^2(\Omega,C)} \\ \|{}^{k+1}x_t - {}^k x_t\|_{L^2(\Omega,C)} &\le (ML^2)^{k-1} \frac{t^{k-1}}{(k-1)!} \|{}^2 x - {}^1 x\|_{L^2(\Omega,C)} \end{aligned}\right\} \tag{1}$$

where M is a "martingale" constant and L is the Lipschitz constant of g.

Take ${}^1 x : [-r,a] \times \Omega \to \mathbf{R}^d$ to be

$${}^1 x(t,\omega) = \begin{cases} \theta(\omega)(0) & t \in [0,a] \\ \theta(\omega)(t) & t \in J \end{cases}$$

a.s., and

$${}^{k+1}x(t,\omega) = \begin{cases} \theta(\omega)(0) + (\omega) \displaystyle\int_0^t g(u, {}^k x_u) dW(\cdot)(u) & t \in [0,a] \\ \theta(\omega)(t) \qquad t \in J \end{cases} \tag{2}$$

a.s.

Since $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_0$-measurable, then ${}^1 x \in L^2(\Omega, C([-r,a], \mathbf{R}^d))$ and is trivially adapted to $(\mathcal{F}_t)_{t \in [0,a]}$. Hence ${}^1 x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_t$-measurable for all $t \in [0,a]$. $P(1)$ (iii) holds trivially.

Now suppose $P(k)$ is satisfied for some $k > 1$. Then by Hypothesis $(E_1)(i), (ii)$ and the continuity of the slicing map (*stochastic memory*), it follows from $P(k)(ii)$ that the process

$$[0, a] \ni u \longmapsto g(u, {}^k x_u) \in L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d))$$

is continuous and adapted to $(\mathcal{F}_t)_{t \in [0,a]}$. $P(k+1)(i)$ and $P(k+1)(ii)$ follow from the continuity and adaptability of the stochastic integral. Check $P(k+1)(iii)$, by using Doob's inequality.

For each $k > 1$, write

$${}^k x = {}^1 x + \sum_{i=1}^{k-1} ({}^{i+1} x - {}^i x).$$

Now $L^2_A(\Omega, C([-r, a], \mathbf{R}^d))$ is closed in $L^2(\Omega, C([-r, a], \mathbf{R}^d))$; so the series

$$\sum_{i=1}^{\infty} ({}^{i+1} x - {}^i x)$$

converges in $L^2_A(\Omega, C([-r, a], \mathbf{R}^d))$ because of (1) and the convergence of

$$\sum_{i=1}^{\infty} \left[(ML^2)^{i-1} \frac{a^{i-1}}{(i-1)!} \right]^{1/2}.$$

Hence $\{{}^k x\}_{k=1}^{\infty}$ converges to some $x \in L^2_A(\Omega, C([-r, a], \mathbf{R}^d))$.

Clearly $x|J = \theta$ and is $\mathcal{F}_0$-measurable, so applying Doob's inequality to the Itô integral of the difference

$$u \longmapsto g(u, {}^k x_u) - g(u, x_u)$$

gives

$$\begin{aligned} E\Big(\sup_{t\in[0,a]} \Big| \int_0^t g(u, {}^k x_u)\, dW(\cdot)(u) - \int_0^t g(u, x_u)\, dW(\cdot)(u) \Big|^2\Big) \\ < ML^2 a \|{}^k x - x\|^2_{L^2(\Omega,C)} \\ \longrightarrow 0 \text{ as } k \to \infty. \end{aligned}$$

Thus viewing the right-hand side of (2) as a process in $L^2(\Omega, C\,([-r,a], \mathbf{R}^d))$ and letting $k \to \infty$, it follows from the above that x must satisfy the sfde (XII) a.s. for all $t \in [-r, a]$.

For uniqueness, let $\tilde{x} \in L^2_A(\Omega, ([-r,a], \mathbf{R}^d))$ be also a solution of (XII) with initial process θ. Then by the Lipschitz condition:

$$\|x_t - \tilde{x}_t\|^2_{L^2(\Omega,C)} < ML^2 \int_0^t \|x_u - \tilde{x}_u\|^2_{L^2(\Omega,C)}\, du$$

for all $t \in [0,a]$. Therefore we must have $x_t - \tilde{x}_t = 0$ for all $t \in [0,a]$; so $x = \tilde{x}$ in $L^2(\Omega, C([-r,a], \mathbf{R}^d))$ a.s. $\square$

Remarks and Generalizations.

(i) In Theorem I.2 replace the process $(t, W(t))$ by a (square integrable) semimartingale $Z(t)$ satisfying appropriate conditions.([Mo], 1984, Chapter II).

(ii) Results on existence of solutions of sfde's driven by white noise were first obtained by Itô and Nisio ([I-N], J. Math. Kyoto University, 1968) and then Kushner (JDE, 197).

(iii) Extensions to sfde's with *infinite* memory. Fading memory case: work by Mizel and Trützer [M-T],JIE, 1984, Marcus and Mizel [M-M], Stochastics, 1988; general infinite memory: Itô and Nisio [I-N], J. Math. Kyoto University, 1968.

(iii) Pathwise local uniqueness holds for sfde's of type (XIII) under a global Lipschitz condition: If coeffcients of two sfde's agree on an open set in C, then the corresponding trajectories leave the open set at the same time and agree almost surely up to the time they leave the open set ([Mo], Pitman Books, 1984, Theorem 4.2, pp. 150-151.)

(iv) Replace the state space C by the Delfour-Mitter Hilbert space $M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d)$ with the Hilbert norm

$$\|(v,\eta)\|_{M_2} = \left(|v|^2 + \int_{-r}^{0} |\eta(s)|^2\, ds\right)^{1/2}$$

for $(v,\eta) \in M_2$ (T. Ahmed, S. Elsanousi and S. Mohammed, 1983).

(v) Have Lipschitz and smooth dependence of ${}^{\theta}x_t$ on the initial process $\theta \in L^2(\Omega, C)$ ([Mo], 1984, Theorems 3.1, 3.2, pp. 41-45).

MARKOV BEHAVIOR

AND THE WEAK GENERATOR

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MARKOV BEHAVIOR AND THE GENERATOR

Consider the sfde

$$\left.\begin{aligned} dx(t) &= H(t, x_t)\, dt + G(t, x_t)\, dW(t), \qquad t > 0 \\ x_0 &= \eta \in C := C([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (XIII)$$

with coefficients $H : [0, T] \times C \to \mathbf{R}^d$, $\quad G : [0, T] \times C \to \mathbf{R}^{d \times m}$, m-dimensional Brownian motion W and trajectory field $\{{}^{\eta}x_t : t \geq 0, \eta \in C\}$.

1. Questions

(i) For the sfde (XIII) does the trajectory field x_t give a diffusion in C (or M_2)?

(ii) How does the trajectory x_t transform under smooth non-linear functionals $\phi : C \to \mathbf{R}$?

(iii) What "diffusions" on C (or M_2) correspond to sfde's on $\mathbf{R}^d$?

We will only answer the first two questions. More details in [Mo], Pitman Books, 1984, Chapter III, pp. 46-112. Third question is OPEN.

Difficulties

(i) Although the current state $x(t)$ is a semimartingale, the trajectory x_t does *not* seem to possess any martingale properties when viewed as C-(or M_2)-valued process: e.g. for Brownian motion W $\big(H \equiv 0, G \equiv 1\big)$:

$$[E(W_t|\mathcal{F}_{t_1})](s) = W(t_1) = W_{t_1}(0), \qquad s \in [-r, 0]$$

whenever $t_1 \leq t - r$.

(ii) Lack of strong continuity leads to the use of weak limits in C which tend to live outside C.

(iii) We will show that x_t is a Markov process in C. However almost all tame functions lie *outside* the domain of the (weak) generator.

(iv) Lack of an Itô formula makes the computation of the generator hard.

Hypotheses (M)

(i) $\mathcal{F}_t :=$ completion of $\sigma\{W(u) : 0 \leq u \leq t\}$, $\quad t \geq 0$.

(ii) H, G are jointly continuous and globally Lipschitz in second variable uniformly wrt the first:

$$|H(t,\eta_1) - H(t,\eta_2)| + \|G(t,\eta_1) - G(t,\eta_2)\| \leq L\|\eta_1 - \eta_2\|_C$$

for all $t \in [0,T]$ and $\eta_1, \eta_2 \in C$.

2. The Markov Property

${}^{\eta}x^{t_1} :=$ solution starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_{t_1})$ at $t = t_1$ for the sfde:

$${}^{\eta}x^{t_1}(t) = \begin{cases} \eta(0) + \int_{t_1}^{t} H(u, x_u^{t_1})\,du + \int_{t_1}^{t} G(u, x_u^{t_1})\,dW(u), & t > t_1 \\ \eta(t - t_1), & t_1 - r \leq t \leq t_1. \end{cases}$$

This gives a two-parameter family of mappings

$$T^{t_1}_{t_2} : L^2(\Omega, C; \mathcal{F}_{t_1}) \to L^2(\Omega, C; \mathcal{F}_{t_2}),\ t_1 \leq t_2,$$

$$T^{t_1}_{t_2}(\theta) := {}^{\theta}x^{t_1}_{t_2}, \qquad \theta \in L^2(\Omega, C; \mathcal{F}_{t_1}). \tag{1}$$

Uniqueness of solutions gives the *two-parameter* semigroup property:

$$T^{t_1}_{t_2} \circ T^0_{t_1} = T^0_{t_2}, \quad t_1 \leq t_2. \tag{2}$$

([Mo], Pitman Books, 1984, Theorem II (2.2), p. 40.)

Theorem II.1 (Markov Property)([Mo], 1984).

In (XIII) suppose Hypotheses (M) hold. Then the trajectory field $\{{}^{\eta}x_t : t \geq 0, \eta \in C\}$ is a Feller process on C with transition probabilities

$$p(t_1, \eta, t_2, B) := P\big({}^{\eta}x^{t_1}_{t_2} \in B\big) \quad t_1 \leq t_2, \quad B \in \text{ Borel } C, \quad \eta \in C.$$

i.e.

$$P\big(x_{t_2} \in B \big| \mathcal{F}_{t_1}\big) = p(t_1, x_{t_1}(\cdot), t_2, B) = P\big(x_{t_2} \in B \big| x_{t_1}\big) \text{ a.s.}$$

Further, if H and G do not depend on t, then the trajectory is time-homogeneous:

$$p(t_1, \eta, t_2, \cdot) = p(0, \eta, t_2 - t_1, \cdot), \quad 0 \leq t_1 \leq t_2, \quad \eta \in C.$$

Proof.

[Mo], 1984, Theorem III.1.1, pp. 51-58. [Mo], 1984, Theorem III.2.1, pp. 64-65. □

3. The Semigroup

In the autonomous sfde

$$\left.\begin{aligned} dx(t) &= H(x_t)\,dt + G(x_t)\,dW(t) \quad t>0 \\ x_0 &= \eta \in C \end{aligned}\right\} \qquad (XIV)$$

suppose the coefficients $H : C \to \mathbf{R}^d$, $\quad G : C \to \mathbf{R}^{d\times m}$ are *globally bounded* and globally Lipschitz.

$C_b :=$ Banach space of all bounded uniformly continuous functions $\phi : C \to \mathbf{R}$, with the sup norm

$$\|\phi\|_{C_b} := \sup_{\eta\in C} |\phi(\eta)|, \quad \phi \in C_b.$$

Define the operators $P_t : C_b \hookrightarrow C_b, t \geq 0$, on C_b by

$$P_t(\phi)(\eta) := E\phi\big({}^\eta x_t\big) \quad t \geq 0,\ \eta \in C.$$

A family $\phi_t, t > 0$, *converges weakly* to $\phi \in C_b$ as $t \to 0+$ if $\lim\limits_{t\to 0+} < \phi_t, \mu > = < \phi, \mu >$ for all finite regular Borel measures μ on C. Write $\phi := w - \lim\limits_{t\to 0+} \phi_t$. This is equivalent to

$$\begin{cases} \phi_t(\eta) \to \phi(\eta) \text{ as } t \to 0+, \text{ for all } \eta \in C \\ \\ \{\|\phi_t\|_{C_b} : t \geq 0\} \text{ is bounded .} \end{cases}$$

(Dynkin, [Dy], Vol. 1, p. 50). Proof uses uniform boundedness principle and dominated convergence theorem.

Theorem II.2([Mo], Pitman Books, 1984)

(i) $\{P_t\}_{t\geq 0}$ *is a one-parameter contraction semigroup on* C_b.

(ii) $\{P_t\}_{t\geq 0}$ *is weakly continuous at* $t=0$*:*

$$\begin{cases} P_t(\phi)(\eta) \to \phi(\eta) \text{ as } t \to 0+ \\ \\ \{|P_t(\phi)(\eta)| : t \geq 0, \eta \in C\} \text{is bounded by } \|\phi\|_{C_b}. \end{cases}$$

(iii) *If* $r>0$, $\{P_t\}_{t\geq 0}$ *is never strongly continuous on* C_b *under the sup norm.*

Proof.

(i) One parameter semigroup property

$$P_{t_2} \circ P_{t_1} = P_{t_1+t_2}, \quad t_1, t_2 \geq 0$$

follows from the continuation property (2) and time-homogeneity of the Feller process x_t (Theorem II.1).

(ii) Definition of P_t, continuity and boundedness of ϕ and sample-continuity of trajectory ${}^\eta x_t$ give weak continuity of $\{P_t(\phi) : t > 0\}$ at $t=0$ in C_b.

(iii) Lack of strong continuity of semigroup:

Define the canonical shift (static) semigroup

$$S_t : C_b \to C_b, \ t \geq 0,$$

by

$$S_t(\phi)(\eta) := \phi(\tilde{\eta}_t), \quad \phi \in C_b, \quad \eta \in C,$$

where $\tilde{\eta} : [-r, \infty) \to \mathbf{R}^d$ is defined by

$$\tilde{\eta}(t) = \begin{cases} \eta(0) & t \geq 0 \\ \eta(t) & t \in [-r, 0). \end{cases}$$

Then P_t is strongly continuous iff S_t is strongly continuous. P_t and S_t have the same "domain of strong continuity" independently of H, G, and W. This follows from the global boundedness of H and G. ([Mo], Theorem IV.2.1, pp. 72-73). Key relation is

$$\lim_{t\to 0+} E\|{}^\eta x_t - \tilde{\eta}_t\|_C^2 = 0$$

uniformly in $\eta \in C$. But $\{S_t\}$ is strongly continuous on C_b iff C is locally compact iff $r = 0$ (no memory) ! ([Mo], Theorems IV.2.1 and IV.2.2, pp.72-73). Main idea is to pick any $s_0 \in [-r, 0)$ and consider the function $\phi_0 : C \to \mathbf{R}$ defined by

$$\phi_0(\eta) := \begin{cases} \eta(s_0) & \|\eta\|_C \leq 1 \\ \dfrac{\eta(s_0)}{\|\eta\|_C} & \|\eta\|_C > 1 \end{cases}$$

Let C_b^0 be the domain of strong continuity of P_t, viz.

$$C_b^0 := \{\phi \in C_b : P_t(\phi) \to \phi \text{ as } t \to 0+ \text{ in } C_b\}.$$

Then $\phi_0 \in C_b$, but $\phi_0 \notin C_b^0$ because $r > 0$. $\square$

4. The Generator

Define the *weak generator* $A : D(A) \subset C_b \to C_b$ by the weak limit

$$A(\phi)(\eta) := w - \lim_{t \to 0+} \frac{P_t(\phi)(\eta) - \phi(\eta)}{t}$$

where $\phi \in D(A)$ iff the above weak limit exists. Hence $D(A) \subset C_0^b$ (Dynkin [Dy], Vol. 1, Chapter I, pp. 36-43). Also $D(A)$ is weakly dense in C_b and A is weakly closed. Further

$$\frac{d}{dt} P_t(\phi) = A(P_t(\phi)) = P_t(A(\phi)), \quad t > 0$$

for all $\phi \in D(A)$ ([Dy], pp. 36-43).

Next objective is to derive a formula for the weak generator A. *We need to augment C by adjoining a canonical d-dimensional direction. The generator A will be equal to the weak generator of the shift semigroup $\{S_t\}$ plus a second order linear partial differential operator along this new direction.* Computation requires the following lemmas.

Let

$$F_d = \{v\chi_{\{0\}} : v \in \mathbf{R}^d\}$$

$$C \oplus F_d = \{\eta + v\chi_{\{0\}} : \eta \in C, v \in \mathbf{R}^d\}, \quad \|\eta + v\chi_{\{0\}}\| = \|\eta\|_C + |v|$$

Lemma II.1.([Mo], Pitman Books, 1984)

Suppose $\phi : C \to \mathbf{R}$ is C^2 and $\eta \in C$. Then $D\phi(\eta)$ and $D^2\phi(\eta)$ have unique weakly continuous linear and bilinear extensions

$$\overline{D\phi(\eta)} : C \oplus F_d \to \mathbf{R}, \quad \overline{D^2\phi(\eta)} : (C \oplus F_d) \times (C \oplus F_d) \to \mathbf{R}$$

respectively.

Proof.

First reduce to the one-dimensional case $d = 1$ by using coordinates.

Let $\alpha \in C^* = [C([-r, 0], \mathbf{R})]^*$. We will show that there is a weakly continuous linear extension $\overline{\alpha} : C \oplus F_1 \to \mathbf{R}$ of α; viz. If $\{\xi^k\}$ is a bounded sequence in C such that $\xi^k(s) \to \xi(s)$ as $k \to \infty$ for all $s \in [-r, 0]$, where $\xi \in C \oplus F_1$, then $\alpha(\xi^k) \to \overline{\alpha}(\xi)$ as $k \to \infty$. By the Riesz representation theorem there is a unique finite regular Borel measure μ on $[-r, 0]$ such that

$$\alpha(\eta) = \int_{-r}^{0} \eta(s)\, d\mu(s)$$

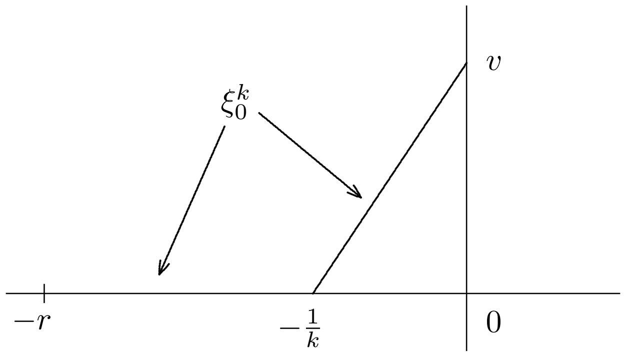
for all $\eta \in C$. Define $\overline{\alpha} \in [C \oplus F_1]^*$ by

$$\overline{\alpha}(\eta + v\chi_{\{0\}}) = \alpha(\eta) + v\mu(\{0\}), \quad \eta \in C, \quad v \in \mathbf{R}.$$

Easy to check that $\overline{\alpha}$ is weakly continuous. (*Exercise:* Use Lebesgue dominated convergence theorem.)

Weak extension $\overline{\alpha}$ is unique because each function $v\chi_{\{0\}}$ can be approximated weakly by a sequence of continuous functions $\{\xi_0^k\}$:

$$\xi_0^k(s) := \begin{cases} (ks+1)v, & -\frac{1}{k} \le s \le 0 \\ 0 & -r \le s < -\frac{1}{k}. \end{cases}$$



Put $\alpha = D\phi(\eta)$ to get first assertion of lemma.

To construct a weakly continuous bilinear extension $\overline{\beta} : (C \oplus F_1) \times (C \oplus F_1) \to \mathbf{R}$ for any continuous bilinear form
$\beta : C \times C \to \mathbf{R}$, use classical theory of vector measures (Dunford and Schwartz, [D-S], Vol. I, Section 6.3). Think of β as a continuos *linear* map $C \to C^*$. Since C^* is weakly complete ([D-S], I.13.22, p. 341), then β is a weakly compact linear operator ([D-S], Theorem I.7.6, p. 494): i.e. it maps norm-bounded sets in C into weakly sequentially compact sets in C^*. By the Riesz representation theorem (for vector measures), there is a unique C^*-valued Borel measure λ on $[-r, 0]$ (of finite semi-variation) such that

$$\beta(\xi) = \int_{-r}^{0} \xi(s)\, d\lambda(s)$$

for all $\xi \in C$. ([D-S], Vol. I, Theorem VI.7.3, p. 493). By the dominated convergence theorem for vector measures ([D-S], Theorem IV.10.10, p. 328), one could reach elements in F_1 using weakly convergent sequences of type $\{\xi_0^k\}$. This gives a unique weakly continuous extension $\hat{\beta} : C \oplus F_1 \to C^*$. Next for each $\eta \in C$, $v \in \mathbf{R}$, extend $\hat{\beta}(\eta + v\chi_{\{0\}}) : C \to \mathbf{R}$ to a weakly continuous linear map $\overline{\hat{\beta}(\eta + v\chi_{\{0\}})} : C \oplus F_1 \to \mathbf{R}$. Thus $\overline{\beta}$ corresponds to the weakly continuous bilinear extension $\overline{\hat{\beta}(\cdot)}(\cdot) : [C \oplus F_1] \times [C \oplus F_1] \to \mathbf{R}$ of β. (Check this as exercise).

Finally use $\beta = D^2\phi(\eta)$ for each fixed $\eta \in C$ to get the required bilinear extension $\overline{D^2\phi(\eta)}$. $\square$

Lemma II.2. ([Mo], Pitman Books, 1984)

For $t > 0$ define $W_t^* \in C$ by

$$W_t^*(s) := \begin{cases} \frac{1}{\sqrt{t}}[W(t+s) - W(0)], & -t \le s < 0, \\ 0 & -r \le s \le -t. \end{cases}$$

Let β be a continuous bilinear form on C. Then

$$\lim_{t \to 0+} \left[\frac{1}{t} E\beta({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t) - E\beta(G(\eta) \circ W_t^*, G(\eta) \circ W_t^*) \right] = 0$$

Proof.

Use

$$\lim_{t \to 0+} E \| \frac{1}{\sqrt{t}} ({}^\eta x_t - \tilde{\eta}_t - G(\eta) \circ W_t^* \|_C^2 = 0.$$

The above limit follows from the Lipschitz continuity of H and G and the martingale properties of the Itô integral. Conclusion of lemma is obtained by a computation using the bilinearity of β, Hölder's inequality and the above limit.([Mo], Pitman Books, 1984, pp. 86-87.) $\square$

Lemma II.3. ([Mo], Pitman Books, 1984)

Let β be a continuous bilinear form on C and $\{e_i\}_{i=1}^m$ be any basis for $\mathbf{R}^m$. Then

$$\lim_{t \to 0+} \frac{1}{t} E\beta({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t) = \sum_{i=1}^m \overline{\beta}\big(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}\big)$$

for each $\eta \in C$.

Proof.

By taking coordinates reduce to the one-dimensional case $d = m = 1$:

$$\lim_{t\to 0+} E\beta(W_t^*, W_t^*) = \overline{\beta}(\chi_{\{0\}}, \chi_{\{0\}})$$

with W one-dimensional Brownian motion. The proof of the above relation is lengthy and difficult. A key idea is the use of the projective tensor product $C \otimes_\pi C$ in order to view the continuous *bilinear* form β as a continuous *linear* functional on $C \otimes_\pi C$. At this level β commutes with the (Bochner) expectation. Rest of computation is effected using Mercer's theorem and some Fourier analysis. See [Mo], 1984, pp. 88-94. □

Theorem II.3.([Mo], Pitman Books, 1984)

In (XIV) suppose H and G are globally bounded and Lipschitz. Let $S : D(S) \subset C_b \to C_b$ be the weak generator of $\{S_t\}$. Suppose $\phi \in D(S)$ is sufficiently smooth (e.g. ϕ is C^2, $D\phi$, $D^2\phi$ globally bounded and Lipschitz). Then $\phi \in D(A)$ and

$$\begin{aligned} A(\phi)(\eta) = S(\phi)(\eta) &+ \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big) \\ &+ \frac{1}{2}\sum_{i=1}^{m} \overline{D^2\phi(\eta)}\big(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}\big). \end{aligned}$$

where $\{e_i\}_{i=1}^m$ is any basis for $\mathbf{R}^m$.

Proof.

Step 1.

For fixed $\eta \in C$, use Taylor's theorem:

$$\phi({}^\eta x_t) - \phi(\eta) = \phi(\tilde{\eta}_t) - \phi(\eta) + D\phi(\tilde{\eta}_t)({}^\eta x_t - \tilde{\eta}_t) + R(t)$$

a.s. for $t > 0$; where

$$R(t) := \int_0^1 (1-u) D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)]({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t)\, du.$$

Take expectations and divide by $t > 0$:

$$\left.\begin{aligned}\frac{1}{t}E[\phi({}^{\eta}x_t) - \phi(\eta)] = \frac{1}{t}[S_t(\phi(\eta) - \phi(\eta)] + D\phi(\tilde{\eta}_t)\Big\{E[\frac{1}{t}({}^{\eta}x_t - \tilde{\eta}_t)]\Big\} \\ + \frac{1}{t}ER(t)\end{aligned}\right\} \tag{3}$$

for $t > 0$.

As $t \to 0+$, the first term on the RHS converges to $S(\phi)(\eta)$, because $\phi \in D(S)$.

Step 2.

Consider second term on the RHS of (3). Then

$$\begin{aligned}\lim_{t\to 0+}\left[E\left\{\frac{1}{t}({}^{\eta}x_t - \tilde{\eta}_t)\right\}\right](s) &= \begin{cases}\lim\limits_{t\to 0+}\dfrac{1}{t}\displaystyle\int_0^t E[H({}^{\eta}x_u)]\,du, & s = 0\\ 0 & -r \leq s < 0.\end{cases}\\ &= [H(\eta)\chi_{\{0\}}](s), \qquad\qquad -r \leq s \leq 0.\end{aligned}$$

Since H is bounded, then $\|E\{\frac{1}{t}({}^{\eta}x_t - \tilde{\eta}_t)\}\|_C$ is bounded in $t > 0$ and $\eta \in C$ (*Exercise*). Hence

$$w - \lim_{t\to 0+}\left[E\left\{\frac{1}{t}({}^{\eta}x_t - \tilde{\eta}_t)\right\}\right] = H(\eta)\chi_{\{0\}} \quad (\notin C).$$

Therefore by Lemma II.1 and the continuity of $D\phi$ at η:

$$\begin{aligned}\lim_{t\to 0+} D\phi(\tilde{\eta}_t)\left\{E\left[\frac{1}{t}({}^{\eta}x_t - \tilde{\eta}_t)\right]\right\} &= \lim_{t\to 0+} D\phi(\eta)\left\{E\left[\frac{1}{t}({}^{\eta}x_t - \tilde{\eta}_t)\right]\right\}\\ &= \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big)\end{aligned}$$

Step 3.

To compute limit of third term in RHS of (3), consider

$$
\begin{aligned}
&\bigg|\frac{1}{t}ED^2\phi[\tilde{\eta}_t+u({}^\eta x_t-\tilde{\eta}_t)]({}^\eta x_t-\tilde{\eta}_t,{}^\eta x_t-\tilde{\eta}_t)\\
&\quad-\frac{1}{t}ED^2\phi(\eta)({}^\eta x_t-\tilde{\eta}_t,{}^\eta x_t-\tilde{\eta}_t)\bigg|\\
&\quad\le (E\|D^2\phi[\tilde{\eta}_t+u({}^\eta x_t-\tilde{\eta}_t)]-D^2\phi(\eta)\|^2)^{1/2}\left[\frac{1}{t^2}E\|{}^\eta x_t-\tilde{\eta}_t\|^4\right]^{1/2}\\
&\quad\le K(t^2+1)^{1/2}[E\|D^2\phi[\tilde{\eta}_t+u({}^\eta x_t-\tilde{\eta}_t)]-D^2\phi(\eta)\|^2]^{1/2}\\
&\quad\to 0
\end{aligned}
$$

as $t\to 0+$, uniformly for $u\in[0,1]$, by martingale properties of the Itô integral and the Lipschitz continuity of $D^2\phi$. Therefore by Lemma II.3

$$
\begin{aligned}
\lim_{t\to 0+}\frac{1}{t}ER(t)&=\int_0^1(1-u)\lim_{t\to 0+}\frac{1}{t}ED^2\phi(\eta)({}^\eta x_t-\tilde{\eta}_t,{}^\eta x_t-\tilde{\eta}_t)\,du\\
&=\frac{1}{2}\sum_{i=1}^m\overline{D^2\phi(\eta)}\big(G(\eta)(e_i)\chi_{\{0\}},G(\eta)(e_i)\chi_{\{0\}}\big).
\end{aligned}
$$

The above is a weak limit since $\phi\in D(S)$ and has first and second derivatives globally bounded on C. □

5. Quasitame Functions

Recall that a function $\phi: C\to\mathbf{R}$ is *tame* (or a *cylinder function*) if there is a finite set $\{s_1<s_2<\cdots<s_k\}$ in $[-r,0]$ and a C^∞-bounded function $f:(\mathbf{R}^d)^k\to\mathbf{R}$ such that

$$\phi(\eta)=f(\eta(s_1),\cdots,\eta(s_k)),\qquad \eta\in C.$$

The set of all tame functions is a weakly dense subalgebra of C_b, invariant under the static shift S_t and generates $Borel\,C$. For $k\ge 2$ the tame function ϕ *lies outside* the domain of strong continuity C_b^0 of P_t, and hence *outside* $D(A)$ ([Mo], Pitman Books, 1984, pp.98-103; see also proof of Theorem IV .2.2, pp. 73-76). To overcome this difficulty we introduce

Definition.

Say $\phi : C \to \mathbf{R}$ is *quasitame* if there are C^∞-bounded maps $h : (\mathbf{R}^d)^k \to \mathbf{R}$, $f_j : \mathbf{R}^d \to \mathbf{R}^d$, and piecewise C^1 functions $g_j : [-r, 0] \to \mathbf{R}, 1 \le j \le k-1$, such that

$$\phi(\eta) = h\Big(\int_{-r}^0 f_1(\eta(s))g_1(s)\,ds, \cdots, \int_{-r}^0 f_{k-1}(\eta(s))g_{k-1}(s)\,ds, \eta(0)\Big) \tag{4}$$

for all $\eta \in C$.

Theorem II.4. ([Mo], Pitman Books, 1984)

The set of all quasitame functions is a weakly dense subalgebra of C_b^0, invariant under S_t, generates Borel C and belongs to $D(A)$. In particular, if ϕ is the quasitame function given by (4), then

$$\begin{aligned} A(\phi)(\eta) = \sum_{j=1}^{k-1} D_j h(m(\eta))\{&f_j(\eta(0))g_j(0) - f_j(\eta(-r))g_j(-r) \\ &- \int_{-r}^0 f_j(\eta(s))g_j'(s)\,ds\} \\ &+ D_k h(m(\eta))(H(\eta)) + \frac{1}{2}\text{trace}[D_k^2 h(m(\eta)) \circ (G(\eta) \times G(\eta))]. \end{aligned}$$

for all $\eta \in C$, where

$$m(\eta) := \Big(\int_{-r}^0 f_1(\eta(s))g_1(s)\,ds, \cdots, \int_{-r}^0 f_{k-1}(\eta(s))g_{k-1}(s)\,ds, \eta(0)\Big).$$

Remarks.

(i) Replace C by the Hilbert space M_2. No need for the weak extensions because M_2 is weakly complete. Extensions of $D\phi(v, \eta)$ and $D^2\phi(v, \eta)$ correspond to partial derivatives in the $\mathbf{R}^d$-variable. *Tame functions do not exist on M_2* but quasitame functions do! (with $\eta(0)$ replaced by $v \in \mathbf{R}^d$).

Analysis of supermartingale behavior and stability of $\phi({}^{\eta}x_t)$ given in Kushner ([Ku], JDE, 1968). Infinite fading memory setting by Mizel and Trützer ([M-T], JIE, 1984) in the weighted state space $\mathbf{R}^d \times L^2((-\infty, 0], \mathbf{R}^d; \rho)$.

(ii) For each quasitame ϕ on C, $\phi({}^{\eta}x_t)$ *is* a semimartingale, and the Itô formula holds:

$$d[\phi({}^{\eta}x_t)] = A(\phi)({}^{\eta}x_t)\,dt + \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big)\,dW(t).$$

REGULARITY:

CLASSIFICATION OF SFDE'S

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III. REGULARITY. CLASSIFICATION OF SFDE'S

Denote the state space by E where $E = C$ or $M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d)$. Most results hold for either choice of state space.

Objectives

To study regularity properties of the trajectory of a sfde as a random field $X := \{{}^\eta x_t : t \geq 0, \eta \in C\}$ in the variables (t,η,ω) $\big(E = C\big)$ or $(t,(v,\eta),\omega)$ $\big(E = M_2\big)$:

(i) Pathwise regularity of trajectories in the time variable.

(ii) Regularity of trajectories (in probability or pathwise) in the initial state $\eta \in C$ or $(v,\eta) \in M_2$.

(iii) Classification of sfde's into regular and singular types.

Denote by $C^\alpha := C^\alpha([-r,0],\mathbf{R}^d)$ the (separable) Banach space of α-Hölder continuous paths $\eta : [-r,0] \to \mathbf{R}^d$ with the Hölder norm

$$\|\eta\|_\alpha := \|\eta\|_C + \sup\left\{\frac{|\eta(s_1) - \eta(s_2)|}{|s_1 - s_2|^\alpha} : s_1, s_2 \in [-r,0], s_1 \neq s_2\right\}.$$

C^α can be constructed in a *separable manner* by completing the space of smooth paths $[-r,0] \to \mathbf{R}^d$ with respect to the above norm (Tromba [Tr], JFA, 1972). First step is to think of ${}^\eta x_t(\omega)$ as a measurable mapping $X : \mathbf{R}^+ \times C \times \Omega \to C$ in the three variables (t,η,ω) simultaneously:

Theorem III.1([Mo], Pitman Books, 1984)

In the sfde

$$\left.\begin{aligned} dx(t) &= H(t,x_t)\,dt + G(t,x_t)\,dW(t) \quad t > 0 \\ x_0 &= \eta \in C \end{aligned}\right\} \qquad (XIII)$$

assume that the coefficients H, G are (jointly) continuous and globally Lipschitz in the second variable uniformly wrt the first. Then *(i) For any* $0 < \alpha < \frac{1}{2}$*, and each initial path* $\eta \in C$*,*

$$P({}^{\eta}x_t \in C^{\alpha}, \textit{for all}\ \ t \geq r) = 1.$$

(ii) the trajectory field has a measurable version

$$X : \mathbf{R}^+ \times C \times \Omega \to C.$$

(iii) The trajectory field ${}^{\eta}x_t, t \geq r,\ \eta \in C$*, admits a measurable version*

$$[r, \infty) \times C \times \Omega \to C^{\alpha}.$$

Remark.

Similar statements hold for $E = M_2$.

Give $L^0(\Omega, E)$ the complete (psuedo)metric

$$d_E(\theta_1, \theta_2) := \inf_{\epsilon > 0}[\epsilon + P(\|\theta_1 - \theta_2\|_E \geq \epsilon)], \quad \theta_1, \theta_2 \in L^0(\Omega, E),$$

(which corresponds to convergence in probability, Dunford and Schwartz [D-S], Lemma III.2.7, p. 104).

Proof of Theorem III.1.

(i) Sufficient to show that

$$P\big({}^{\eta}x|[0, a] \in C^{\alpha}([0, a], \mathbf{R}^d)\big) = 1$$

by using the estimate

$$P\left(\sup_{0 \leq t_1, t_2 \leq a, t_1 \neq t_1} \frac{|{}^{\eta}x(t_1) - {}^{\eta}x(t_2)|}{|t_1 - t_2|^{\alpha}} \geq N\right) \leq C_k^1(1 + \|\eta\|_C^{2k})\frac{1}{N^{2k}},$$

for all integers $k > (1 - 2\alpha)^{-1}$, and the Borel-Cantelli lemma. Above estimate is proved using Gronwall's lemma, Chebyshev's inequality, and Garsia-Rodemick-Rumsey lemma ([Mo], Pitman

Books, 1984, Theorem 4.1, p. 150; [Mo], Pitman Books, 1984, Theorem 4.4, pp.152-154.)

(ii) By mean-square Lipschitz dependence ([Mo], Pitman Books, 1984, Theorem 3.1, p. 41), the trajectory

$$\begin{aligned} [0,a] \times C &\to L^2(\Omega, C) \subset L^0(\Omega, C) \\ (t, \eta) &\mapsto {}^{\eta}x_t \end{aligned}$$

is globally Lipschitz in η uniformly wrt t in compact sets, and is continuous in t for fixed η. Therefore it is jointly continuous in (t, η) as a map

$$[0,a] \times C \ni (t, \eta) \mapsto {}^{\eta}x_t \in L^0(\Omega, C).$$

Then apply the Cohn-Hoffman-Jϕrgensen theorem:
If T, E are complete separable metric spaces, then each Borel map $X : T \to L^0(\Omega, E; \mathcal{F})$ admits a measurable version

$$T \times \Omega \to E$$

to the trajectory field to get measurability in (t, η). (Take $T = [0,a] \times C,\ E = C$ ([Mo], Pitman Books, 1984, p. 16).)

(iii) Use the estimate

$$P\big(\|{}^{\eta_1}x_t - {}^{\eta_2}x_t\|_{C^\alpha} \geq N\big) \leq \frac{C_k^2}{N^{2k}} \|\eta_1 - \eta_2\|_C^{2k}$$

for $t \in [r,a], N > 0$, ([Mo], 1984, Theorem 4.7, pp.158-162) to prove joint continuity of the trajectory

$$\begin{aligned} [r,a] \times C &\to L^0(\Omega, C^\alpha) \\ (t, \eta) &\mapsto {}^{\eta}x_t \end{aligned}$$

([Mo], Theorem 4.7, pp. 158-162) viewed as a process with values in the separable Banach space C^α. Again apply the Cohn-Hoffman-Jϕrgensen theorem. $\square$

As we have seen in Lecture I, the trajectory of a sfde possesses good regularity properties *in the mean-square.* The following theorem shows good behavior in distribution.

Theorem III.2. ([Mo], Pitman Books, 1984)

Suppose the coefficients H, G are globally Lipschitz in the second variable uniformly with respect to the first. Let $\alpha \in (0, 1/2)$ and k be any integer such that $k > (1-2\alpha)^{-1}$. Then there are positive constants C_k^3, C_k^4, C_k^5 such that

$$d_C({}^{\eta_1}x_t, {}^{\eta_2}x_t) \leq C_k^3 \|\eta_1 - \eta_2\|_C^{2k/(2k+1)} \qquad t \in [0, a]$$

$$d_{C^\alpha}({}^{\eta_1}x_t, {}^{\eta_2}x_t) \leq C_k^4 \|\eta_1 - \eta_2\|_C^{2k/(2k+1)} \qquad t \in [r, a]$$

$$P\Big(\|{}^\eta x_t\|_{C^\alpha} \geq N\Big) \leq C_k^5 (1 + \|\eta\|_C^{2k}) \frac{1}{N^{2k}}, \qquad t \in [r, a], \quad N > 0.$$

In particular the transition probabilities

$$\begin{aligned} [r, a] \times C &\to \mathcal{M}_p(C) \\ (t, \eta) &\mapsto p(0, \eta, t, \cdot) \end{aligned}$$

take bounded sets into relatively weak compact sets in the space $\mathcal{M}_p(C)$ of probability measures on C.*

Proof of Theorem III.2.

Proofs of the estimates use Gronwall's lemma, Chebyshev's inequality, and Garsia-Rodemick-Rumsey lemma ([Mo], 1984, Theorem 4.1, p. 150; [Mo], 1984, Theorem 4.7, pp.159-162.) The weak* compactness assertion follows from the last estimate, Prohorov's theorem and the compactness of the embedding $C^\alpha \hookrightarrow C$ ([Mo], 1984, Theorem 4.6, pp. 156-158). $\square$

Erratic Behavior. The Noisy Loop Revisited

Definition.

A sfde is *regular* with respect to M_2 if its trajectory random field $\{(x(t), x_t) : (x(0), x_0) = (v, \eta) \in M_2,\ t \geq 0\}$ admits a $(Borel\,\mathbf{R}^+ \otimes Borel\,M_2 \otimes \mathcal{F},\ Borel\,M_2)$-measurable version $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ with a.a. sample functions continuous on $\mathbf{R}^+ \times M_2$. The sfde is said to be *singular* otherwise. Similarly for regularity with respect to C.

Consider the one-dimensional linear sdde with a *positive delay* r

$$\left.\begin{aligned} dx(t) &= \sigma x(t-r))\, dW(t), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R} \times L^2([-r, 0], \mathbf{R}), \end{aligned}\right\} \qquad (I)$$

driven by a Wiener process W.

Theorem III.3 below implies that (I) is singular with respect to M_2 (and C).(See also [Mo], Stochastics, 1986).

Consider the regularity of the more general one-dimen-sional linear sfde:

$$\left.\begin{aligned} dx(t) &= \int_{-r}^{0} x(t+s) d\nu(s)\, dW(t), \qquad t > 0 \\ (x(0), x_0) &\in M_2 := \mathbf{R} \times L^2([-r, 0], \mathbf{R}) \end{aligned}\right\} \qquad (II')$$

where W is a Wiener process and ν is a fixed finite real-valued Borel measure on $[-r, 0]$.

Exercise:

(II$'$) is regular if ν has a C^1 (or even L_1^2) density with respect to Lebesgue measure on $[-r, 0]$. (Hint: Use integration by parts to eliminate the Itô integral!)

The following theorem gives conditions on the measure ν under which (II′) is singular.

Theorem III.3 ([M-S], II, 1996)

Let $r > 0$, and suppose that there exists $\epsilon \in (0, r)$ such that supp $\nu \subset [-r, -\epsilon]$. *Suppose $0 < t_0 \leq \epsilon$. For each $k \geq 1$, set*

$$\nu_k := \sqrt{t_0}\left|\int_{[-r,0]} e^{2\pi iks/t_0}\, d\nu(s)\right|.$$

Assume that

$$\sum_{k=1}^{\infty} \nu_k x^{1/\nu_k^2} = \infty \qquad (1)$$

for all $x \in (0, 1)$. Let $Y : [0, \epsilon] \times M_2 \times \Omega \to \mathbf{R}$ be any Borel-measurable version of the solution field $\{x(t) : 0 \leq t \leq \epsilon,\ (x(0), x_0) = (v, \eta) \in M_2\}$ of (II′). Then for a.a. $\omega \in \Omega$, the map $Y(t_0, \cdot, \omega) : M_2 \to \mathbf{R}$ is unbounded in every neighborhood of every point in M_2, and (hence) non-linear.

Corollary. ([Mo], Pitman Books, 1984)

Suppose $r > 0, \sigma \neq 0$ *in (I). Then the trajectory* $\{^\eta x_t : 0 \leq t \leq r, \eta \in C\}$ *of (I) has a measurable version* $X : \mathbf{R}^+ \times C \times \Omega \to C$ *s.t. for every* $t \in (0, r]$

$$P\Big(X(t, \eta_1 + \lambda\eta_2, \cdot) = X(t, \eta_1, \cdot) + \lambda X(t, \eta_2, \cdot) \quad \text{for all } \lambda \in \mathbf{R}, \eta_1, \eta_2 \in C \Big) = 0.$$

But

$$P\Big(X(t, \eta_1 + \lambda\eta_2, \cdot) = X(t, \eta_1, \cdot) + \lambda X(t, \eta_2, \cdot) \Big) = 1.$$

for all $\lambda \in \mathbf{R}, \eta_1, \eta_2 \in C$.

Remark.

(i) Condition (1) of the theorem is implied by

$$\lim_{k\to\infty} \nu_k \sqrt{\log k} = \infty.$$

(ii) For the delay equation (I), $\nu = \sigma\delta_{-r},\ \epsilon = r$. In this case condition (1) is satisfied for *every* $t_0 \in (0, r]$.

(iii) Theorem III.3 also holds for state space C since every bound-ed set in C is also bounded in $L^2([-r, 0], \mathbf{R})$.

Proof of Theorem III.3.

Joint work with V. Mizel.

Main idea is to track the solution random field of (a complexified version of) (II′) along the classical Fourier basis

$$\eta_k(s) = \mathrm{e}^{2\pi iks/t_0}\ , \quad -r \leq s \leq 0, \quad k \geq 1 \tag{2}$$

in $L^2([-r,0],\mathbf{C})$. On this basis, the solution field gives an infinite family of independent Gaussian random variables. This allows us to show that no Borel measurable version of the solution field can be bounded with positive probability on an arbitrarily small neighborhood of 0 in M_2, and hence on any neighborhood of any point in M_2 (cf. [Mo], Pitman Books, 1984; [Mo], Stochastics, 1986). For simplicity of computations, complexify the state space in (II′) by allowing (v,η) to belong to $M_2^C := \mathbf{C}\times L^2([-r,0],\mathbf{C})$. Thus consider the sfde

$$\left.\begin{aligned} dx(t) &= \int_{[-r,0]} x(t+s)d\nu(s)\, dW(t), t>0,\\ (x(0),x_0) &= (v,\eta)\in M_2^C \end{aligned}\right\} \qquad (II'-C))$$

where $x(t)\in\mathbf{C}$, $t\geq -r$, and ν, W are real-valued.

Use contradiction. Let $Y:[0,\epsilon]\times M_2\times\Omega\to\mathbf{R}$ be any Borel-measurable version of the solution field $\{x(t):0\leq t\leq\epsilon,\ (x(0),x_0)=(v,\eta)\in M_2\}$ of (II′). Suppose, if possible, that there exists a set $\Omega_0\in\mathcal{F}$ of positive P-measure, $(v_0,\eta_0)\in M_2$ and a positive δ such that for all $\omega\in\Omega_0$, $Y(t_0,\cdot,\omega)$ is bounded on the open ball $B((v_0,\eta_0),\delta)$ in M_2 of center (v_0,η_0) and radius δ. Define the complexification $Z(\cdot,\omega):M_2^C\to\mathbf{C}$ of $Y(t_0,\cdot,\omega):M_2\to\mathbf{R}$ by

$$Z(\xi_1+i\xi_2,\omega) := Y(t_0,\xi_1,\omega)+i\,Y(t_0,\xi_2,\omega), \qquad i=\sqrt{-1},$$

for all $\xi_1,\xi_2\in M_2$, $\omega\in\Omega$. Let $(v_0,\eta_0)^C$ denote the complexification $(v_0,\eta_0)^C := (v_0,\eta_0)+i(v_0,\eta_0)$. Clearly $Z(\cdot,\omega)$ is bounded on the complex ball $B((v_0,\eta_0)^C,\delta)$ in M_2^C for all $\omega \in \Omega_0$. Define the sequence of complex random variables $\{Z_k\}_{k=1}^{\infty}$ by

$$Z_k(\omega) := Z((\eta_k(0),\eta_k),\omega) - \eta_k(0), \qquad \omega \in \Omega, \quad k \geq 1.$$

Then

$$Z_k = \int_0^{t_0} \int_{[-r,-\epsilon]} \eta_k(u+s)\, d\nu(s)\, dW(u), \quad k \geq 1.$$

By standard properties of the Itô integral, and Fubini's theorem,

$$EZ_k\overline{Z_l} = \int_{[-r,-\epsilon]} \int_{[-r,-\epsilon]} \int_0^{t_0} \eta_k(u+s)\overline{\eta_l(u+s')}\, du\, d\nu(s)\, d\nu(s') = 0$$

for $k \neq l$, because

$$\int_0^{t_0} \eta_k(u+s)\overline{\eta_l(u+s')}\; du = 0$$

whenever $k \neq l$, for all $s, s' \in [-r,0]$. Furthermore

$$\int_0^{t_0} \eta_k(u+s)\overline{\eta_k(u+s')}\; du = t_0 \mathrm{e}^{2\pi i k(s-s')/t_0}$$

for all $s, s' \in [-r,0]$. Hence

$$\begin{aligned} E|Z_k|^2 &= \int_{[-r,-\epsilon]} \int_{[-r,-\epsilon]} t_0 \mathrm{e}^{2\pi i k(s-s')/t_0}\;\, d\nu(s)\, d\nu(s') \\ &= t_0 \left| \int_{[-r,0]} \mathrm{e}^{2\pi i k s/t_0}\;\, d\nu(s) \right|^2 \\ &= \nu_k^2. \end{aligned}$$

$Z(\cdot,\omega) : M_2^C \to \mathbf{C}$ is bounded on $B((v_0,\eta_0)^C,\delta)$ for all $\omega \in \Omega_0$, and $\|(\eta_k(0),\eta_k)\| = \sqrt{r+1}$ for all $k \geq 1$. By the linearity property

$$\begin{aligned} &Z\left((v_0,\eta_0)^C + \frac{\delta}{2\sqrt{r+1}}(\eta_k(0),\eta_k), \cdot\right) \\ &\qquad = Z((v_0,\eta_0)^C,\cdot) + \frac{\delta}{2\sqrt{r+1}} Z((\eta_k(0),\eta_k),\cdot), \; k \geq 1, \end{aligned}$$

a.s., it follows that

$$P\big(\sup_{k\geq 1}|Z_k|<\infty\big)>0. \tag{3}$$

It is easy to check that $\{ReZ_k,\, ImZ_k : k\geq 1\}$ are independent $\mathcal{N}(0,\nu_k^2/2)$-distributed Gaussian random variables. Get a contradiction to (3):

For each integer $N\geq 1$,

$$\begin{aligned} P\bigg(\sup_{k\geq 1}|Z_k|<N\bigg) &\leq \prod_{k\geq 1} P\bigg(|ReZ_k|<N\bigg)\\ &=\prod_{k\geq 1}\bigg[1-\frac{2}{\sqrt{2\pi}}\int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty} \mathrm{e}^{-x^2/2}\,dx\bigg]\\ &\leq \exp\bigg\{-\frac{2}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty}\mathrm{e}^{-x^2/2}\,dx\bigg\}. \end{aligned} \tag{4}$$

There exists $N_0>1$ (independent of $k\geq 1$) such that

$$\int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty}\mathrm{e}^{-x^2/2}\,dx\geq \frac{\nu_k}{2\sqrt{2}N}\mathrm{e}^{-\frac{N^2}{\nu_k^2}} \tag{5}$$

for all $N\geq N_0$ and all $k\geq 1$.

Combine (4) and (5) and use hypothesis (1) of the theorem to get

$$P\bigg(\sup_{k\geq 1}|Z_k|<N\bigg)=0$$

for all $N\geq N_0$. Hence

$$P\bigg(\sup_{k\geq 1}|Z_k|<\infty\bigg)=0.$$

This contradicts (3)(cf. Dudley [Du], JFA, 1967).

Since $Y(t_0,\cdot,\omega)$ is locally unbounded, it must be non-linear because of Douady's Theorem:

Every Borel measurable linear map between two Banach spaces is continuous.

(Schwartz [Sc], Radon Measures, Part II, 1973, pp. 155-160). □

Note that the pathological phenomenon in Theorem III.3 is peculiar to the delay case $r > 0$. The proof of the theorem suggests that this pathology is due to the *Gaussian nature* of the Wiener process W coupled with the *infinite-dimensionality* of the state space M_2. Because of this, one may expect similar difficulties in certain types of linear spde's driven by *multi-dimensional* white noise (Flandoli and Schaumlöffel [F-S], Stochastics, 1990).

Problem.

Classify all finite signed measures ν on $[-r, 0]$ for which (II') is regular.

Note that (I) automatically satisfies the conditions of Theorem III.3, and hence its trajectory field *explodes on every small neighborhood of* $0 \in M_2$. Because of the singular nature of (I), it is surprising that the maximal exponential growth rate of the trajectory of (I) is *negative* for small σ and is bounded away from zero *independently of the choice of the initial path* in M_2. This will be shown later in Lecture V (Theorem V.1).

Regular Linear Systems. White Noise

SDE's on $\mathbf{R}^d$ driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$, with smooth coefficients.

$$\left.\begin{aligned} dx(t) = H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\ + \sum_{i=1}^{m} g_i x(t)\, dW_i(t), \quad t > 0 \\ (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (VIII)$$

(VIII) is defined on

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) =$ canonical complete filtered Wiener space:

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R}^+ \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F} :=$ completed Borel σ-field of Ω;

$\mathcal{F}_t :=$ completed sub-σ-field of $\mathcal{F}$ generated by the evaluations $\omega \to \omega(u)$, $0 \le u \le t$, $\quad t \ge 0$;

$P :=$ Wiener measure on Ω;

$dW_i(t) =$ Itô stochastic differentials, $1 \le i \le m$.

Several finite delays $0 < d_1 < d_2 < \cdots < d_N \le r$ in drift term; *no delays in diffusion coefficient.*

$H : (\mathbf{R}^d)^{N+1} \times L^2([-r,0], \mathbf{R}^d) \to \mathbf{R}^d$ is a fixed continuous linear map; g_i, $i = 1, 2, \ldots, m$, fixed (deterministic) $d \times d$-matrices.

Theorem III.4.([Mo], Stochastics, 1990])

(VIII) is regular with respect to the state space $M_2 = \mathbf{R}^d \times \mathbf{L}^2([-r,0], \mathbf{R}^d)$. There is a measurable version $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ of the trajectory field $\{(x(t), x_t) : t \in \mathbf{R}^+, (x(0), x_0) = (v, \eta) \in M_2\}$ with the following properties:

(i) *For each $(v, \eta) \in M_2$ and $t \in \mathbf{R}^+$, $X(t, (v, \eta), \cdot) = (x(t), x_t)$ a.s., is $\mathcal{F}_t$-measurable and belongs to $L^2(\Omega, M_2; P)$.*

(ii) *There exists $\Omega_0 \in \mathcal{F}$ of full measure such that, for all $\omega \in \Omega_0$, the map $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times M_2 \to M_2$ is continuous.*

(iii) *For each $t \in \mathbf{R}^+$ and every $\omega \in \Omega_0$, the map $X(t, \cdot, \omega) : M_2 \to M_2$ is continuous linear; for each $\omega \in \Omega_0$, the map $\mathbf{R}^+ \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ is measurable and locally bounded in the uniform operator norm on $L(M_2)$. The map $[r, \infty) \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ is continuous for all $\omega \in \Omega_0$.*

(iv) *For each $t \ge r$ and all $\omega \in \Omega_0$, the map*

$$X(t, \cdot, \omega) : M_2 \to M_2$$

is compact.

Proof uses variational technique to reduce the problem to the solution of a random family of classical integral equations involving *no stochastic integrals.*

Compactness of semi-flow for $t \geq r$ will be used later to define hyperbolicity for (VIII) and the associated exponential dichotomies (Lecture IV).

Regular Linear Systems. Semimartingale Noise

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ a complete filtered probability space satisfying the usual conditions.

Linear systems driven by semimartingale noise, and memory driven by a measure-valued process
$\nu : \mathbf{R} \times \Omega \to \mathcal{M}([-r,0], \mathbf{R}^{d\times d})$, where $\mathcal{M}([-r,0], \mathbf{R}^{d\times d})$ is the space of all $d \times d$-matrix-valued Borel measures on $[-r,0]$ (or $R^{d\times d}$-valued functions of bounded variation on $[-r,0]$). This space is given the σ-algebra generated by all evaluations. The space $\mathbf{R}^{d\times d}$ of all $d\times d$-matrices is given the Euclidean norm $\|\cdot\|$.

$$\left.\begin{aligned} dx(t) = \Big\{\int_{[-r,0]} \nu(t)(ds)\, x(t+s)\Big\}\, dt + dN(t) \int_{-r}^{0} K(t)(s)\, x(t+s)\, ds \\ + dL(t)\, x(t-), \qquad t > 0 \\ (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (IX)$$

Hypotheses (R)

(i) The process $\nu : \mathbf{R}\times\Omega \to \mathcal{M}([-r,0], \mathbf{R}^{d\times d})$ is measurable and $(\mathcal{F}_t)_{t\geq 0}$)-adapted. For each $\omega \in \Omega$ and $t \geq 0$ define the positive measure $\bar{\nu}(t,\omega)$ on $[-r,\infty)$ by

$$\bar{\nu}(t,\omega)(A) := |\nu|(t,\omega)\{(A-t)\cap[-r,0]\}$$

for all Borel subsets A of $[-r, \infty)$, where $|\nu|$ is the total variation measure of ν wrt the Euclidean norm on $\mathbf{R}^{d\times d}$. Therefore the equation

$$\mu(\omega)(\cdot) := \int_0^\infty \bar{\nu}(t, \omega)(\cdot)\, dt$$

defines a positive measure on $[-r, \infty)$. For each $\omega \in \Omega$ suppose that $\mu(\omega)$ has a density wrt Lebesgue measure which is locally essentially bounded.
(*Exercise:* This condition is automatically satisfied if $\nu(t, \omega)$ is independent of (t, ω).)

(ii) $K : \mathbf{R} \times \Omega \to L^\infty([-r, o], \mathbf{R}^{d\times d})$ is measurable and $(\mathcal{F}_t)_{t\geq 0}$- adapted. Define the random field $\tilde{K}(t, s, \omega)$ by
$\tilde{K}(t, s, \omega) := K(t, \omega)(s - t)$ for $t \geq 0$, $-r \leq s - t \leq 0$. Assume that $\tilde{K}(t, s, \omega)$ is absolutely continuous in t for Lebesgue a.a. s and all $\omega \in \Omega$. For every $\omega \in \Omega$, $\dfrac{\partial \tilde{K}}{\partial t}(t, s, \omega)$ and $\tilde{K}(t, s, \omega)$ are locally essentially bounded in (t, s). $\dfrac{\partial \tilde{K}}{\partial t}(t, s, \omega)$ is jointly measurable.

(iii) $L = M + V$, M continuos local martingale, V B.V. process.

Theorem III.5. ([M-S], I, AIHP, 1996)

Under hypotheses (R), equation (IX) is regular w.r.t. M_2 with a measurable flow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$. This flow satisfies Theorem III.4.

Proof.

This is achieved via a construction in ([M-S], I, AIHP, 1996) which reduces (IX) to a random linear integral equation with *no stochastic integrals* ([M-S], AIHP, 1996, pp. 85-96). Do a complicated pathwise analysis on the integral equation to establish existence and regularity properties of the semiflow. $\square$

Regular Non-linear Systems

(a) SFDE's with Ordinary Diffusion Coefficients

In the sfde,

$$\left.\begin{aligned} dx(t) &= H(x_t)dt + \sum_{i=1}^{m} g_i(x(t))dW_i(t) \\ x_0 &= \eta \in C \end{aligned}\right\} \qquad (XV)$$

let $H : C \to \mathbf{R}^d$ be globally Lipschitz and $g_i : \mathbf{R}^d \to \mathbf{R}^d$ C^2-bounded maps satisfying the Frobenius condition (vanishing Lie brackets):

$$Dg_i(v)g_j(v) = Dg_j(v)g_i(v), \quad 1 \le i, j \le m, \ v \in \mathbf{R}^d;$$

and $W := (W_1, W_2, \cdots, W_m)$ is m-dimensional Brownian motion. Note that the diffusion coefficient in (XV) has no memory.

Theorem III.6 ([Mo], Pitman Books, 1984)

Suppose the above conditions hold. Then the trajectory field $\{^\eta x_t : t \ge 0, \eta \in C\}$ of (XV) has a measurable version $X : \mathbf{R}^+ \times C \times \Omega \to C$ satisfying the following properties. For each $\alpha \in (0, 1/2)$, there is a set $\Omega_\alpha \subset \Omega$ of full measure such that for every $\omega \in \Omega_\alpha$

(i) $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times C \to C$ *is continuous;*

(ii) $X(\cdot, \cdot, \omega) : [r, \infty) \times C \to C^\alpha$ *is continuous;*

(iii) *for each* $t \ge r$, $X(t, \cdot, \omega) : C \to C$ *is compact;*

(iv) *for each* $t \ge r$, $X(t, \cdot, \omega) : C \to C^\alpha$ *is Lipschitz on every bounded set in* C, *with a Lipschitz constant independent of* t *in compact sets. Hence each map* $X(t, \cdot, \omega) : C \to C$ *is compact: viz. takes bounded sets into relatively compact sets.*

Proof of Theorem III.6.

([Mo], Pitman Books, 1984, Theorem (2.1), Chapter (V), §2, p. 121). This latter result is proved using a non-linear variational method originally due to Sussman ([Su], Ann. Prob., 1978) and Doss ([Do], AIHP, 1977) in the non-delay case $r = 0$. Write $g := (g_1, g_2, \cdots, g_m) : \mathbf{R}^d \to \mathbf{R}^{d\times m}$. By the Frobenius condition, there is a C^2 map $F : \mathbf{R}^m \times \mathbf{R}^d \to \mathbf{R}^d$ such that $\{F(\underline{t}, \cdot) : \underline{t} \in \mathbf{R}^m\}$ is a group of C^2 diffeomorphisms $\mathbf{R}^d \to \mathbf{R}^d$ satisfying

$$\begin{aligned} D_1 F(\underline{t}, x) &= g(F(\underline{t}, x)), \\ F(\underline{0}, x) &= x \end{aligned}$$

for all $\underline{t} \in \mathbf{R}^m, x \in \mathbf{R}^d$.

Define

$$W^0(t) := \begin{cases} W(t) - W(0), & t \geq 0 \\ 0 & -r \leq t < 0 \end{cases}$$

and $\tilde{H} : \mathbf{R}^+ \times C \times \Omega \to \mathbf{R}^d$, by

$$\begin{aligned} \tilde{H}(t, \eta, \cdot) :=& D_2 F(W^0(t), \eta(0))^{-1} \big\{ H[F \circ (W_t^0, \eta)] \\ & - \frac{1}{2} \mathrm{trace}\big(Dg[F(W^0(t), \eta(0))] \circ g[F(W^0(t), \eta(0))] \big) \big\} \end{aligned}$$

where the expression under the "trace" is viewed as a bilinear form $\mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^d$, and the trace has values in $\mathbf{R}^d$. Then for each ω, $\tilde{H}(t, \eta, \omega)$ is jointly continuous, Lipschitz in η in bounded subsets of C uniformly for t in compact sets, and satisfies a global linear growth condition in η ([Mo], Pitman Books, 1984, pp. 114-126).

Therefore solve the fde

$$\begin{aligned} {}^\eta\xi'_t &= \tilde{H}(t, {}^\eta\xi_t, \cdot) \qquad t \geq 0 \\ {}^\eta\xi_0 &= \eta. \end{aligned}$$

Define the semiflow

$$X(t, \eta, \omega) = F \circ \big(W_t^0(\omega), {}^{\eta}x_t(\omega)\big).$$

Check that X satisfies all assertions of theorem ([Mo], 1984, pp.126-133). $\square$

(b) SFDE's with Smooth Memory

$$\left.\begin{aligned} dx(t) &= H(dt, x(t), x_t) + G(dt, x(t), g(x_t)), \quad t > 0 \\ (x(0), x_0) &= (v, eta) \in M_2 \end{aligned}\right\} \qquad (XVI)$$

Coefficients H and G in (XVI) are semimartingale-valued random fields on $M_2 = \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$ and $\mathbf{R}^d \times \mathbf{R}^m$, respectively. The memory is driven by a functional $g : L^2([-r, 0], \mathbf{R}^d) \to \mathbf{R}^m$ with the smoothness property that the process $t \mapsto g(x_t)$ has absolutely continuous paths for each adapted process x. Under (technical) but general regularity and boundedness conditions on the characteristics of H and G, equation (XVI) is regular:

Theorem III.7 ([M-S], 1996)

Let

$$\Delta := \{(t_0, t) \in \mathbf{R}^2 : t_0 \leq t\}.$$

Under suitable regularity conditions on H, G, g in (XVI), there exists a random field $X : \Delta \times M_2 \times \Omega \to M_2$ satisfying the following properties:

(i) *For each $(v, \eta) \in M_2$, $(t_0, t) \in \Delta$, $X(t_0, t, (v, \eta), \cdot) = (x^{t_0,(v,\eta)}(t), x_t^{t_0,(v,\eta)})$ a.s., where $x^{t_0,(v,\eta)}$ is the unique solution of (XVI) with $x_{t_0}^{t_0,(v,\eta)} = (v, \eta)$.*

(ii) *For each $(t_0, t, \omega) \in \Delta \times \Omega$, the map*

$$X(t_0, t, \cdot, \omega) : M_2 \to M_2$$

is C^∞.

(iii) *For each $\omega \in \Omega$ and $(t_0, t) \in \Delta$ with $t > t_0 + r$, the map*

$$X(t_0, t, \cdot, \omega) : M_2 \to M_2$$

carries bounded sets into relatively compact sets.

SFDE'S
AS DYNAMICAL SYSTEMS:
THE LINEAR CASE

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1. Regular Linear SFDE's-Ergodic Theory.

Linear sfde's on $\mathbf{R}^d$ driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$.

$$\left.\begin{aligned} dx(t) &= H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\ &\qquad + \sum_{i=1}^{m} g_i x(t)\, dW_i(t), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \tag{I}$$

(I) is defined on

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}}, P) =$ canonical complete filtered Wiener space.

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R} \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F} :=$(completed) Borel σ-field of Ω;

$\mathcal{F}_t :=$ (completed) sub-σ-field of $\mathcal{F}$ generated by the evaluations $\omega \to \omega(u), \; u \leq t, \quad t \in \mathbf{R}$.

$P :=$ Wiener measure on Ω.

$dW_i(t) =$ Itô stochastic differentials.

Several finite delays $0 < d_1 < d_2 < \cdots < d_N \leq r$ in drift term; *no delays in diffusion coefficient.*

$H : (\mathbf{R}^d)^{N+1} \times L^2([-r,0], \mathbf{R}^d) \to \mathbf{R}^d$ is a fixed continuous linear map, g_i, $i = 1, 2, \ldots, m$, fixed (deterministic) $d \times d$-matrices.

2. Plan

Use state space $M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d)$. For (I) consider the following themes:

I) Existence of a "perfect" cocycle on M_2-a modification of the trajectory field $(x(t), x_t) \in M_2$.

II) Existence of almost sure Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|(x(t), x_t)\|_{M_2}$$

Multiplicative ergodic theorem and *hyperbolicity* of cocycle.

III) *"Random Saddle-Point Property"* in hyperbolic case.

3. Regularity

Say SFDE (I) is *regular* (wrt. M_2) if trajectory $\{(x(t), x_t) : (x(0), x_0) = (v, \eta) \in M_2\}$ admits a measurable modification $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ such that $X(\cdot, \cdot, \omega)$ is continuous for a.a. $\omega \in \Omega$.

Theorem 1.([Mo], 1990])

(I) is regular with respect to state space $M_2 = \mathbf{R}^d \times \mathbf{L}^2([-r, 0], \mathbf{R}^d)$*. There is a measurable version* $X : \mathbf{R}^+ \times$

$M_2 \times \Omega \to M_2$ *of the trajectory field* $\{(x(t), x_t) : t \in \mathbf{R}^+,\ (x(0), x_0) = (v, \eta) \in M_2\}$ *of (I) with the following properties:*

(i) *For each* $(v, \eta) \in M_2$ *and* $t \in \mathbf{R}^+$, $X(t, (v, \eta), \cdot) = (x(t), x_t)$ *a.s., is* $\mathcal{F}_t$*-measurable and belongs to* $L^2(\Omega, M_2; P)$.

(ii) *There exists* $\Omega_0 \in \mathcal{F}$ *of full measure such that, for all* $\omega \in \Omega_0$*, the map* $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times M_2 \to M_2$ *is continuous.*

(iii) *For each* $t \in \mathbf{R}^+$ *and every* $\omega \in \Omega_0$*, the map* $X(t, \cdot, \omega) : M_2 \to M_2$ *is continuous linear; for each* $\omega \in \Omega_0$*, the map* $\mathbf{R}^+ \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ *is measurable and locally bounded in the uniform operator norm on* $L(M_2)$*. The map* $[r, \infty) \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ *is continuous for all* $\omega \in \Omega_0$.

(iv) For each $t \geq r$ and all $\omega \in \Omega_0$, the map

$$X(t, \cdot, \omega) : M_2 \to M_2$$

is compact.

Compactness of semi-flow for $t \geq r$ will be used to define hyperbolicity for (I) and the associated exponential dichotomies.

Example: $dx(t) = x(t-1)\,dW(t)$ *is not regular (singular).*

4. Lyapunov Exponents. Hyperbolicity

Version X of the trajectory field of (I) (in Theorem 1) is a multiplicative $L(M_2)$-valued linear cocycle over the canonical Brownian shift $\theta : \mathbf{R} \times \Omega \to \Omega$ on Wiener space:

$$\theta(t, \omega)(u) := \omega(t+u) - \omega(t), \quad u, t \in \mathbf{R}, \quad \omega \in \Omega.$$

I.e.

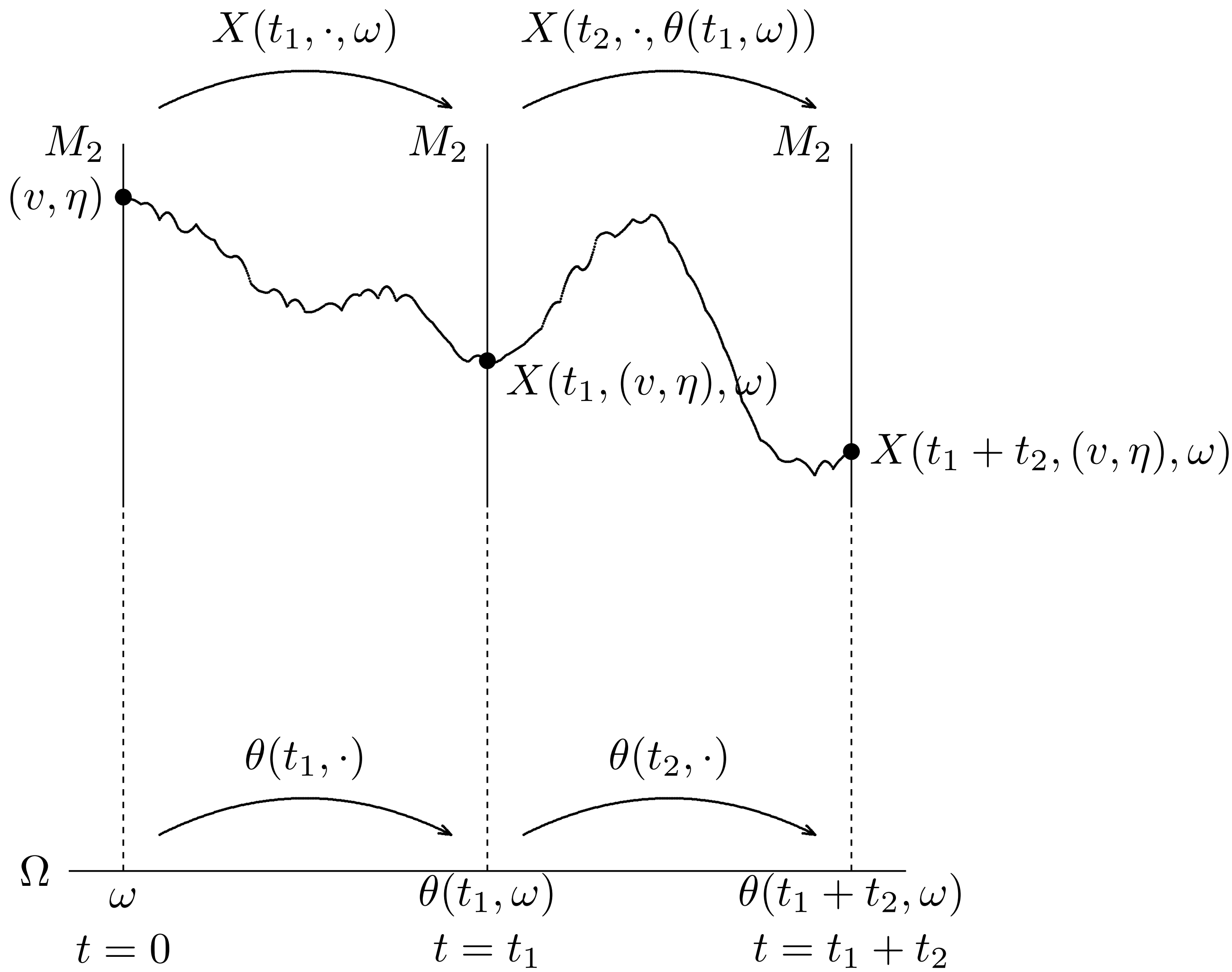
Theorem 2([Mo], 1990)

There is an $\mathcal{F}$-measurable set $\hat{\Omega}$ of full P-measure such that $\theta(t,\cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$ for all $t \geq 0$ and

$$X(t_2, \cdot, \theta(t_1, \omega)) \circ X(t_1, \cdot, \omega) = X(t_1 + t_2, \cdot, \omega)$$

for all $\omega \in \hat{\Omega}$ and t_1, $t_2 \geq 0$.

The Cocycle Property



Vertical solid lines represent random fibers: copies of M_2. (X, θ) is a "vector-bundle morphism".

Proof of Theorem 2. (Sketch)

For simplicity consider case of a single delay d_1; i.e. $N = 1$ in (I).

First step.

Approximate the Brownian motion W in (I) by smooth adapted processes $\{W^k\}_{k=1}^{\infty}$:

$$W^k(t) := k\int_{t-(1/k)}^{t} W(u)\,du - k\int_{-(1/k)}^{0} W(u)\,du, \qquad t \geq 0,\ k \geq 1. \tag{1}$$

Then each W^k is a *helix* (i.e. has stationary increments):

$$W^k(t_1+t_2,\omega) - W^k(t_1,\omega) = W^k(t_2,\theta(t_1,\omega)), \qquad t_1,t_2 \in \mathbf{R},\ \omega \in \Omega. \tag{2}$$

Let $X^k : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ be the stochastic (semi)flow of the random fde's:

$$\left.\begin{aligned} dx^k(t) &= H(x^k(t-d_1), x^k(t), x^k_t)dt \\ &\qquad + \sum_{i=1}^m g_i x(t)(W_i^k)'(t)\,dt - \frac{1}{2}\sum_{i=1}^m g_i^2 x^k(t)\,dt \qquad t>0 \\ (x^k(0), x^k_0) &= (v,\eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d) \end{aligned}\right\} \tag{$I-k$}$$

If $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ is the flow of (I) constructed in Theorem 1, then

$$\lim_{k\to\infty} \sup_{0\le t\le T} \|X^k(t,\cdot,\omega) - X(t,\cdot,\omega)\|_{L(M_2)} = 0 \tag{3}$$

for every $0 < T < \infty$ and all ω in a Borel set $\hat{\Omega}$ of full Wiener measure which is invariant under $\theta(t,\cdot)$ for all $t \ge 0$ ([Mo], Stochastics, 1990). Prove (3) by stochastic variation:

Let $\phi : \mathbf{R}^+ \times \Omega \to \mathbf{R}^{d\times d}$ be the $d \times d$-matrix-valued solution of the linear Itô sode (without delay):

$$\left.\begin{aligned} d\phi(t) &= \sum_{i=1}^{m} g_i \phi(t)\, dW_i(t) \qquad\quad t > 0 \\ \phi(0,\omega) &= I \in \mathbf{R}^{d\times d} \qquad\quad \text{a.a. } \omega \end{aligned}\right\} \tag{4}$$

Denote by $\phi^k : \mathbf{R}^+ \times \Omega \to \mathbf{R}^{d\times d}$, $k \geq 1$, the $d \times d$-matrix solution of the random family of linear ode's:

$$\left.\begin{aligned} d\phi^k(t) &= \sum_{i=1}^{m} g_i \phi^k(t) (W_i^k)'(t) - \frac{1}{2} \sum_{i=1}^{m} g_i^2 \phi^k(t)\, dt \qquad t > 0 \\ \phi^k(0,\cdot) &= I \in \mathbf{R}^{d\times d}. \end{aligned}\right\} \tag{4$'$}$$

Let $\hat{\Omega}$ be the sure event of all $\omega \in \Omega$ such that

$$\phi(t,\omega) := \lim_{k\to\infty} \phi^k(t,\omega) \tag{5}$$

exists uniformly for t in compact subsets of $\mathbf{R}^+$. Each ϕ^k is an $\mathbf{R}^{d\times d}$-valued *cocycle over* θ, viz.

$$\phi^k(t_1+t_2,\omega) = \phi^k(t_2,\theta(t_1,\omega))\phi^k(t_1,\omega) \qquad (6)$$

for all $t_1, t_2 \in \mathbf{R}^+$ and $\omega \in \Omega$. By definition of $\hat{\Omega}$ and passing to the limit in (6) as $k \to \infty$, conclude that $\{\phi(t,\omega) : t > 0,\ \omega \in \Omega\}$, is an $\mathbf{R}^{d\times d}$-valued *perfect* cocycle over θ, viz.

(i) $P(\hat{\Omega}) = 1$;

(ii) $\theta(t,\cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$ for all $t \geq 0$;

(iii) $\phi(t_1+t_2,\omega) = \phi(t_2,\theta(t_1,\omega))\phi(t_1,\omega)$ for all $t_1, t_2 \in \mathbf{R}^+$ and *every* $\omega \in \hat{\Omega}$;

(iv) $\phi(\cdot,\omega)$ is continuous for every $\omega \in \hat{\Omega}$.

Alternatively use perfection theorem in ([M-S], AIHP, 1996, Theorem 3.1, p. 79-82) for crude cocycles with values in a metrizable second countable topological group. Observe that $\phi(t,\omega) \in GL(\mathbf{R}^d)$.

Define $\hat{H} : \mathbf{R}^+ \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ by

$$\hat{H}(t, v_1, v, \eta, \omega)$$

$$:= \phi(t,\omega)^{-1}[H(\phi_t(\cdot,\omega)(-d_1, v_1), \phi(t,\omega)(v), \phi_t(\cdot,\omega) \circ (id_J, \eta))] \tag{7}$$

for $\omega \in \Omega, t \geq 0,\ v, v_1 \in \mathbf{R}^d,\ \eta \in L^2([-r,0], \mathbf{R}^d),$ where

$$\phi_t(\cdot,\omega)(s,v) = \begin{cases} \phi(t+s,\omega)(v) & t+s \geq 0 \\ \\ v & -r \leq t+s < 0 \end{cases}$$

and

$$(id_J, \eta)(s) = (s, \eta(s)), \qquad s \in J.$$

Define $\hat{H}^k : \mathbf{R}^+ \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ by a relation similar to (7) with ϕ replaced by ϕ^k. Then the random fde's

$$\left.\begin{aligned} y'(t) &= \hat{H}(t, y(t-d_1), y(t), y_t, \omega), \quad t > 0 \\ (y(0), y_0) &= (v, \eta) \in M_2 \end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned} {y^k}'(t) &= \hat{H}^k(t, y^k(t-d_1), y^k(t), y^k_t, \omega), \quad t > 0 \\ (y^k(0), y^k_0) &= (v, \eta) \in M_2 \end{aligned}\right\} \qquad (9)$$

have unique *non-explosive* solutions

$$y,\, y^k : [-r, \infty) \times \Omega \to \mathbf{R}^d$$

([Mo], Stochastics, 1990, pp. 93-98). Itô's formula implies that

$$X(t, v, \eta, \omega) = (\phi(t, \omega)(y(t, \omega)), \phi_t(\cdot, \omega) \circ (id_J, y_t)) \quad (10)$$

The chain rule gives a similar relation for X^k with ϕ replaced by ϕ^k ([Mo], Stochastics, 1990, pp. 96-97).

Get the convergence

$$\lim_{k\to\infty} |\hat{H}^k(t, v_1, v, \eta, \omega) - \hat{H}(t, v_1, v, \eta, \omega)| = 0 \qquad (11)$$

uniformly for (t, v_1, v, η) in bounded sets of $\mathbf{R}^+ \times \mathbf{R}^d \times M_2$. Use Gronwall's lemma and (11) to deduce (3).

Second step.

Fix $\omega \in \hat{\Omega}$ and use uniqueness of solutions to the approximating equation (I-k) and the helix property (2) of W^k to obtain the cocycle property for (X^k, θ):

$$X^k(t_2, \cdot, \theta(t_1, \omega)) \circ X^k(t_1, \cdot, \omega) = X^k(t_1 + t_2, \cdot, \omega)$$

for all $\omega \in \hat{\Omega}$ and $t_1,\ t_2 \geq 0,\ k \geq 1$.

Third step.

Pass to limit as $k \to \infty$ in the above identity and use the convergence (3) *in operator norm* to get the perfect cocycle property for X. $\square$

The a.s. Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v(\omega),\eta(\omega)),\omega)\|_{M_2},$$

(for a.a. $\omega \in \Omega$, $(v,\eta) \in L^2(\Omega, M_2)$) of the system (I) are characterized by the following "spectral theorem". Each $\theta(t,\cdot)$ is ergodic and preserves Wiener measure P. The proof of Theorem 3 below uses compactness of $X(t,\cdot,\omega) : M_2 \to M_2$, $t \geq r$, together with an infinite-dimensional version of Oseledec's multiplicative ergodic theorem due to Ruelle (1982).

Theorem 3. ([Mo], 1990)

Let $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ be the flow of (I) given in Theorem 1. Then there exist

(a) an $\mathcal{F}$-measurable set $\Omega^* \subseteq \Omega$ such that $P(\Omega^*) = 1$ and $\theta(t,\cdot)(\Omega^*) \subseteq \Omega^*$ for all $t \geq 0$,

(b) *a fixed (non-random) sequence of real numbers* $\{\lambda_i\}_{i=1}^{\infty}$*, and*

(c) *a random family* $\{E_i(\omega) : i \geq 1, \omega \in \Omega^*\}$ *of (closed) finite-codimensional subspaces of* M_2*, with the following properties:*

(i) If the **Lyapunov spectrum** $\{\lambda_i\}_{i=1}^{\infty}$ *is infinite, then* $\lambda_{i+1} < \lambda_i$ *for all* $i \geq 1$ *and* $\lim\limits_{i \to \infty} \lambda_i = -\infty$*; otherwise there is a fixed (non-random) integer* $N \geq 1$ *such that* $\lambda_N = -\infty < \lambda_{N-1} < \cdots < \lambda_2 < \lambda_1$*;*

(ii) each map $\omega \mapsto E_i(\omega)$, $i \geq 1$, *is* $\mathcal{F}$*-measurable into the Grassmannian of* M_2*;*

(iii) $E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2$, $i \geq 1$, $\omega \in \Omega^*$*;*

(iv) for each $i \geq 1$*, codim* $E_i(\omega)$ *is fixed independently of* $\omega \in \Omega^*$*;*

(v) for each $\omega \in \Omega^*$ *and* $(v, \eta) \in E_i(\omega) \backslash E_{i+1}(\omega)$,

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} = \lambda_i, \ i \geq 1;$$

(vi) **Top Exponent***:*

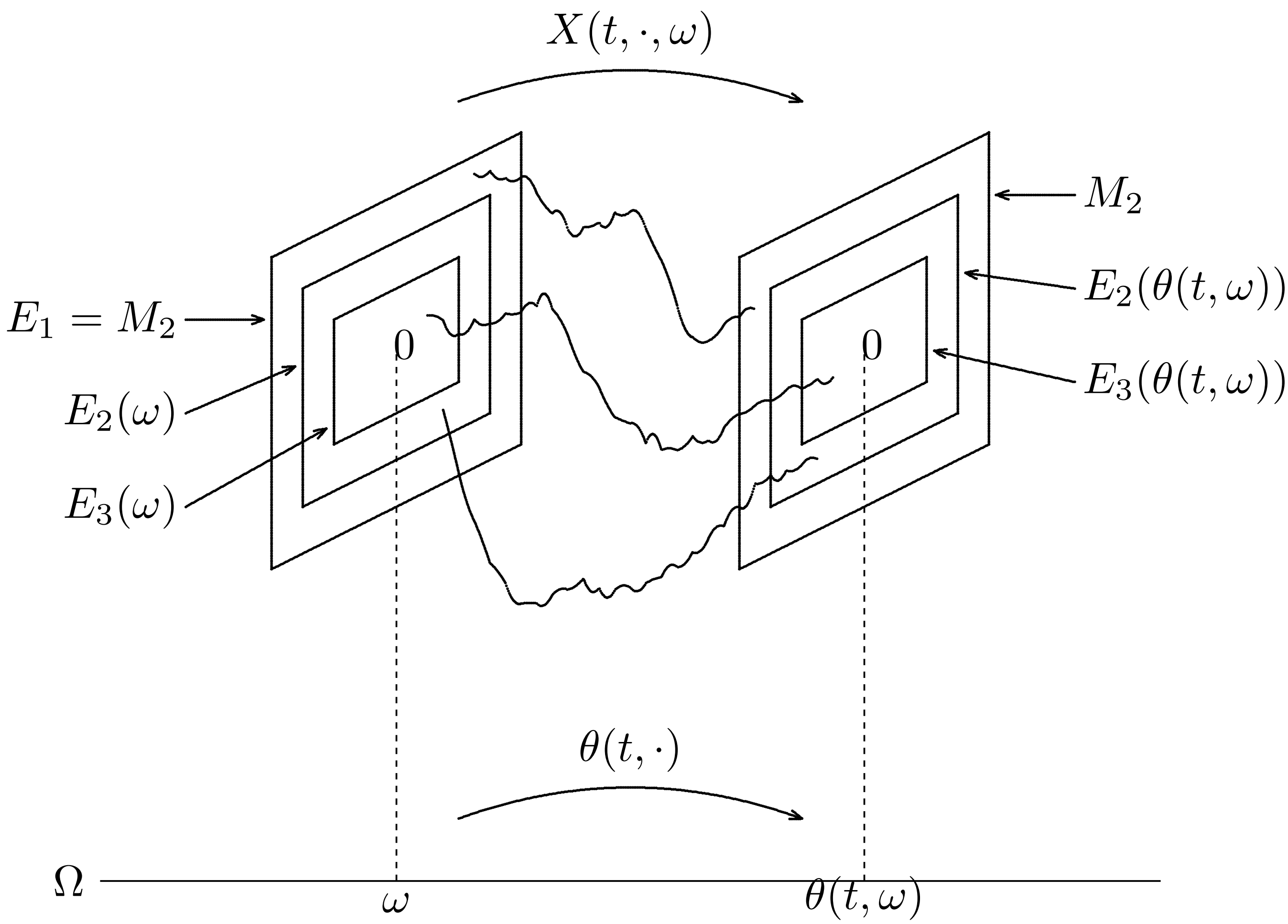
$$\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \log \|X(t, \cdot, \omega)\|_{L(M_2)} \quad \text{for all } \omega \in \Omega^*;$$

(vii) **Invariance***:*

$$X(t, \cdot, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

for all $\omega \in \Omega^*, \ t \geq 0, \ i \geq 1.$

Spectral Theorem



Proof of Theorem 3 is based on Ruelle's discrete version of Oseledec's multiplicative ergodic theorem in Hilbert space ([Ru], Ann. of Math. 1982, Theorem (1.1), p. 248 and Corollary (2.2), p. 253):

Theorem 4 ([Ru], 1982)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\tau : \Omega \to \Omega$ a P-preserving transformation. Assume that H is a separable Hilbert space and $T : \Omega \to L(H)$ a measurable map (w.r.t. the Borel field on the space of all bounded linear operators $L(H)$). Suppose that $T(\omega)$ is compact for almost all $\omega \in \Omega$, and $E \log^+ \|T(\cdot)\| < \infty$. Define the family of linear operators $\{T^n(\omega) : \omega \in \Omega,\ n \geq 1\}$ by

$$T^n(\omega) := T(\tau^{n-1}(\omega)) \circ \cdots T(\tau(\omega)) \circ T(\omega)$$

for $\omega \in \Omega,\ n \geq 1$.

Then there is a set $\Omega_0 \in \mathcal{F}$ of full P-measure such that $\tau(\Omega_0) \subseteq \Omega_0$, and for each $\omega \in \Omega_0$, the limit

$$\lim_{n\to\infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm and is a positive compact self-adjoint operator on H. Furthermore, each $\Lambda(\omega)$ has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

where the μ_i's are distinct. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at $\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigen-space $F_i(\omega)$, with $m_i(\omega) := dim F_i(\omega)$. Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \quad E_\infty(\omega) := ker\, \Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = H$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|T^n(\omega)x\|_H = \begin{cases} \mu_i(\omega), & \textit{if} \quad x \in E_i(\omega) \backslash E_{i+1}(\omega) \\ -\infty & \textit{if} \quad x \in \textit{ker}\ \Lambda(\omega). \end{cases}$$

Proof.

[Ru], Ann. of Math., 1982, pp. 248-254.

□

The following "perfect" version of Kingman's subadditive ergodic theorem is also used to construct the shift invariant set Ω^* appearing in Theorem 3 above.

Theorem 5([M], 1990)("Perfect" Subadditive Ergodic Theorem)

Let $f : \mathbf{R}^+ \times \Omega \to \mathbf{R} \cup \{-\infty\}$ *be a measurable process on the complete probability space* $(\Omega, \mathcal{F}, P)$ *such that*

(i) $E \sup\limits_{0 \le u \le 1} f^+(u, \cdot) < \infty$, $E \sup\limits_{0 \le u \le 1} f^+(1-u, \theta(u, \cdot)) < \infty$;

(ii) $f(t_1+t_2, \omega) \le f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ *for all* $t_1, t_2 \ge 0$ *and* **every** $\omega \in \Omega$.

Then there exist a set $\hat{\hat{\Omega}} \in \mathcal{F}$ *and a measurable* $\tilde{f} : \Omega \to \mathbf{R} \cup \{-\infty\}$ *with the properties:*

(a) $P(\hat{\hat{\Omega}}) = 1$, $\theta(t, \cdot)(\hat{\hat{\Omega}}) \subseteq \hat{\hat{\Omega}}$ *for all* $t \ge 0$;

(b) $\tilde{f}(\omega) = \tilde{f}(\theta(t, \omega))$ *for all* $\omega \in \hat{\hat{\Omega}}$ *and all* $t \ge 0$;

(c) $\tilde{f}^+ \in \mathbf{L}^1(\Omega, \mathbf{R}; P)$;

(d) $\lim\limits_{t \to \infty} (1/t) f(t, \omega) = \tilde{f}(\omega)$ *for every* $\omega \in \hat{\hat{\Omega}}$.

If θ *is ergodic, then there exist* $f^* \in \mathbf{R} \cup \{-\infty\}$ *and* $\tilde{\tilde{\Omega}} \in \mathcal{F}$ *such that* *(a)'* $P(\tilde{\tilde{\Omega}}) = 1, \theta(t,\cdot)(\tilde{\tilde{\Omega}}) \subseteq \tilde{\tilde{\Omega}}$, $t \geq 0$;

(b)' $\tilde{f}(\omega) = f^* = \lim_{t\to\infty}(1/t)f(t,\omega)$ *for every* $\omega \in \tilde{\tilde{\Omega}}$.

Proof.

[Mo], Stochastics, 1990, Lemma 7, pp. 115–117. □

Proof of Theorem 3 is an application of Theorem 4. Requires Theorem 5 and the following sequence of lemmas.

Lemma 1

For each integer $k \geq 1$ *and any* $0 < a < \infty$,

$$E \sup_{0\leq t\leq a} \|\phi(t,\omega)^{-1}\|^{2k} < \infty;$$

$$E \sup_{0\leq t_1,t_2\leq a} \|\phi(t_2,\theta(t_1,\cdot))\|^{2k} < \infty.$$

Proof.

Follows by standard sode estimates, the cocycle property for ϕ and Hölder's inequality. ([Mo], pp. 106-108). $\square$

The next lemma is a crucial estimate needed to apply Ruelle-Oseledec theorem (Theorem 4).

Lemma 2

$$E \sup_{0\le t_1, t_2 \le r} \log^+ \|X(t_2, \cdot, \theta(t_1, \cdot))\|_{L(M_2)} < \infty.$$

Proof.

If $y(t,(v,\eta),\omega)$ is the solution of the fde (8), then using Gronwall's inequality, taking

$E \sup_{0\le t_1,t_2\le r} \log^+ \sup_{\|(v,\eta)\|\le 1}$ and applying Lemma 1, gives

$$E \sup_{0\le t_1,t_2\le r} \log^+ \sup_{\|(v,\eta)\|\le 1} \|(y(t_2,(v,\eta),\theta(t_1,\cdot)), y_{t_2}(\cdot,(v,\eta),\theta(t_1,\cdot)))\|_{M_2} < \infty.$$

Conclusion of lemma now follows by replacing ω' with $\theta(t_1, \omega)$ in the formula

$$X(t_2, (v, \eta), \omega')$$
$$= (\phi(t_2, \omega')(y(t_2, (v, \eta), \omega')), \phi_{t_2}(\cdot, \omega') \circ (id_J, y_{t_2}(\cdot, (v, \eta), \omega'))$$

and Lemma 1. $\square$

The existence of the Lyapunov exponents is obtained by interpolating the discrete limit

$$\frac{1}{r} \lim_{k \to \infty} \frac{1}{k} \log \|X(kr, (v(\omega), \eta(\omega)), \omega)\|_{M_2}, \qquad (12)$$

a.a. $\omega \in \Omega$, $(v, \eta) \in L^2(\Omega, M_2)$, between delay periods of length r. This requires the next two lemmas.

Lemma 3

Let $h : \Omega \to \mathbf{R}^+$ be $\mathcal{F}$-measurable and suppose $E \sup_{0 \le u \le r} h(\theta(u, \cdot)$ is finite. Then

$$\Omega_1 := \big(\lim_{t \to \infty} \frac{1}{t} h(\theta(t, \cdot) = 0\big)$$

is a sure event and $\theta(t, \cdot)(\Omega_1) \subseteq \Omega_1$ for all $t \geq 0$.

Proof.

Use interpolation between delay periods and the discrete ergodic theorem applied to the L^1 function

$$\hat{h} := \sup_{0 \le u \le r} h(\theta(u, \cdot).$$

([Mo], Stochastics, 1990, Lemma 5, pp. 111-113.) $\square$

Lemma 4

Suppose there is a sure event Ω_2 such that $\theta(t,\cdot)(\Omega_2) \subseteq \Omega_2$ for all $t \geq 0$, and the limit (12) exists (or equal to $-\infty$) for all $\omega \in \Omega_2$ and all $(v,\eta) \in M_2$. Then there is a sure event Ω_3 such that $\theta(t,\cdot)(\Omega_3) \subseteq \Omega_3$ and

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2} = \frac{1}{r} \lim_{k\to\infty} \frac{1}{k} \log \|X(kr,(v,\eta),\omega)\|_{M_2}, \tag{13}$$

for all $\omega \in \Omega_3$ and all $(v,\eta) \in M_2$.

Proof:

Take $\Omega_3 := \hat{\Omega} \cap \Omega_1 \cap \Omega_2$. Use cocycle property for X, Lemma 2 and Lemma 3 to interpolate. ([Mo], Stochastics 1990, Lemma 6, pp. 113-114.)

$\square$

Proof of Theorem 3. (Sketch)

Apply Ruelle-Oseledec Theorem (Theorem 4) with

$T(\omega) := X(r, \omega) \in L(M_2)$, compact linear for $\omega \in \hat{\Omega}$;

$\tau : \Omega \to \Omega; \quad \tau := \theta(r, \cdot)$.

Then cocycle property for X implies

$$
\begin{aligned}
X(kr, \omega, \cdot) &= T(\tau^{k-1}(\omega)) \circ T(\tau^{k-2}(\omega)) \circ \cdots \circ T(\tau(\omega)) \circ T(\omega) \\
&:= T^k(\omega)
\end{aligned}
$$

for all $\omega \in \hat{\Omega}$.

Lemma 2 implies

$$
E \log^+ \|T(\cdot)\|_{L(M_2)} < \infty.
$$

Theorem 4 gives a random family of compact self-adjoint positive linear operators $\{\Lambda(\omega) : \omega \in \Omega_4\}$ such that

$$\lim_{n\to\infty}[T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm for $\omega \in \Omega_4$, a (continuous) shift-invariant set of full measure. Furthermore each $\Lambda(\omega)$ has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

where the μ_i's are distinct, with no accumulation points except possibly $-\infty$. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at $\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigenspace $F_i(\omega)$, with $m_i(\omega) := dim F_i(\omega)$. Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \quad E_\infty(\omega) := \ker \Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2.$$

Note that $\operatorname{codim} E_i(\omega) = \sum_{j=1}^{i-1} m_j(\omega) < \infty$. Also

$$\lim_{k\to\infty} \frac{1}{k} \log \|X(kr,(v,\eta),\omega)\|_{M_2} = \begin{cases} \mu_i(\omega), & \text{if } (v,\eta) \in E_i(\omega)\backslash E_{i+1}(\omega) \\ -\infty & \text{if } (v,\eta) \in \ker \Lambda(\omega). \end{cases}$$

The functions

$$\omega \mapsto \mu_i(\omega), \quad \omega \mapsto m_i(\omega), \quad \omega \mapsto N(\omega)$$

are invariant under the ergodic shift $\theta(r,\cdot)$. Hence they take the fixed values $\mu_i,\ m_i,\ N$ almost surely, respectively.

Lemma 4 gives a continuous-shift-invariant sure event $\Omega^* \subseteq \Omega_4$ such that

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2} &= \frac{1}{r} \lim_{k\to\infty} \frac{1}{k} \log \|X(kr,(v,\eta),\omega)\|_{M_2} \\ &= \frac{\mu_i}{r} =: \lambda_i, \end{aligned}$$

for $(v, \eta) \in E_i(\omega) \backslash E_{i+1}(\omega), \ \omega \in \Omega^*, i \geq 1$.

$\{\lambda_i := \dfrac{\mu_i}{r} : i \geq 1\}$ is the *Lyapunov spectrum* of (I).

Since Lyapunov spectrum is discrete with no finite accumulation points, then $\{\lambda_i : \lambda_i > \lambda\}$ is finite for all $\lambda \in \mathbf{R}$.

To prove invariance of the Oseledec space $E_i(\omega)$ under the cocycle (X, θ) use the random field

$$\lambda((v, \eta), \omega) := \lim_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2}, \ (v, \eta) \in M_2, \ \omega \in \Omega^*$$

and the relations

$$E_i(\omega) := \{(v, \eta) \in M_2 : \lambda((v, \eta), \omega) \leq \lambda_i\},$$

$$\lambda(X(t, (v, \eta), \omega), \theta(t, \omega)) = \lambda((v, \eta), \omega), \qquad \omega \in \Omega^*, \ t \geq 0$$

([Mo], Stochastics 1990, p. 122). $\square$

Lyapunov exponents $\{\lambda_i\}_{i=1}^{\infty}$ of (I) are non-random because θ is ergodic. Say (I) is *hyperbolic* if $\lambda_i \neq 0$ for all $i \geq 1$. When (I) is hyperbolic the flow satisfies a *stochastic saddle-point property* (or exponential dichotomy) (cf. the deterministic case with $E = C([-r, 0], \mathbf{R}^d)$, $g_i \equiv 0$, $i = 1, \ldots, m$, in Hale [H], Theorem 4.1, p. 181).

Theorem 6 *(Random Saddles)*([Mo], 1990)

Suppose the sfde (I) is hyperbolic. Then there exist

(a) a set $\tilde{\Omega}^* \in \mathcal{F}$ *such that* $P(\tilde{\Omega}^*) = 1$, *and* $\theta(t, \cdot)(\tilde{\Omega}^*) = \tilde{\Omega}^*$ *for all* $t \in \mathbf{R}$,

and

(b) a measurable splitting

$$M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \tilde{\Omega}^*,$$

with the following properties:

(i) $\mathcal{U}(\omega)$, $\mathcal{S}(\omega)$, $\omega \in \tilde{\Omega}^*$, *are closed linear subspaces of* M_2, $\dim \mathcal{U}(\omega)$ *is finite and fixed independently of* $\omega \in \tilde{\Omega}^*$.

(ii) *The maps* $\omega \mapsto \mathcal{U}(\omega)$, $\omega \mapsto \mathcal{S}(\omega)$ *are* $\mathcal{F}$*-measurable into the Grassmannian of* M_2.

(iii) *For each* $\omega \in \tilde{\Omega}^*$ *and* $(v, \eta) \in \mathcal{S}(\omega)$ *there exists* $\tau_1 = \tau_1(v, \eta, \omega) > 0$ *and a positive* δ_1, *independent of* (v, η, ω) *such that*

$$\|X(t, (v, \eta), \omega)\|_{M_2} \leq \|(v, \eta)\|_{M_2} e^{-\delta_1 t}, \quad t \geq \tau_1.$$

(iv) *For each* $\omega \in \tilde{\Omega}^*$ *and* $(v, \eta) \in \mathcal{U}(\omega)$ *there exists* $\tau_2 = \tau_2(v, \eta, \omega) > 0$ *and a positive* δ_2, *independent of* (v, η, ω) *such that*

$$\|X(t, (v, \eta), \omega)\|_{M_2} \geq \|(v, \eta)\|_{M_2} e^{\delta_2 t}, \quad t \geq \tau_2.$$

(v) For each $t \geq 0$ and $\omega \in \tilde{\Omega}^*$,

$$X(t,\omega,\cdot)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t,\omega)),$$

$$X(t,\omega,\cdot)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)).$$

In particular, the restriction

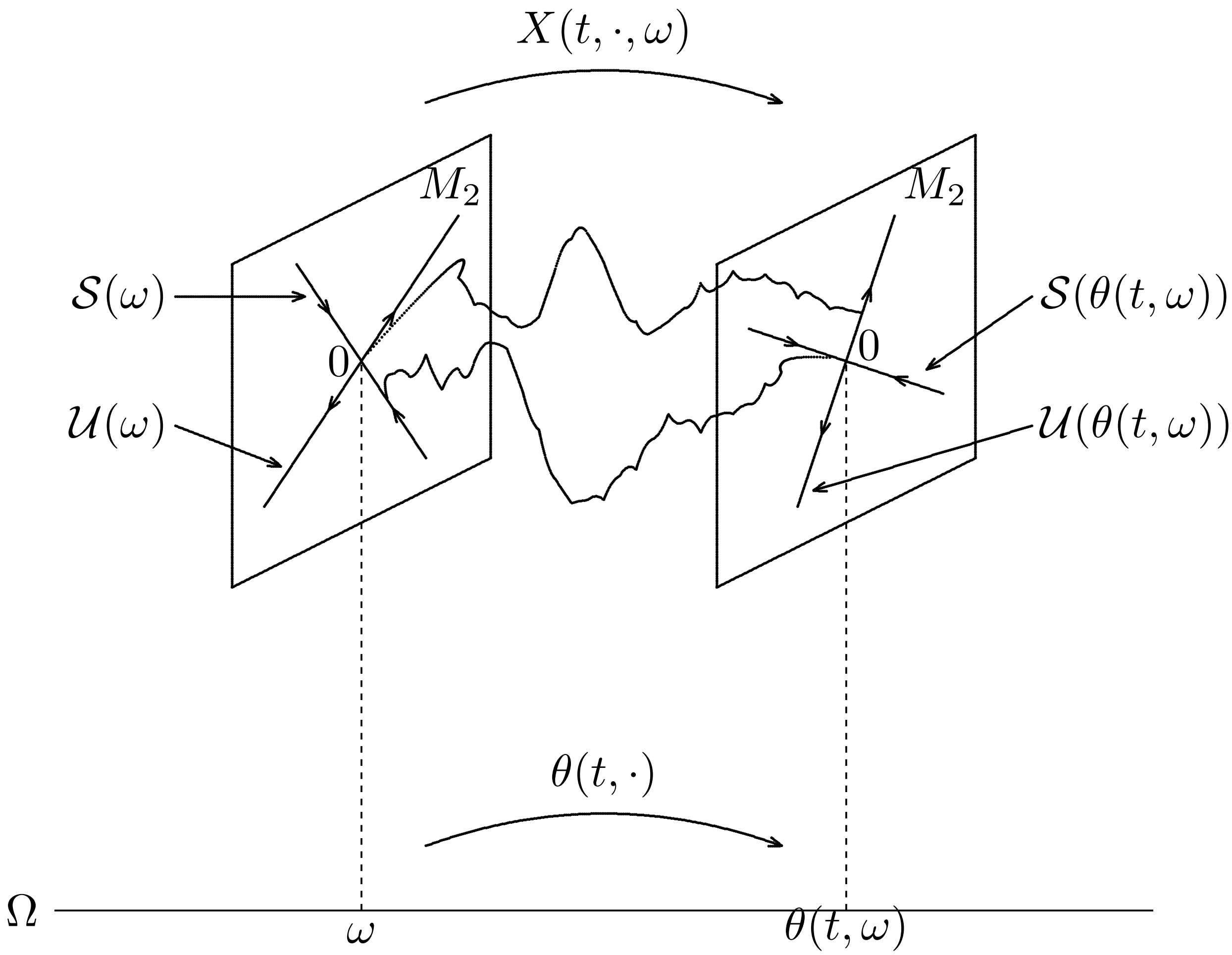
$$X(t,\omega,\cdot)\,|\,\mathcal{U}(\omega) : \mathcal{U}(\omega) \to \mathcal{U}(\theta(t,\omega))$$

is a linear homeomorphism onto.

Proof.

[Mo], Stochastics, 1990, Corollary 2, pp. 127-130. $\square$

The Saddle-Point Property



STABILITY:

EXAMPLES AND CASE STUDIES

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STABILITY. EXAMPLES AND CASE STUDIES

1. Plan.

I) Estimates on the "maximal exponential growth rate" for the singular noisy feedback loop. Use of Lyapunov functionals.

II) Examples and case studies of linear sfde's: Existence of the stochastic semiflow and its Lyapunov spectrum.

III) Study almost sure asymptotic stability via upper bounds on the top Lyapunov exponent λ_1.

IV) Lyapunov spectrum for sdde's with Poisson noise.

Lyapunov exponents for linear sode's (without memory): studied by many authors: e.g. Arnold, Kliemann and Oeljeklaus, 1989, Arnold, Oeljeklaus and Pardoux, 1986, Baxendale, 1985, Pardoux and Wihstutz[PW1], 1988, Pinsky and Wihstutz [PW2], 1988, and the references therein.

Asymptotic stability of sfde's: treated in Kushner [K], JDE, 1968, Mizel and Trutzer [MT],1984, Mohammed [M1]-[M4], 1984, 1986, 1990, 1992, Mohammed and Scheutzow [MS], 1996, Scheutzow [S], 1988, Kolmanovskii and Nosov [KN], 1986. Mao ([Ma], 1994, Chapter 5) gives several results concerning top exponential growth rate for sdde's driven by $\mathcal{C}$-valued semimartingales. Assumes that second-order characteristics of the driving semimartingales are *time-dependent* and *decay to zero exponentially fast in time, uniformly in the space variable.*

2. Noisy Feedback Loop Revisited Once More!

Noisy feedback loop is modelled by the one-dimensional linear sdde

$$\left.\begin{aligned} dx(t) &= \sigma x(t-r))\, dW(t), \quad t>0 \\ (x(0), x_0) &= (v,\eta) \in M_2 := \mathbf{R} \times L^2([-r,0],\mathbf{R}), \end{aligned}\right\} \qquad (I)$$

driven by a Wiener process W with a *positive delay* r.

(I) is singular with respect to M_2 (Theorem III.3).

Consider the more general one-dimensional linear sfde:

$$\left.\begin{aligned} dx(t) &= \int_{-r}^{0} x(t+s) d\nu(s)\, dW(t), \quad t>0 \\ (x(0), x_0) &\in M_2 := \mathbf{R} \times L^2([-r,0],\mathbf{R}) \end{aligned}\right\} \qquad (II')$$

where W is a Wiener process and ν is a fixed finite real-valued Borel measure on $[-r, 0]$.

(II') is regular if ν has a C^1 (or even L_1^2) density with respect to Lebesgue measure on $[-r, 0]$ ([M-S], I, 1996). If ν satisfies Theorem III.3, then (II') is singular.

In the singular case, there is no stochastic flow (Theorem III.3) and we do not know whether a (discrete) set of Lyapunov exponents

$$\lambda((v,\eta),\cdot) := \lim_{t\to\infty} \frac{1}{t} \log \|(x(t,(v,\eta)), x_t(\cdot,(v,\eta)))\|_{M_2}, \qquad (v,\eta) \in M_2$$

exists. Existence of Lyapunov exponents for singular equations is hard. But can still define the *maximal exponential growth rate*

$$\overline{\lambda}_1 := \sup_{(v,\eta)\in M_2} \limsup_{t\to\infty} \frac{1}{t} \log \|(x(t,(v,\eta)), x_t(\cdot,(v,\eta)))\|_{M_2}$$

for the trajectory random field $\{(x(t,(v,\eta)), x_t(\cdot,(v,\eta))) : t \geq 0, (v,\eta) \in M_2\}$. $\overline{\lambda}_1$ may depend on $\omega \in \Omega$. But $\overline{\lambda}_1 = \lambda_1$ in the regular case.

Inspite of the *extremely erratic dependence on the initial paths* of solutions of (I), it is shown in Theorem V.1 that for small noise variance, *uniform almost sure global asymptotic stability* still persists. For small σ, $\overline{\lambda}_1 \leq -\sigma^2/2 + o(\sigma^2)$ uniformly in the initial path (Theorem V.1, and Remark (iii)). For large $|\sigma|$ and $\nu = \delta_{-r}$,

$$\frac{1}{2r} \log|\sigma| + o(\log|\sigma|) \leq \overline{\lambda}_1 \leq \frac{1}{r} \log|\sigma|$$

([M-S], II, 1996, Remark (ii) after proof ofTheorem 2.3). This result is in sharp contrast with the non-delay case ($r = 0$), where $\lambda_1 = -\sigma^2/2$ *for all* values of σ. Proofs of Theorems V.1, V.2

involve very delicate constructions of new types of Lyapunov functionals on the underlying state space.

Theorem V.1.([M-S], II, 1996).

Let ν be a probability measure on $[-r,0]$, $r>0$, and consider the sfde

$$\left.\begin{aligned} dx(t) &= \sigma\left(\int_{[-r,0]} x(t+s)\,d\nu(s)\right) dW(t), \quad t\geq 0 \\ (x(0),x_0) &= (v,\eta)\in M_2 \end{aligned}\right\} \qquad (II')$$

with $\sigma\in\mathbf{R}$, $(v,\eta)\in M_2$, W standard Brownian motion, and $x(\cdot,(v,\eta))$ the solution of (II') through $(v,\eta)\in M_2$. Then there exists $\sigma_0>0$ and a continuous strictly negative nonrandom function $\phi:(-\sigma_0,\sigma_0)\to\mathbf{R}^-$ (independent of $(v,\eta)\in M_2$ and ν) such that

$$P\left(\limsup_{t\to\infty}\frac{1}{t}\log\|(x(t,(v,\eta)),x_t(\cdot,(v,\eta)))\|_{M_2}\leq\phi(\sigma)\right)=1.$$

for all $(v, \eta) \in M_2$ *and all* $-\sigma_0 < \sigma < \sigma_0$.

Remark:

Theorem also holds for state space C with $\|\cdot\|_\infty$.

Proof of Theorem V.1. (Sketch)

Sufficient to consider (II') on $C \equiv C([-r, 0], \mathbf{R})$, because C is continuously embedded in M_2. W.l.o.g., assume that $\sigma > 0$.

- Use Lyapunov functional $V : C \to \mathbf{R}^+$

$$V(\eta) := (R(\eta) \vee |\eta(0)|)^\alpha + \beta R(\eta)^\alpha, \quad \eta \in C.$$

where $R(\eta) := \overline{\eta} - \underline{\eta}$, the diameter of the range of η, $\overline{\eta} := \sup_{-r \leq s \leq 0} \eta(s)$ and $\underline{\eta} := \inf_{-r \leq s \leq 0} \eta(s)$.

- Fix $0 < \alpha < 1$ and *arrange* for $\beta = \beta(\sigma)$ for sufficiently small σ such that

$$E(V({}^{\eta}x_r)) \leq \delta(\sigma)V(\eta), \quad \eta \in C, \tag{1}$$

  and $\delta(\sigma) \in (0,1)$ is a continuous function of σ defined near 0. There is a positive $K = K(\alpha)$ (independent of η, ν) such that $\delta(\sigma) \sim (1 - K\sigma^2)$. Set

$$\phi(\sigma) := \frac{1}{\alpha}\log\delta(\sigma).$$

  Estimate (1) is hard ([M-S], II, 1996, pp. 12-18).

- $\{{}^{\eta}x_{nr}\}_{n=1}^{\infty}$ is a Markov process in C. So (1) implies that $\delta(\sigma)^{-n}V({}^{\eta}x_{nr})$, $n \geq 1$, is a non-negative $(\mathcal{F}_{nr})$ supermartingale.

- There exists $Z : \Omega \to [0,\infty)$ such that

$$\lim_{n\to\infty}\frac{V({}^{\eta}x_{nr})}{\delta(\sigma)^n} = Z \qquad \text{a.s.} \tag{2}$$

- Form of V and (2) imply

$$\begin{aligned}\overline{\lim_{t\to\infty}}\frac{1}{t}\log|x(t)| &\le \overline{\lim_{n\to\infty}}\frac{1}{nr}\log[|x(nr)|+R(x_{nr})]\\ &= \frac{1}{\alpha}\overline{\lim_{n\to\infty}}\frac{1}{nr}\log V(x_{nr}) \le \frac{1}{\alpha}\log\delta(\sigma)=\phi(\sigma)<0.\end{aligned}$$

- $\delta(\sigma)$, $\phi(\sigma)$ independent of η, ν. "Domain" of ϕ also independent of η, ν. $\square$

Remarks.

(i) Choice of σ_0 in Theorem V.1 depends on r. In (I) the scaling $t \mapsto t/r$ has the effect of replacing r by 1 and σ by $\sigma\sqrt{r}$. If $\overline{\lambda}_1(r,\sigma)$ is the maximal exponential growth rate of (I), then $\overline{\lambda}_1(r,\sigma) = \frac{1}{r}\overline{\lambda}_1(1,\sigma\sqrt{r})$ (*Exercise*). Hence σ_0 decreases (like $\frac{1}{\sqrt{r}}$) as r increases . Thus (for a fixed σ), a *small delay* r tends to *stabilize* equation (I). A *large delay* in (I) has a *destabilizing* effect (Theorem V.2 below).

(ii) Using a Lyapunov function(al) argument, Theorem V.2 below shows that for sufficiently large σ, the singular delay equation (I) is unstable. Result is in sharp contrast with the non-delay case $r = 0$, where

$$\lim_{t\to\infty} \frac{1}{t} \log |x(t)| = -\sigma^2/2 < 0$$

for *all* $\sigma \in \mathbf{R}$ (even when σ is large).

(iii) The growth rate function ϕ in Theorem V.1 satisfies

$$\phi(\sigma) = -\sigma^2/2 + o(\sigma^2)$$

as $\sigma \to 0^+$. Agrees with non-delay case $r = 0$. Above relation follows by modifying proof of Theorem V.1.

Theorem V.2.

Consider the equation

$$\left.\begin{aligned} dx(t) &= \sigma x(t-1)\,dW(t), \quad t>0 \\ (x(0),x_0) &= (v,\eta)\in M_2 := \mathbf{R}\times L^2([-r,0],\mathbf{R}), \end{aligned}\right\} \qquad (I)$$

driven by a standard Wiener process W with a positive delay r and $\sigma \in \mathbf{R}$. Then there exists a continuous function $\psi : (0,\infty) \to \mathbf{R}$ which is increasing to infinity such that

$$P\left(\liminf_{t\to\infty} \frac{1}{t} \log \|(x(t,(v,\eta)), x_t(\cdot,(v,\eta))\|_{M_2} \geq \psi(|\sigma|)\right) = 1,$$

for all $(v,\eta) \in M_2\backslash\{0\}$ and all sufficiently large $|\sigma|$. The function ψ is independent of the choice of $(v,\eta) \in M_2\backslash\{0\}$.

Remarks.

(i) $\|\cdot\|_{M_2}$ can be replaced by the sup-norm on C.

(ii) Proof shows $\psi(\sigma) \sim \frac{1}{2}\log\sigma$ for large σ.

Proof of Theorem V.2.

Use the continuous Lyapunov functional

$$V : M_2\backslash\{0\} \to [0,\infty)$$

$$V((v,\eta)) := \left(v^2 + |\sigma| \int_{-1}^{0} \eta^2(s)\, ds\right)^{-1/4}$$

[M-S], Part II, 1996, pp. 20-24. □

3. Regular one-dimensional linear sfde's

To outline a general scheme for obtaining estimates on the top Lyapunov exponent for a class of one-dimensional regular linear sfde's. Then apply scheme to specific examples within the above class.

Scheme applies to multidimensional linear equations with multiple delays.

Note: Approach in ([Ku], JDE, 1968) uses Lyapunov functionals and yields strictly weaker estimates in all cases.

Consider the class of one-dimensional linear sfde's

$$\left.\begin{aligned} dx(t) = \Big\{\nu_1 x(t) + \mu_1 x(t-r) + \int_{-r}^{0} x(t+s)\sigma_1(s)\,ds\Big\}\,dt \\ + \Big\{\nu_2 x(t) + \int_{-r}^{0} x(t+s)\sigma_2(s)\,ds\Big\}\,dM(t), \end{aligned}\right\} \qquad (XVII)$$

where $r > 0, \sigma_1, \sigma_2 \in C^1([-r,0], \mathbf{R})$, and M is a continuous helix local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with (stationary) ergodic increments. Ergodic theorem gives the a.s. deterministic limit $\beta := \lim_{t\to\infty} \frac{\langle M\rangle(t)}{t}$. Assume that $\beta < \infty$ and $\langle M\rangle(1) \in L^\infty(\Omega, \mathbf{R})$.

Hence (XVII) is regular with respect to M_2 and has a sample-continuous stochastic semiflow

$X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ (Theorem III.5). The stochastic semiflow X has a fixed (non-random) Lyapunov spectrum (Theorem IV.7). Let λ_1 be its top exponent. We wish to develop an upper bound for λ_1. By the spectral theorem (Theorem IV.7, cf. Theorem IV.2), there is a shift-invariant set $\Omega^* \in \mathcal{F}$ of full P-measure and a measurable random field $\lambda : M_2 \times \Omega \to \mathbf{R} \cup \{-\infty\}$,

$$\lambda((v,\eta),\omega) := \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2}, \quad (v,\eta) \in M_2,\ \omega \in \Omega^*, \tag{1}$$

giving the Lyapunov spectrum of (XVII).

Introduce family of equivalent norms

$$\|(v,\eta)\|_\alpha := \left\{ \alpha v^2 + \int_{-r}^0 \eta(s)^2\, ds \right\}^{1/2}, \quad (v,\eta) \in M_2, \quad \alpha > 0, \tag{2}$$

on M_2. Then

$$\lambda((v,\eta),\omega) = \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_\alpha, \quad (v,\eta) \in M_2,\ \omega \in \Omega^* \tag{3}$$

for all $\alpha > 0$; i.e. the Lyapunov spectrum of (XVII) with respect to $\|\cdot\|_\alpha$ is independent of $\alpha > 0$.

Let x be the solution of (XVII) starting at $(v,\eta) \in M_2$. Define

$$\rho_\alpha(t)^2 := \|X(t)\|_\alpha^2 = \alpha x(t)^2 + \int_{t-r}^{t} x(u)^2\,du, \quad t > 0, \quad \alpha > 0. \tag{4}$$

For each fixed $(v,\eta) \in M_2$, define the set $\Omega_0 \in \mathcal{F}$ by $\Omega_0 := \{\omega \in \Omega : \rho_\alpha(t,\omega) \neq 0 \text{ for all } t > 0\}$. If $P(\Omega_0) = 0$, then by uniqueness there is a random time τ_0 such that a.s. $X(t,(v,\eta),\cdot) = 0$ for all $t \geq \tau_0$.

Hence $\lambda_1 = -\infty$. So suppose that $P(\Omega_0) > 0$. Itô's formula implies

$$\log \rho_\alpha(t) = \log \rho_\alpha(0) + \int_0^t Q_\alpha(a(u), b(u), I_1(u))\, du + \int_0^t \tilde{Q}_\alpha(a(u), I_2(u))\, d\langle M\rangle(u) + \int_0^t R_\alpha(a(u), I_2(u))\, dM(u), \tag{5}$$

for $t > 0$, a.s. on Ω_0, where

$$\left.\begin{aligned} Q_\alpha(z_1, z_2, z_3) &:= \nu_1 z_1^2 + \sqrt{\alpha}\, \mu_1 z_1 z_2 + \sqrt{\alpha}\, z_1 z_3 + \tfrac{1}{2}\frac{z_1^2}{\alpha} - \tfrac{1}{2} z_2^2 \\ \tilde{Q}_\alpha(z_1, z_3') &:= \alpha(\tfrac{1}{2} - z_1^2)\left(\frac{\nu_2}{\sqrt{\alpha}}\, z_1 + z_3'\right)^2 \\ R_\alpha(z_1, z_3') &:= \nu_2 z_1^2 + \sqrt{\alpha} z_1 z_3', \quad \|\sigma_i\|_2 := \left\{\int_{-r}^0 \sigma_i(s)^2 ds\right\}^{1/2}, \end{aligned}\right\} \tag{6}$$

$i = 1, 2$, and

$$a(t) := \frac{\sqrt{\alpha} x(t)}{\rho_\alpha(t)}, \quad b(t) := \frac{x(t-r)}{\rho_\alpha(t)}, \quad I_i(t) := \frac{\int_{-r}^0 x(t+s)\sigma_i(s)\, ds}{\rho_\alpha(t)} \tag{7}$$

for $i = 1, 2$, $t > 0$, a.s. on Ω_0.

Since

$$|I_i(t)| \le \frac{1}{\rho_\alpha(t)} \left(\int_{-r}^{0} x(t+s)^2 \, ds \right)^{1/2} \|\sigma_i\|_2 = \sqrt{1 - a^2(t)}\, \|\sigma_i\|_2,$$

$i = 1, 2,$ a.s. on Ω_0 the variables z_1, z_2, z_3, z_3' in (6) must satisfy

$$|z_1| \le 1, \;\; z_2 \in \mathbf{R}, \;\; |z_3|^2 \le (1 - z_1^2)\|\sigma_1\|_2^2, \;\; |z_3'|^2 \le (1 - z_1^2)\|\sigma_2\|_2^2.$$

Let $\tau_1 := \inf\{t > 0 : \rho_\alpha(t) = 0\}$. Then the local martingale

$$\int_0^{t \wedge \tau_1} R_\alpha(a(u), I_2(u)) \, dM(u), \; t > 0,$$

is a time-changed (possibly stopped) Brownian motion. Since $|R_\alpha(a(u), I_2(u))| \le |\nu_2| + \sqrt{\alpha}\|\sigma_2\|_2$ for all $u \in [0, \tau_1)$, a.s., then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^{t \wedge \tau_1} R_\alpha(a(u), I_2(u)) \, dM(u) = 0 \qquad \text{a.s.} \qquad (8)$$

Divide (5) by t, let $t \to \infty$, to get

$$\lambda((v,\eta),\omega) \le \limsup_{t\to\infty} \frac{1}{t} \int_0^t Q_\alpha(a(u), b(u), I_1(u))\, du + \limsup_{t\to\infty} \frac{1}{t} \int_0^t \tilde{Q}_\alpha(a(u), I_2(u))\, d\langle M\rangle(u). \tag{9}$$

a.s. on Ω_0, for all $\alpha > 0$.

Wish to develop upper bounds on λ_1 in the following cases.

One-dimensional linear sfde (smooth memory in white-noise term):

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}\, dt + \left\{\int_{-r}^0 x(t+s)\sigma_2(s)\, ds\right\} dW(t), \qquad t > 0 \tag{VII}$$

with real constants ν_1, μ_1 and $\sigma_2 \in C^1([-r,0],\mathbf{R})$. It is a special case of (XVII). Hence (VII) is regular with respect to M_2. The process $\int_{-r}^0 x(t+s)\sigma_2(s)\, ds$

has C^1 paths in t. Hence the stochastic differential dW in (VII) may be interpreted in the Itô or Stratonovich sense *without changing the solution* x.

Theorem V.3.

Suppose λ_1 *is the top a.s. Lyapunov exponent of (VII). Define the function*

$$\theta(\delta,\alpha) := -\delta + \left(\nu_1+\delta+\frac{1}{2}\alpha\mu_1^2 e^{2\delta r}+\frac{1}{2\alpha}\right)\vee\left(\frac{\alpha}{2}\|\sigma_2\|_2^2 e^{2\delta^+ r}\right)$$

for all $\alpha\in\mathbf{R}^+, \delta\in\mathbf{R}$, *where* $\delta^+ := \max\{\delta, 0\}$.

Then

$$\lambda_1 \le \inf\{\theta(\delta,\alpha) : \delta\in\mathbf{R},\ \alpha\in\mathbf{R}^+\}. \tag{10}$$

Proof.

Maximize the integrand on the right-hand-side of (9) (with $M = W$); then use exponential shift by δ to refine the resulting estimate. Then minimize over α, δ ([M-S], II, 1996, pp. 34-35).

$\square$

Corollary below shows that the estimate in Theorem V.3 reduces to well-known estimate in deterministic case $\sigma_2 \equiv 0$ (Hale [Ha], pp.17-18).

Corollary V.3.1.

In (VII), suppose $\mu_1 \neq 0$ and let δ_0 be the unique real solution of the transcendental equation

$$\nu_1 + \delta + |\mu_1| e^{\delta r} = 0. \tag{11}$$

Then

$$\lambda_1 \leq -\delta_0 + \frac{1}{2} \frac{\|\sigma_2\|_2^2}{|\mu_1|} e^{|\delta_0| r}. \tag{12}$$

If $\mu_1 = 0$ *and* $\nu_1 \geq 0$, *then* $\lambda_1 \leq \frac{1}{2}\big(\nu_1 + \sqrt{\nu_1^2 + \|\sigma_2\|_2^2}\big)$. *If* $\mu_1 = 0$ *and* $\nu_1 < 0$, *then* $\lambda_1 \leq \nu_1 + \frac{1}{2}\|\sigma_2\|_2 e^{-\nu_1 r}$.

Proof.

Suppose $\mu_1 \neq 0$. Denote by $f(\delta)$, $\delta \in \mathbf{R}$, the left-hand-side of (11). Then $f(\delta)$ is an increasing function of δ. f has a unique real zero δ_0. Using (10), we may put $\delta = \delta_0$ and $\alpha = |\mu_1|^{-1} e^{-\delta_0 r}$ in the expression for $\theta(\delta, \alpha)$. This gives (12).

Suppose $\mu_1 = 0$. Put $\delta = (-\nu_1)^+$ in $\theta(\delta, \alpha)$ and minimize the resulting expression over all $\alpha > 0$. This proves the last two assertions of the corollary ([M-S], II, 1996, pp. 35-36). $\square$

Remarks.

(i) Upper bounds for λ_1 in Theorem (V.3) and Corollary V.3.1 agree with corresponding bounds in the deterministic case (for $\mu_1 \geq 0$), but are not optimal when $\mu_1 = 0$ and σ_2 is strictly positive and sufficiently small; cf. Theorem V.1 for small $\|\sigma_2\|_2$.

(ii) *Problem:* What are the asymptotics of λ_1 for small delays $r \downarrow 0$?

Our second example is the stochastic delay equation

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}\, dt + x(t)dM(t), \qquad t > 0, \tag{XVIII}$$

where M is the helix local martingale appearing in (XVII) and satisfying the conditions therein. Hence (XVIII) is regular with respect to M_2. Theorem below gives estimate on its top exponent.

Theorem V.4.

In (XVIII) define δ_0 as in Corollary V.3.1. Then the top a.s. Lyapunov exponent λ_1 of (XVIII) satisfies

$$\lambda_1 \leq -\delta_0 + \frac{\beta}{16}. \tag{13}$$

Proof.

Maximize the following functions separately over their appropriate ranges:

$$\begin{gathered} Q_\alpha(z_1, z_2) := \nu_1 z_1^2 + \sqrt{\alpha}\,\mu_1 z_1 z_2 + \tfrac{1}{2}\frac{z_1^2}{\alpha} - \tfrac{1}{2} z_2^2, \\ \tilde{Q}_\alpha(z_1) := (\tfrac{1}{2} - z_1^2) z_1^2, \quad |z_1| \leq 1,\, z_2 \in \mathbf{R}. \end{gathered}$$

Then use an exponential shift of the Lyapunov spectrum by an amount δ. Minimize the resulting bound over all α (for fixed δ) and then over all $\delta \in \mathbf{R}$. This minimum is attained if δ solves the transcendental equation (11). Hence the conclusion of the theorem ([M-S], II, 1996, pp. 36-37).

□

Remark.

The above estimate for λ_1 is sharp in the deterministic case $\beta = 0$ and $\mu_1 \geq 0$, but is not sharp when $\beta \neq 0$; e.g. $M = W$, one-dimensional standard Brownian motion in the non-delay case ($\mu_1 = 0$). When $M = \nu_2 W$ for a fixed real ν_2, the above bound may be considerably sharpened as in Theorem V.5 below. The sdde in this theorem is a model of dye circulation in the blood stream (cf. Bailey and Williams [B-W], 1996; Lenhart and Travis, 1986).

Theorem V.5. *([M-S], II, 1996).*

For the equation

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}dt + \nu_2 x(t)\, dW(t) \quad (VI)$$

set

$$\phi(\delta) := -\delta + \frac{1}{4\nu_2^2}\left[\left(|\mu_1|e^{\delta r} + \nu_1 + \delta + \frac{1}{2}\nu_2^2\right)^+\right]^2, \quad (14)$$

for $\nu_2 \neq 0$. *Then*

$$\lambda_1 \leq \inf_{\delta \in \mathbf{R}} \phi(\delta). \quad (15)$$

In particular, if δ_0 *is the unique solution of the equation*

$$\nu_1 + \delta + |\mu_1|e^{\delta r} + \frac{1}{2}\nu_2^2 = 0, \quad (16)$$

then $\lambda_1 \leq -\delta_0$.

Proof.

Maximize

$$Q_\alpha(z_1, z_2, 0) + \tilde{Q}_\alpha(z_1, 0) = \left(\nu_1 + \frac{1}{2\alpha} + \frac{\nu_2^2}{2}\right)z_1^2 + \sqrt{\alpha}\,\mu_1 z_1 z_2 - \frac{1}{2}z_2^2 - \nu_2^2 z_1^4 \quad (17)$$

for $|z_1| \leq 1, z_2 \in \mathbf{R}$ and then minimize the resulting bound for λ_1 over $\alpha > 0$. Get

$$\lambda_1 \leq \frac{1}{16\nu_2^2}\left[(2\nu_1 + 2|\mu_1| + \nu_2^2)^+\right]^2.$$

The first assertion of the theorem follows from above estimate by applying an exponential shift to (VI). Last assertion of the theorem is obvious ([M-S], II, 1996, pp. 38-39.) $\square$

Problem: Is $\lambda_1 = \inf_{\delta \in \mathbf{R}} \phi(\delta)$?

Remark.

Estimate in Theorem V.5 agrees with the non-delay case $\mu_1 = 0$ whereby $\lambda_1 = \nu_1 - \frac{1}{2}\nu_2^2 = \inf\limits_{\delta \in \mathbf{R}} \phi(\delta)$. Cf. also [AOP], 1986, [B], 1985, and [AKO], 1989.

4. SDDE with Poisson Noise.

Consider the one-dimensional linear delay equation

$$\left.\begin{aligned} dx(t) &= x((t-1)-)\,dN(t) \quad t>0 \\ x_0 &= \eta \in D := D([-1,0],\mathbf{R}). \end{aligned}\right\} \qquad (V)$$

The process $N(t) \in \mathbf{R}$ is a Poisson process with i.i.d. inter-arrival times $\{T_i\}_{i=1}^{\infty}$ which are exponentially distributed with the same parameter μ. The jumps $\{Y_i\}_{i=1}^{\infty}$ of N are i.i.d. and independent of all the T_i's. Let

$$j(t) := \sup\left\{ j \geq 0 : \sum_{i=1}^{j} T_i \leq t \right\}.$$

Then

$$N(t) = \sum_{i=1}^{j(t)} Y_i.$$

Equation (V) can be solved a.s. in forward steps of lengths 1, using the relation

$$x^{\eta}(t) = \eta(0) + \sum_{i=1}^{j(t)} Y_i x\left((\sum_{j=1}^{i} T_j - 1)- \right) \qquad \text{a.s.}$$

Trajectory $\{x_t : t \geq 0\}$ is a Markov process in the state space D (with the supremum norm $\|\cdot\|_\infty$). Furthermore, the above relation implies that (V) is regular in D; i.e., it admits a measurable flow $X : \mathbf{R}^+ \times D \times \Omega \to D$ with $X(t, \cdot, \omega) = {}^\eta x_t(\cdot, \omega)$, continuous linear in η for all $t \geq 0$ and a.a. $\omega \in \Omega$ (cf. the singular equation (I)).

The a.s. Lyapunov spectrum of (V) may be characterized directly (without appealing to the Oseledec Theorem) by interpolating between the sequence of random times:

$$\tau_0(\omega) := 0,$$

$$\tau_1(\omega) := \inf\left\{ n \geq 1 : \sum_{j=1}^{k} T_j \notin [n-1, n] \quad \text{for all } k \geq 1 \right\},$$

$$\tau_{i+1}(\omega) := \inf\left\{ n > \tau_i(\omega) : \sum_{j=1}^{k} T_j \notin [n-1, n] \text{ for all } k \geq 1 \right\}, \quad i \geq 1.$$

It is easy to see that $\{\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \cdots\}$ are i.i.d. and $E\tau_1 = \mathrm{e}^{\mu}$.

Theorem V.6. *([M-S], II, 1996)*

Let $\xi \in D$ be the constant path $\xi(s) = 1$ for all $s \in [-1, 0]$. Suppose $E \log \|X(\tau_1(\cdot), \xi, \cdot)\|_\infty$ exists (possibly $= +\infty$ or $-\infty$). Then the a.s. Lyapunov spectrum

$$\lambda(\eta) := \lim_{t \to \infty} \frac{1}{t} \log \|X(t, \eta, \omega)\|_\infty, \quad \eta \in D, \ \omega \in \Omega$$

of (V) is $\{-\infty, \lambda_1\}$ where

$$\lambda_1 = e^{-\mu} \, E \log \|X(\tau_1(\cdot), \xi, \cdot)\|_\infty.$$

In fact,

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t, \eta, \omega)\|_\infty = \begin{cases} \lambda_1 & \eta \notin \operatorname{Ker} X(\tau_1(\omega), \cdot, \omega) \\ -\infty & \eta \in \operatorname{Ker} X(\tau_1(\omega), \cdot, \omega). \end{cases}$$

Proof.

The i.i.d. sequence

$$S_i := \frac{\|(X(\tau_i, \xi, \cdot))\|}{\|(X(\tau_{i-1}, \xi, \cdot))\|} \qquad i = 1, 2, \ldots$$

and the LLN give

$$\lim_{n\to\infty} \frac{1}{\tau_n} \log \ \|(X(\tau_n, \xi, \omega))\| = e^{-\mu}(E \log S_1)$$

for a.a. $\omega \in \Omega$.

Interpolate between the times $\tau_1, \tau_2, \tau_3, \cdots$ to get the continuos limit ([M-S], II, 1996, pp. 27-28). $\square$

SOME NUMERICS OF STOCHASTIC SYSTEMS WITH MEMORY

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Outline

- Strong Euler scheme for (general) SFDE's. Order of convergence 0.5.
- Strong Milstein scheme for SDDE's. Order of convergence 1.
- For Milstein scheme, use infinite dimensional Itô formula for "tame" functions acting on segment process of solution of SDDE. Presence of memory in SDDE requires use of Malliavin calculus + anticipating stochastic analysis of Nualart and Pardoux.
- *Conjecture*: Milstein scheme works for mixed discrete and continuous memory. *Open*: for general SFDE's?

Types of SFDE's

Suppose rate of change of physical system depends on *present state* and some noisy input. Model by SODE.

Rate of change depends on *present* and *past* states of the system: Model by SDDE or SFDE.

$\mathbf{R}^m :=$ m-dimensional Euclidean space.

Euclidean norm:

$$|x| := \sqrt{x_1^2 + \cdots + x_m^2}, \qquad x = (x_1, \cdots, x_m) \in \mathbf{R}^m.$$

$T := [0, a],\ J := [-r, 0],\ r, a > 0.$

$C := C(J; \mathbf{R}^m)$; sup norm:

$$\|\eta\|_C := \sup_{-r \le s \le 0} |\eta(s)|, \qquad \eta \in C := C([-r, 0], \mathbf{R}^m).$$

$W :=$ d-dimensional Brownian motion.

SDDE:

$$X(t) = \begin{cases} \eta(0) + \displaystyle\int_0^t g(s, \Pi_1(X_s))\, dW(s) \\ \qquad\qquad + \displaystyle\int_0^t h(s, \Pi_2(X_s))\, ds, \quad t \in [0, a] \\ \eta(t), \quad -r \le t < 0. \end{cases}$$

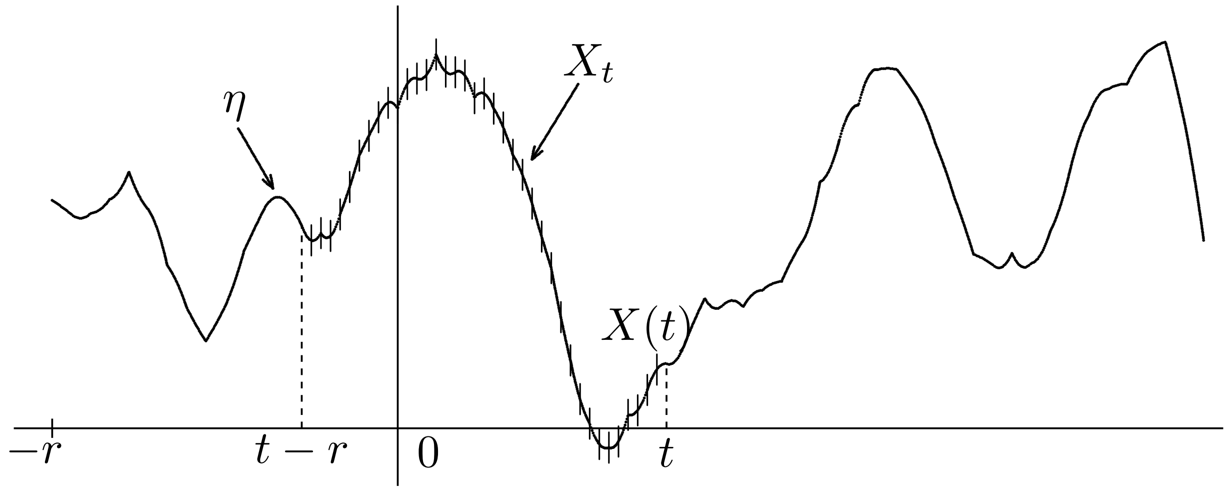
$\Pi_i : C \to \mathbf{R}^{mk_i}$, $i = 1, 2$, two projections of discrete type based on $s_{1,1}, \cdots, s_{1,k_1} \in [-r, 0]$ and $s_{2,1}, \cdots, s_{2,k_2} \in [-r, 0]$:

$$\Pi_i(\eta) := (\eta(s_{i,1}), \cdots, \eta(s_{i,k_i})) \in \mathbf{R}^{mk_i}, \quad \eta \in C, \ i = 1, 2.$$

Segment process X_t, $t \in [0, a]$:

$$X_t(s) = X(t + s), \quad t \in [0, a], \quad s \in [-r, 0].$$

$$g : T \times \mathbf{R}^{mk_1} \to L(\mathbf{R}^d, \mathbf{R}^m), \quad h : T \times \mathbf{R}^{mk_2} \to \mathbf{R}^m.$$



SFDE with mixed discrete and continuous memory:

$$
\begin{aligned}
X(t) &= \eta(0) + \int_0^t g(s, \Pi_1(X_s), Q_1(X_s))\, dW(s) \\
&\qquad + \int_0^t h(s, \Pi_2(X_s), Q_2(X_s))\, ds, \qquad t \in [0, a], \\
X_0 &= \eta \in C = C(J; \mathbf{R}^m), J := [-r, 0].
\end{aligned}
$$

$g : T \times \mathbf{R}^{mk_1} \times \mathbf{R}^{m_1} \to L(\mathbf{R}^d, \mathbf{R}^m), \quad h : T \times \mathbf{R}^{mk_2} \times \mathbf{R}^{m_2} \to \mathbf{R}^m.$

Π_1, Π_2 two projections of discrete type;

Q_1, Q_2 two projections of continuous type:

$$
\begin{aligned}
Q_i(\eta) &:= (Q_{i,1}(\eta), \cdots, Q_{i,m_i}(\eta)), \quad i = 1, 2, \\
Q_{ij}(\eta) &:= \int_{-r}^{0} \phi_{ij}(\eta(s)) a_{ij}(s)\, ds\,, \quad j = 1, \cdots, m_i.
\end{aligned}
$$

$a_{ij} : J \to \mathbf{R}$ and $\phi_{ij} : \mathbf{R}^m \to \mathbf{R}$ sufficiently regular,

$i = 1, 2$, $j = 1, \cdots, m_i$.

General SFDE:

$$
X(t) = \begin{cases} \eta(0) + \displaystyle\int_0^t G(s, X_s)\, dW(s) \\ \qquad\qquad + \displaystyle\int_0^t H(s, X_s)\, ds, \quad t \in [0, a] \\ \eta(t), \quad -r \le t < 0. \end{cases}
$$

$G : T \times C \to L(\mathbf{R}^d, \mathbf{R}^m)$, $\quad H : T \times C \to \mathbf{R}^m$.

Numerical Schemes

SDDE's and SFDE's cannot be solved explicitly: Need effective numerical techniques.

Numerical methods for SODE's : well developed; Kloeden and Platen, Kloeden, Platen and Schurz, McShane, Chapters 5 and 6), Hu, Talay, Protter, etc..

Cauchy-Maruyama scheme for SFDE's with continuous memory: On Delfour-Mitter state space $\mathbf{R}^m \times L^2([-r, 0], \mathbf{R}^m)$ developed by Ahmed, Elsanousi and Mohammed (Ahmed, M.Sc. thesis, Khartoum 1983), Baker and Buckwar, 2000. See also [M], 1984, p. 227, and Hu-Mohammed, 1997.

Aims.

- *Strong Euler schemes* for general SFDE's. Allows for multiple delays and continuous memory. Estimates in supremum norm on $C([-r,0],\mathbf{R}^m)$ (cf. [A]).
- *Strong Milstein scheme* for SDDE's. Solution of SDDE is non-anticipating. But need methods from *anticipating* stochastic analysis and Malliavin calculus to derive Itô's formula for segment process. Itô's formula needed for convergence of Milstein scheme.

Preliminaries

Recall *segment process* X_t, $t \in [0,a]$:

$$X_t(s) = X(t+s), \quad t \in [0,a], \quad s \in [-r,0].$$

for continuous m-dimensional process $\{X(t)\}_{t\in[-r,a]}$.

$\{X_t\}$ is a C-valued or $L^2(J;\mathbf{R}^m)$-valued process.

Distinguish between finite-dimensional current state $X(t)$ *and infinite-dimensional segment* X_t, $t \in [0,a]$.

Itô SFDE:

$$X(t) = \begin{cases} \eta(0) + \displaystyle\int_0^t G(s, X_s)\, dW(s) \\ \qquad\quad + \displaystyle\int_0^t H(s, X_s)\, ds, \quad t \in [0,a] \\ \eta(t), \quad -r \leq t < 0. \end{cases}$$

Coefficients: $G : T \times C([-r,0],\mathbf{R}^m) \to L(\mathbf{R}^d;\mathbf{R}^m)$ and $H : T \times C([-r,0],\mathbf{R}^m) \to \mathbf{R}^m$.

$\{W(t) := (W^1(t),\cdots,W^d(t)) : t \geq 0\}$, d-dimensional standard Brownian motion on $(\Omega,\mathcal{F},P)$.

$(\mathcal{F}_t)_{t\geq 0}$ = Brownian filtration.

$\eta \in C([-r,0];\mathbf{R}^m)$ = random initial path independent of $\{W(t) : t \geq 0\}$.

Lipschitz Condition:

$$\|G(t,\eta) - G(t,\xi)\| + |H(t,\eta) - H(t,\xi)| \leq L\|\eta - \xi\|_C$$

for all $t \in T, \eta, \xi \in C$; $L > 0$ constant.

Boundedness Condition:

$$\sup_{0\leq t\leq a} \left[\|G(t,0)\| + |H(t,0)|\right] < \infty.$$

Lipschitz + bounded conditions imply SFDE has unique strong solution such that for each $q \geq 1$, there exists a constant $C = C(q, L, a) > 0$ with

$$E||X_t||_C^{2q} \leq C(1 + E||\eta||_C^{2q})$$

for all $\eta \in C, t \in [0, a]$ ([M], 1984).

Segment X_t, $t \geq 0$, is a C-valued Markov prcess.

Qualitative theory of SFDE's: [M], 1984, 1996, + references therein.

Strong versus Weak:

SFDE's do not lead to diffusions on Euclidean space. (*Highly degenerate infinite-dimensional diffusions on C.*) Hence no natural link to deterministic PDE's. Strong schemes give information on sample paths dynamics, a.s. financial option-pricing formulas with delays (Arriojas and Mohammed, 2001).

Strong Euler Scheme

Develop Euler scheme for general SFDE's (include discrete and/or continuous memory).

n, l positive integers, $T := [0, a]$, $a > 0$, $J := [-r, 0]$.

$\pi_n : t_{-l} < t_{-l+1} < \cdots < 0 = t_0 < t_1 < t_2 < \cdots < t_n = a$,

partition of $[-r, a]$.

$|\pi_n| := \max\limits_{-l \le i \le n-1} (t_{i+1} - t_i)$, mesh of π_n.

$X^n := X^{\pi_n}$.

SFDE:

$$X(t) = \begin{cases} \eta(0) + \displaystyle\int_0^t G(s, X_s))\, dW(s) \\ \qquad\quad + \displaystyle\int_0^t H(s, X_s)\, ds, \quad t \in [0, a] \\ \eta(t), \quad -r \le t < 0. \end{cases}$$

Euler scheme for SFDE:

$$X^n(t) = \begin{cases} X^n(t_i) + G(t_i, X^n_{t_i})(W(t) - W(t_i)) \\ +H(t_i, X^n_{t_i})(t - t_i),\ t \in (t_i, t_{i+1}], \quad t_i \in (0, a] \\ \eta^n(t), \qquad -r \le t \le 0 \end{cases}$$

Approx. initial path $\eta^n \in C(J, \mathbf{R}^m)$ is prescribed (e.g. a piece-wise linear approximation of η using partition points $\{t_{-l}, \cdots, t_0\}$).

Error function Z^n:

$$\begin{cases} Z^n(t) := X^n(t) - X(t), \quad 0 \le t \le a, \\ Z^n_0 := \eta^n - \eta. \end{cases}$$

Euler scheme for SFDE's has strong order of convergence 0.5 (as in SODE).

Theorem 1.

Assume that the coefficients $G : T \times C([-r,0],\mathbf{R}^m) \to L(\mathbf{R}^d;\mathbf{R}^m)$ *and* $H : T \times C([-r,0],\mathbf{R}^m) \to \mathbf{R}^m$ *in SFDE satisfy the following Lipschitz and regularity conditions:*

$$\|G(t,\eta) - G(t,\xi)\| + |H(t,\eta) - H(t,\xi)| \leq L\|\eta - \xi\|_C,\ t \in T$$

$$\sup_{0\leq t\leq a}\left[\|G(t,0)\| + |H(t,0)|\right] < \infty$$

$$\|G(s,\eta) - G(t,\eta)\| \leq L_1(1 + \|\eta\|_C)|s-t|^\gamma,\ s,t \in T$$

$$|H(s,\eta) - H(t,\eta)| \leq L_1(1 + \|\eta\|_C)|s-t|^\gamma,\ s,t \in T$$

for all $\eta, \xi \in C([-r,0],\mathbf{R}^m)$, *where* L *and* L_1 *are positive constants. Fix any integer* $q \geq 2$. *Suppose that* $\eta : [-r,0] \to L^q(\Omega,\mathbf{R}^m)$ *is independent of*

W and Hölder continuous with exponent $\gamma \in (0,1]$, i.e., there is a positive constant K such that

$$E|\eta(s) - \eta(t)|^q \leq K|s-t|^{\gamma q}$$

for all $s,t \in [-r,0]$. Suppose also that there is a positive constant $C' := C'(q)$ such that

$$E||\eta^n - \eta||_C^q \leq C'|\pi_n|^{\gamma q}.$$

Then there is a constant $C'' := C''(q,a) > 0$, depending on a and q, such that

$$E \sup_{0 \leq t \leq a} ||Z_t^n||_C^q \leq C''|\pi_n|^{\tilde{\gamma} q}$$

where $\tilde{\gamma} := \gamma \wedge (1/2)$.

Proof of Theorem 1.

Based on moment estimates:

$$E||X_t||_C^{2q} \leq C(1 + E||\eta||_C^{2q}),\ q \geq 1$$

for all $\eta \in C, t \in [0, a]$ ([M], 1984), and Burkholder's inequality. $\square$

Theorem 1 applies to SDDE's under Lipschitz and boundedness conditions. Also to SFDE's with mixed discrete and continuous memory:

$$
\begin{aligned}
X(t) = \eta(0) + \int_0^t g(s, \Pi_1(X_s), Q_1(X_s))\, dW(s) \\
+ \int_0^t h(s, \Pi_2(X_s), Q_2(X_s))\, ds, \quad t \in [0, a], \\
X_0 = \eta \in C = C(J; \mathbf{R}^m)
\end{aligned}
$$

Π_1, Π_2 two projections of discrete type;

Q_1, Q_2 two projections of continuous type:

$$
\begin{aligned}
Q_i(\eta) &:= (Q_{i,1}(\eta), \cdots, Q_{i,m_i}(\eta)), \quad i = 1, 2, \\
Q_{ij}(\eta) &:= \int_{-1}^0 \phi_{ij}(\eta(s)) a_{ij}(s)\, ds\,, \quad j = 1, \cdots, m_i.
\end{aligned}
$$

$a_{ij} \in C^{\frac{1}{2}}(J)$, and $\phi_{ij} : \mathbf{R}^m \to \mathbf{R}$, $i = 1, 2$, $j = 1, \cdots, m_i$, satisfy Lipschitz and linear growth conditions.

Euler scheme for SFDE with mixed discrete and continuous memory:

$$
\begin{aligned}
X^n(t) &= X^n(t_i) + g(t_i, \Pi_1(X^n_{t_i}), Q^n_1(X^n_{t_i}))(W(t) - W(t_i)) \\
&\qquad + h(t_i, \Pi_2(X^n_{t_i}), Q^n_2(X^n_{t_i}))(t - t_i), \qquad t \in (t_i, t_{i+1}], \\
X^n(t) &= \eta^n(t), \qquad -r \leq t \leq 0,
\end{aligned}
$$

where $Q^n_i(\eta), i = 1, 2$, are approximations of $Q_i(\eta)$ using partial sums of Riemann integral. Strong order of convergence 0.5 under Lipschitz and regularity conditions as in Theorem 1.

Example: Exact convergence rate.

One-dimensional SDDE:

$$\begin{cases} dX(t) = b(t)X(t-1)\,dW(t)\,, & 0 < t \le a \\ X(t) = \eta(t)\,, & -1 \le t \le 0. \end{cases}$$

Use partitions $\{\pi_n(h)\}$ of $[-1, a]$ generated by a continuous (strictly positive) function $h : [0, a] \to (0, \infty)$. For each integer n, choose partition points $t_{k,n} \equiv t_k$ of $\pi_n(h)$ in $[0, a]$ such that

$$t_0 = 0, \quad \int_{t_k}^{t_{k+1}} h(s)\,ds = \frac{1}{n}, \quad k = 0, 1, \cdots, n-1.$$

i.e. subdivide interval in such a way that the areas under h over each subinterval are all equal to $1/n$. Then

$$\lim_{n\to\infty, t_k \to t} n(t_{k+1} - t_k) = 1/h(t).$$

e.g. $h(t) \equiv 1 \implies (t_{k+1} - t_k) = 1/n,\ k = 0, 1, \cdots, n-1.$

Euler scheme gives

$$X^{\pi_n}(t) = \begin{cases} X^{\pi_n}(t_k) + b(t_k)X^{\pi_n}(t_k - 1)(W(t) - W(t_k)), \\ \qquad\qquad\qquad\qquad\qquad t_k \leq t < t_{k+1}, \\ \eta(t), \quad t \in J := [-1, 0], \end{cases}$$

for $0 \leq k \leq n-1$. By Theorem 1, there is a positive constant C (independent of n) such that

$$nE|X(t) - X^{\pi_n}(t)|^2 \leq C,$$

for all $n \geq 1, t \in [0, a]$. Theorem 2 (below) shows that the left hand side of the above inequality has a limit (as $n \to \infty$) satisfying a *deterministic* DDE.

Theorem 2.

Suppose $\eta \in C^{\gamma}(J, \mathbf{R}^m), 1/2 < \gamma \leq 1$. *Let* $a \geq 1$. *Suppose* $b : [0, a] \to \mathbf{R}$ *satisfies*

$$|b(t) - b(s)| \leq K|t - s|^{(1/2)+\alpha}$$

for all $s, t \in [0, a]$ *and some* $K, \alpha > 0$. *Let* X *be the solution of the SDDE and* X^{π_n} *its Euler approximation. Then* $\mathcal{Z}(t) := \lim_{n\to\infty} n\ E|X(t) - X^{\pi_n}(t)|^2$ *exists for each* $t \in [-1, a]$. *Furthermore,* $\mathcal{Z}(t)$ *satisfies the following deterministic linear DDE*

$$\mathcal{Z}'(t) = b^2(t)\mathcal{Z}(t-1) + b^2(t)b^2(t-1)EX^2(t-2)/h(t),\ 1 < t < a,$$

$$\mathcal{Z}(t) = 0, \qquad -1 \leq t \leq 1,$$

where $EX^2(t)$ *is given by the integral equation*

$$EX^2(t) = \begin{cases} \eta(0)^2 + \int_0^t b^2(s)EX^2(s-1)\,ds, & t \in [0, a], \\ \eta(t)^2, & t \in [-1, 0). \end{cases}$$

Milstein Scheme

Strong second order scheme for SDDE:

$$X(t) = \begin{cases} \eta(0) + \displaystyle\int_0^t g(s, \Pi_1(X_s))\, dW(s) \\ \qquad\qquad + \displaystyle\int_0^t h(s, \Pi_2(X_s))\, ds, \; t \in T := [0, a] \\ \eta(t), \qquad -r \leq t < 0. \end{cases}$$

$$g : T \times \mathbf{R}^{mk_1} \to L(\mathbf{R}^d, \mathbf{R}^m), \qquad h : T \times \mathbf{R}^{mk_2} \to \mathbf{R}^m.$$

Requires infinite-dimensional Itô formula for "tame" functions of segments of semimartingales or (anticipating) processes. Proof based on Nualart-Pardoux anticipating calculus techniques.

$l, n =$ positive integers, $T := [0, a]$, $a > 0$, $J := [-r, 0]$.

Partitions: $\pi_n := \{t_i : -l \leq i \leq n\}$ of $[-r, a]$, mesh $|\pi_n|$. $\quad X^n := X^{\pi_n}$; $\quad (x_{i_1 j_1}) \in \mathbf{R}^{mk_1}$.

Milstein approximations for SDDE:

$$
\begin{aligned}
X^{i,n}(t) &= X^{i,n}(t_k) + h^i(t_k, \Pi_2(X^n_{t_k}))(t - t_k) \\
&\qquad + \sum_j g^{ij}(t_k, \Pi_1(X^n_{t_k}))(W^j(t) - W^j(t_k)) \\
&+ \sum_{i_1, j_1, j} \frac{\partial g^{ij}}{\partial x_{i_1 j_1}}(t_k, \Pi_1(X^n_{t_k}))g^{i_1 j_1}(t_k + s_{1,j_1}, \Pi_1(X^n_{t_k + s_{1,j_1}}))\times \\
&\qquad \times 1_{[0,T]}(t_k + s_{1,j_1}) \times I_{j,j_1}(t_k + s_{1,j_1}, t + s_{1,j_1}; s_{1,j_1}),
\end{aligned}
$$

for $t_k < t \leq t_{k+1}$, $i, i_1 = 1, 2, \cdots, m$, $1 \leq j \leq d$, $1 \leq j_1 \leq k_1$, where

$$
I_{j,j_1}(t_k + s_{1,j_1}, t + s_{1,j_1}; s_{1,j_1}) := \int_{t_k}^{t} \int_{t_k + s_{1,j_1}}^{t_1 + s_{i,j_1}} \circ dW^j(t_2) \circ dW^{j_1}(t_1).
$$

X^i, h^i, g^{ij} = coordinates of X, h and g with respect to standard bases in Euclidean space.

Milstein scheme has strong order of convergence 1.

Theorem 3.

Consider the Milstein scheme for the SDDE. Let $0 < \gamma \leq 1$. *Suppose that* $\eta : [-r, 0] \to L^2(\Omega, \mathbf{R}^m)$ *is Hölder continuous with exponent* $\frac{\gamma}{2}$, *i.e. there is a positive constant* K *such that*

$$E|\eta(s) - \eta(t)|^2 \leq K|s - t|^\gamma$$

for all $s, t \in J$. *Suppose that* $g \in C^{1,2}(T \times \mathbf{R}^{mk_1}, L(\mathbf{R}^d, \mathbf{R}^m))$, $h \in C^{1,2}(T \times \mathbf{R}^{mk_2}, \mathbf{R}^m)$ *and have bounded first and second spatial derivatives. Let*

$$\begin{cases} Z^n(t) := X^n(t) - X(t), & 0 \leq t \leq a, \\ Z_0^n := \eta_0^n - \eta. \end{cases}$$

Assume that

$$\sup_{-r \le s \le 0} E(|Z^n(s)|^2) \le C'|\pi_n|^{2\gamma}$$

for some positive constant C'*. Then there exists a constant* $C > 0$ *(depending on* a *and independent of* π_n*) such that*

$$\sup_{-r \le t \le a} E|Z^n(t)|^2 \le C|\pi_n|^{2\gamma}$$

for any $n \ge 1$*.*

Surprise! Proof requires use of *anticipating* calculus techniques:

Example:

One-dimensional SDDE:

$$dX(t) = g(X(t-1), X(t))\, dW(t), \quad t \geq 0$$
$$X(t) = W(t), \quad -1 \leq t < 0.$$

$g : \mathbf{R}^2 \to \mathbf{R}$ smooth function. For second-order scheme, formally seek a stochastic differential of the coefficient $g(X(t-1), X(t))$ on RHS of SDDE.

For $t \in (0,1]$, formally expect something like:

$$
\begin{aligned}
& dg(W(t-1), W(t)) \\
& = \frac{\partial g}{\partial x_2}(W(t-1), W(t))\, dW(t) \\
& + \frac{\partial g}{\partial x_1}(W(t-1), W(t))\, dW(t-1) \;\; (\textit{anticipating!}) \\
& + \frac{1}{2}\left(\frac{\partial^2 g}{\partial x_1^2}(W(t-1), W(t))\, dt + \frac{\partial^2 g}{\partial x_2^2}(W(t-1), W(t))\, dt \right) \\
& + \frac{1}{2}\frac{\partial^2 g}{\partial x_1 \partial x_2}(W(t-1), W(t))\, dW(t-1)\, dW(t) (= 0!)
\end{aligned}
$$

- *LHS is adapted but anticipating integral on RHS.*
- $(\mathcal{F}_t)_{0 \le t \le 1}$-adapted process

$$[0,1] \ni t \to (X(t-1), X(t)) \in \mathbf{R}^2$$

is not a semimartingale with respect to any natural filtration.

- The components $X(t-1)$ and $X(t)$ are not independent. Existing anticipating versions of Itô's formula do not apply (cf. [AN], [AP] and [NP]). Hence need new Itô formula for tame functions:

$$g(W(t-1), W(t)) = g(W_t(-1), W_t(0)).$$

- Last second-order Itô integral on RHS is zero:

Proof.

$(\Omega, \mathcal{F}, (\mathcal{F}_t), P) :=$ filtered probability space.

$\pi := \{t_i\}$ any partition of $[0, T]$, f any $(\mathcal{F}_t)$-adapted (a.s. bounded) process on $[0, T]$. Then

$$\int_0^T f(t)\, dW(t-1)\, dW(t) = \lim_{|\pi| \to 0} \sum_i f(t_i) \Delta_i W(\cdot - 1) \Delta_i W$$

where

$$\Delta_i W := W(t_{i+1}) - W(t_i),$$

$$\Delta_i W(\cdot - 1) := W(t_{i+1} - 1) - W(t_i - 1)$$

$$E|\sum_i f(t_i)\Delta_i W(\cdot - 1)\Delta_i W|^2 = \sum_{i,j} EX_{i,j}$$

$$X_{i,j} := f(t_i)f(t_j)\Delta_i W(\cdot - 1)\Delta_j W(\cdot - 1)\Delta_i W \Delta_j W$$

For $i < j$,

$$E(X_{i,j}) = E\{E(X_{i,j}|\mathcal{F}_{t_j})\}$$

and

$$E(X_{i,j}|\mathcal{F}_{t_j})$$

$$= f(t_i)f(t_j)\Delta_i W(\cdot - 1)\Delta_j W(\cdot - 1)\Delta_i W \cdot E(\Delta_j W|\mathcal{F}_{t_j})$$

$$= 0$$

By symmetry,

$$\begin{aligned}\sum_{i,j} EX_{i,j} &= \sum_i EX_{i,i} \\ &= \sum_i Ef(t_i)^2[\Delta_i W(\cdot - 1)]^2[\Delta_i W]^2 \\ &\leq K \sum_i E[\Delta_i W(\cdot - 1)]^2 \cdot E[\Delta_i W]^2 \\ &\leq KT|\pi|\end{aligned}$$

Hence

$$E\left|\int_0^T f(t)\, dW(t-1)\, dW(t)\right|^2 = \lim_{|\pi|\to 0} \sum_{i,j} EX_{i,j} = 0. \qquad \square$$

Shorthand:

	$dW(t-1)$	$dW(t)$
$dW(t-1)$	dt	0
$dW(t)$	0	dt

Proof.

Exercise.

Projection $\Pi : C \to \mathbf{R}^{mk}$ associated with $s_1, \cdots, s_k \in [-r, 0]$:

$$\Pi(\eta) := (\eta(s_1), \cdots, \eta(s_k)) \in \mathbf{R}^{mk}, \quad \eta \in C$$

Definition.

$\Phi \in C(T \times C(J; \mathbf{R}^m); \mathbf{R})$ is *tame* if there exist $\phi \in C(T \times \mathbf{R}^{mk}, \mathbf{R})$ and a projection Π such that

$$\Phi(t, \eta) = \phi(t, \Pi(\eta)).$$

for all $t \in T$ and $\eta \in C$.

Proof (Milstein Scheme).

Itô's formula for "tame" functionals

$$T \times C(J, \mathbf{R}^m) \to \mathbf{R}$$

of the segment X_t. Use formula + moment estimates on weak derivatives of X to get global error estimate for the Milstein approximations. $\square$

$W(t) := (W^1(t), \cdots, W^d(t))$, $t \geq 0 :=$ d-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, P)$.

$D := (D_1, \cdots, D_d) :=$ Malliavin differentiation operator associated with $\{W(t) : t \geq 0\}$.

Pathwise-continuous process:

$$X(t) := \begin{cases} \eta(0) + \int_0^t u(s)\, dW(s) + \int_0^t v(s)\, ds, & t > 0, \\ \eta(t), & -r \leq t \leq 0, \end{cases}$$

Skorohod integral. $\eta \in C$, BV.

$u = (u^1, \cdots, u^m)^T$, $u^i \in \mathbb{L}^{2,4}_{d,loc}$;

$v = (v^1, \cdots, v^m)^T$, $v^i \in \mathbb{L}^{1,4}_{loc}$ ([Nualart]).

u and v may not be adapted to the Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$. Set $u(t) := 0$ for $t < 0$ or $t > a$,

$$v(t) := \begin{cases} 0, & t > a \\ \eta'(t), & -r \leq t \leq 0. \end{cases}$$

$W(t) := 0$ if $t < 0$ or $t > a$.

$$U(t) := \int_0^t u(s)\, dW(s),\ V(t) := \begin{cases} \eta(0) + \int_0^t v(s)\, ds,\ t > 0 \\ \eta(t), \quad -r \leq t \leq 0. \end{cases}$$

Then

$$\begin{aligned} D_s X(t) = u(s) 1_{[0,a]}(t-s) + D_s \eta(0) + \int_0^t D_s v(r')\, dr' \\ + \int_0^t D_s u(r')\, dW(r'), \quad t > 0 \end{aligned}$$

$\Pi :=$ projection associated with $s_1, \cdots, s_k \in J$.

Cannot apply multi-dimensional Itô formula to $\phi(t, \Pi(X_t))$ because $\Pi(U_t)$ is of the form

$$\Big(\int_0^t u(s+s_1)\, dW(s+s_1), \cdots, \int_0^t u(s+s_k)\, dW(s+s_k)\Big),$$

and the components $(W(t+s_1), \cdots, W(t+s_k))$ are not independent. Use anticipating calculus (Nualart-Pardoux) to derive an Itô formula for $\phi(t, \Pi(X_t))$.

Assume $\phi \in C^{1,2}(T \times \mathbf{R}^{mk})$, $\vec{x} = (\vec{x}_1, \cdots, \vec{x}_m)$, $\vec{x}_i = (x_{i1}, \cdots, x_{ik}) \in \mathbf{R}^k$. Write

$$\phi(t, \vec{x}) = \phi(t, \vec{x}_1, \cdots, \vec{x}_m).$$

Theorem 4. *(Itô's formula).*

Suppose X *satisfies above conditions and let* $\phi \in C^{1,2}(T \times \mathbf{R}^{mk}, \mathbf{R})$. *Then*

$$\begin{aligned}
&\phi(t, \Pi(X_t)) - \phi(0, \Pi(X_0)) \\
&= \int_0^t \frac{\partial \phi}{\partial s}(s, \Pi(X_s))\, ds + \int_0^t \frac{\partial \phi}{\partial \vec{x}}(s, \Pi(X_s))\, d(\Pi(X_s)) + \\
&\frac{1}{2} \sum_{i,j=1}^{k} \sum_{i_1, j_1 = 1}^{m} \int_0^t \frac{\partial^2 \phi}{\partial x_{i,i_1} \partial x_{j,j_1}}(s, \Pi(X_s)) u^{i_1}(s+s_i) D_{s+s_i} X^{j_1}(s+s_j)\, ds
\end{aligned}$$

a.s. for all $t \in T$.

Example (Revisited)

$$
\begin{aligned}
g(W(t-1), W(t)) - g(W(-1), W(0)) \\
&= \int_0^t \frac{\partial g}{\partial x_1}(W(s-1), W(s))\, dW(s) \\
&+ \int_0^t \frac{\partial g}{\partial x_2}(W(s-1), W(s))\, dW(s-1) \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x_1^2}(W(s-1), W(s)) ds \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x_2^2}(W(s-1), W(s)) 1_{(1,\infty)}(s)\, ds
\end{aligned}
$$

for $t > 0$.

Weak differentiability of solutions of SDDE's.

Cf. Bell and Mohammed, Nualart.

$\mathbb{D}_m^{k,\infty} := \cap_{p\geq 2}\mathbb{D}_m^{k,p},\ k \in N.$

$D_u^l, 1 \leq l \leq d$, $0 \leq u \leq a$, weak differentiation with respect to l-th component of W.

Proposition.

In the Itô SDDE, assume that $g \in C_b^{0,1}(T \times \mathbf{R}^{k_1 m}; L(\mathbf{R}^d, \mathbf{R}^m)$ *and* $h \in C_b^{0,1}(T \times \mathbf{R}^{k_2 m}; \mathbf{R}^m)$*. Let* X *be the solution of the SDDE. Then* $X(t) \in \mathbb{D}_m^{1,\infty}$ *for all* $t \in T$*, and*

$$\sup_{0\leq u\leq a} E(\sup_{u\leq s\leq a} |D_u X(s)|^p) < \infty$$

for all $p \geq 2$*. Furthermore, the "partial" weak derivatives* $D_u^l X^j(t)$ *with respect to the* l*-th coordinate of* W *satisfy* *the following linear SDDE's a.s.:*

$$
\begin{aligned}
D_u^l X^j(t) = g^{jl}(u, \Pi_1(X_u^j)) + & \\
& \int_u^t \sum_{i=1}^{k_1} \frac{\partial g^{jl}}{\partial \vec{x}_i}(s, \Pi_1(X_s)) D_u^l X^j(s + s_{1,i})\, dW^l(s) \\
& + \int_0^t \sum_{i=1}^{k_2} \frac{\partial h^j}{\partial \vec{x}_i}(s, \Pi_2(X_s)) D_u^l X^j(s + s_{2,i})\, ds, \qquad t \geq u, \\
& = 0, \qquad\qquad t < u,\ \ l = 1, \cdots, d,\ \ j = 1, \cdots, m
\end{aligned}
$$

$g^{jl} = (j, l)$ *entry of the* $m \times d$ *matrix* g*,*

$h^j = j$*-th coordinate of* h*.*

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