Random Dynamics (SIUC 2006 Outstanding Scholar Public Lecture)

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RANDOM DYNAMICS

Salah Mohammed [a]

http://sfde.math.siu.edu/

Public Lecture: 4:00pm, November 7, 2006

Life Science III Auditorium

Southern Illinois University

Carbondale, Illinois, USA

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Acknowledgment

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The Plan

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- Examples of random systems with memory: from feedback control to stock market fluctuations.

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- State space is BIG, with infinitely many dimensions.
- View dynamics/time-evolution of slice within the state space.
- Mathematics gets harder but is still “doable”.

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- Equilibria: probabilistically stationary states.
- Stability of equilibria.
- Random dynamics near equilibria: structure within chaos.
- Existence of non-linear stable/unstable “smooth portions" of the state space near equilibria. Such smooth portions are called manifolds.

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- Supported by government scholarships through school, university and graduate school in the UK. Further details in web-site http://sfde.math.siu.edu/.

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- Enormous support from family throughout.

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A sample point ω could be any of the numbers

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Glossary-contd

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A_3 is a sure event, A_5 is an impossible event.

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Random variables may have values in more general spaces than the real numbers $\mathbf{R}$.

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Roll two dice. The sum X of their faces is a random variable on the sample space

$$\Omega = \{(i,j) : i,j = 1,2,3,4,5,6\}$$

$$\omega = (i,j),\ X(\omega) = i + j, i,j = 1,2,3,4,5,6.$$

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Distribution function of X is

$$F(x) := P(X \leq x)$$

where x runs through all possible values of X.

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A random variable $X : \Omega \to \mathbf{R}$ has normal distribution $N(\mu, \sigma^2)$ if

$$P(X \leq x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} dy$$

for all real x. Exponential-base $e = 2.71828$ approx.

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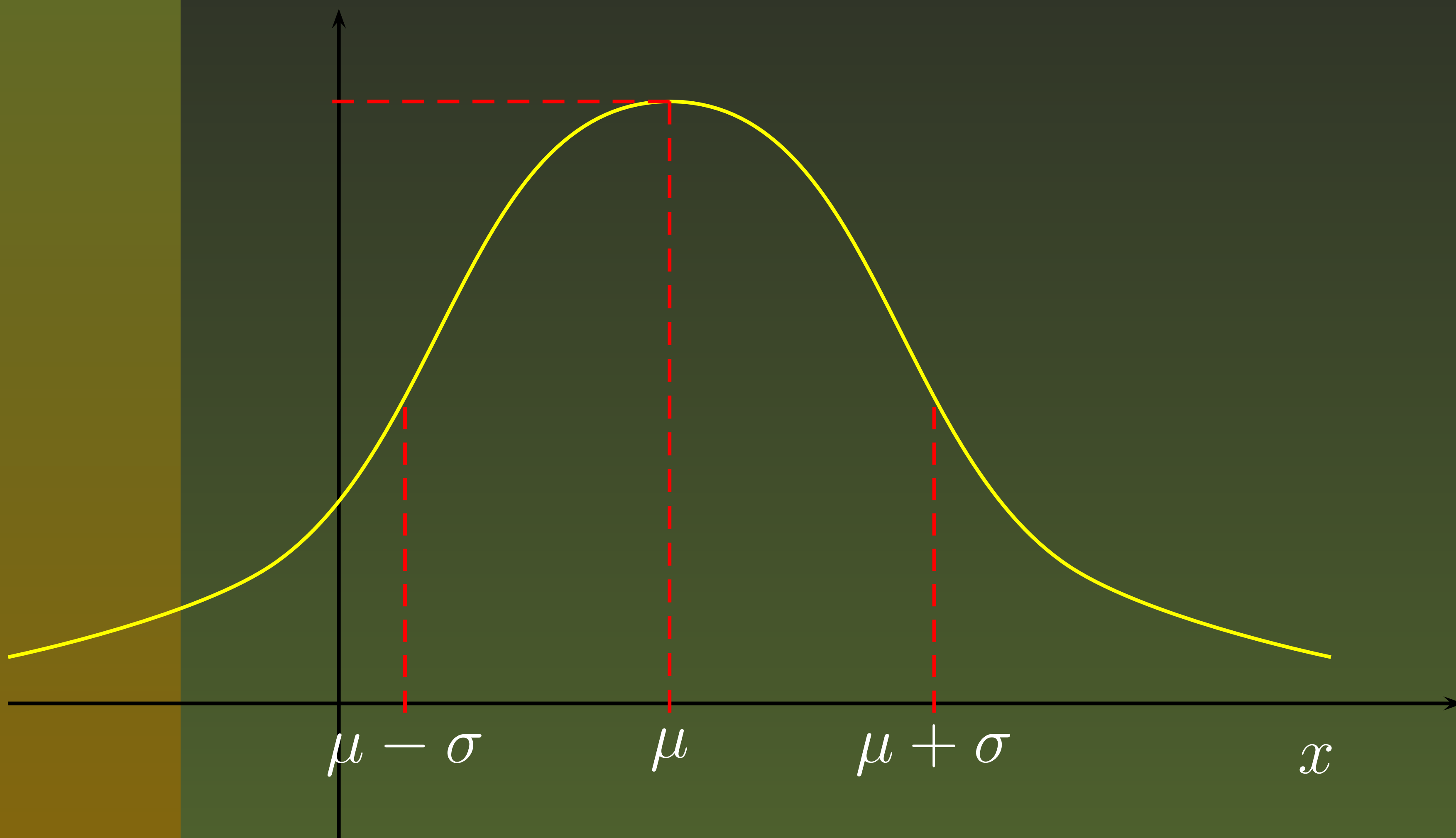
Normal distributions are important building blocks for modelling random evolution.

Normal Density



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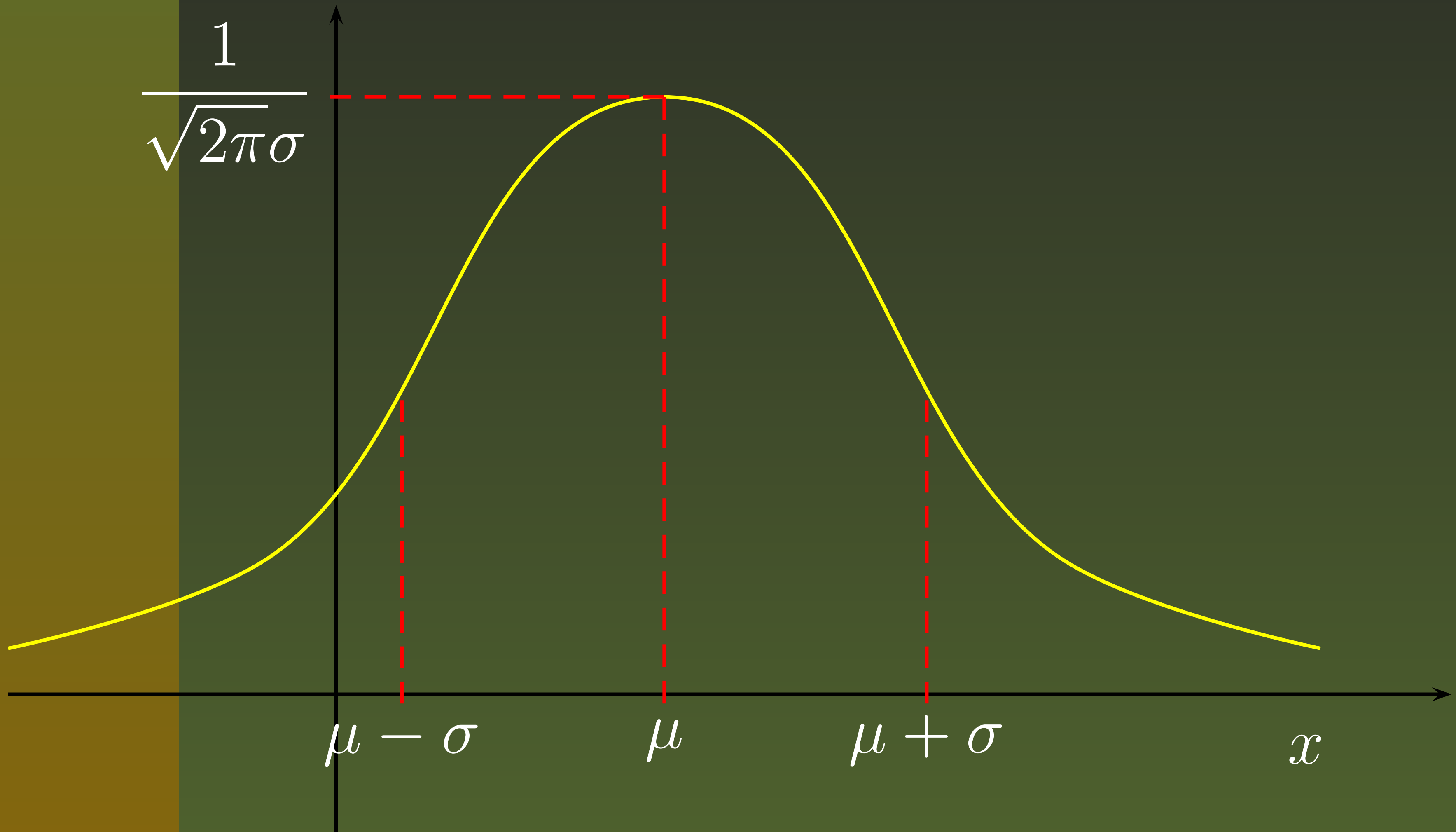

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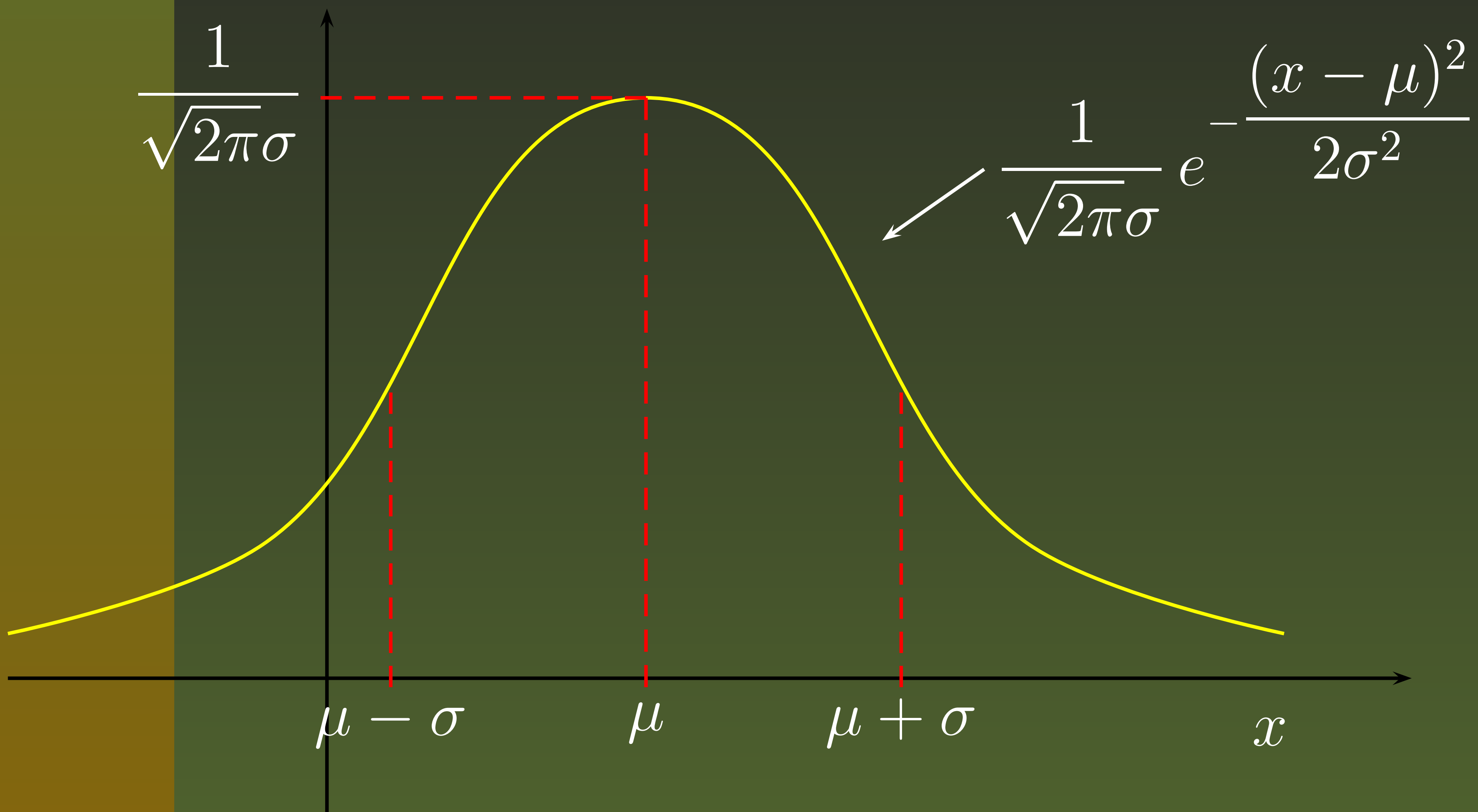

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Two random variables $X_1, X_2 : \Omega \to \mathbf{R}$ are independent if the probability of the event

$$(a_1 < X_1 < b_1 \text{ and } a_2 < X_2 < b_2)$$

is equal to the product

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$$X(t) : \Omega \to \mathbf{R}$$

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A random process is a family of random variables

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indexed by real t (usually time). View a random process as a function $X(t, \omega)$ of time t and chance ω.

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- Each increment $W(t_2) - W(t_1)$ is normal with mean zero and variance $t_2 - t_1$.

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"Markov" means that distributionally speaking, the future states of W are independent of their past history.

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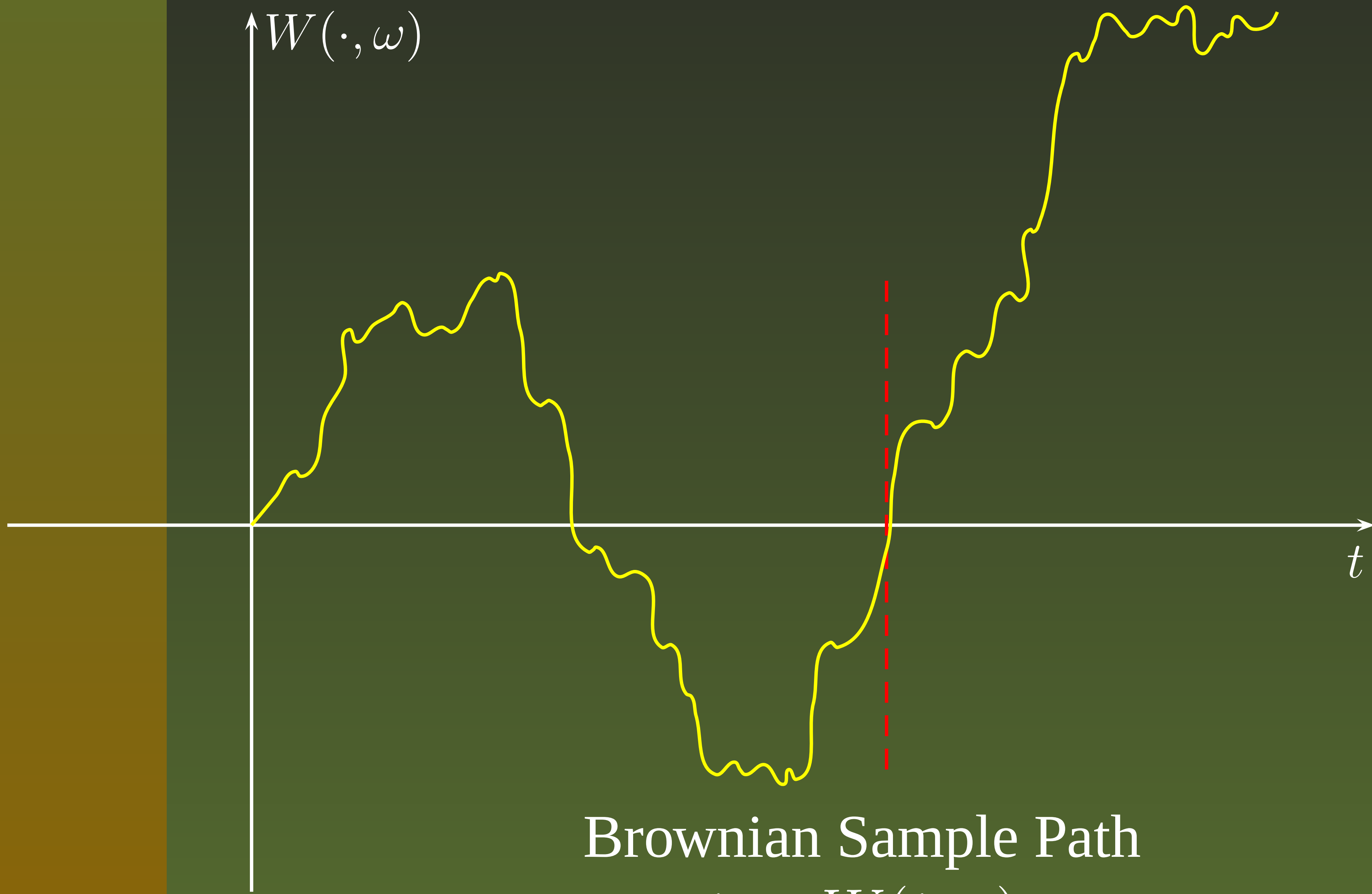
Nevertheless, I will go ahead and show you one!

Brownian Sample Path



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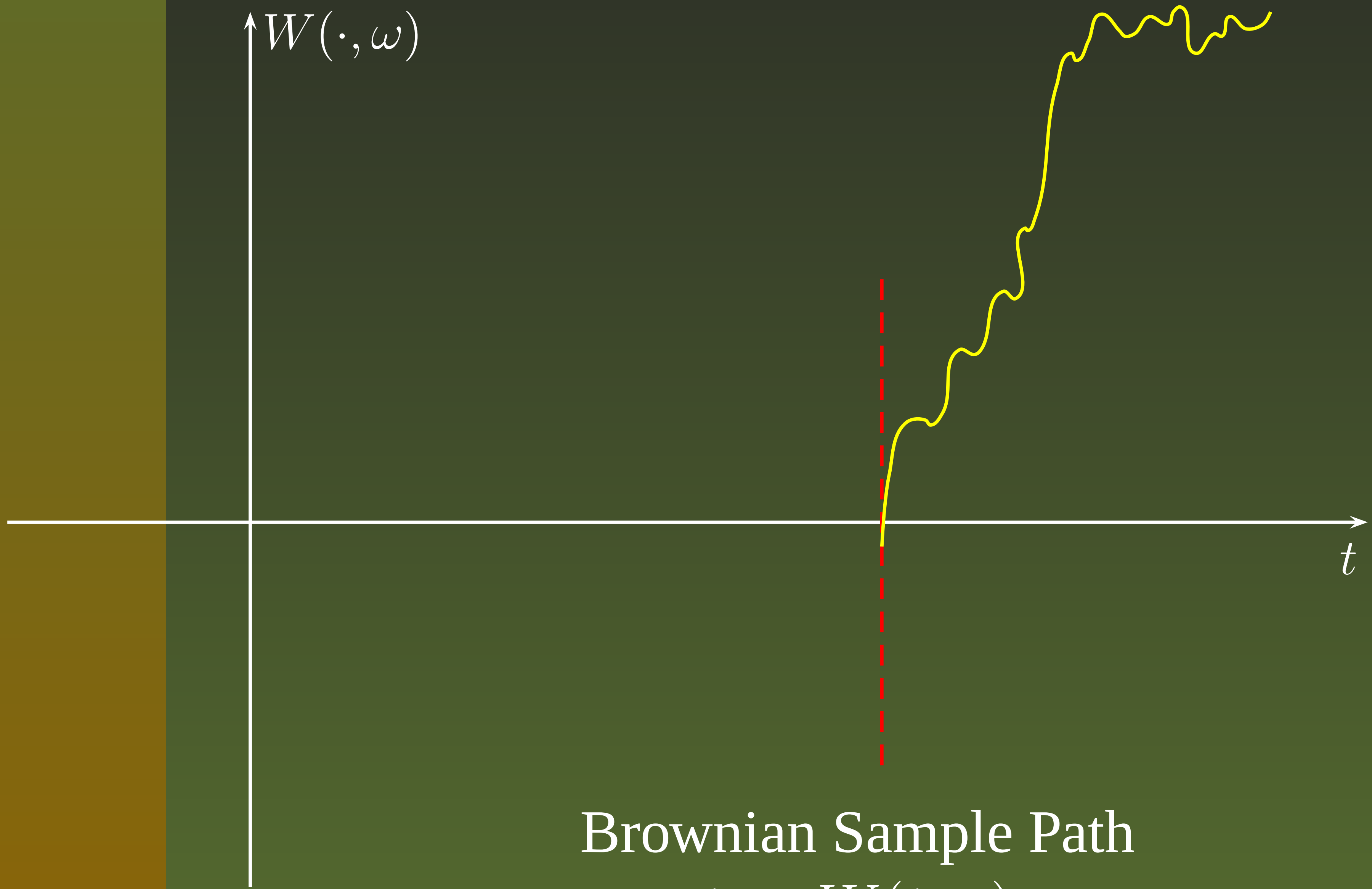

Brownian Sample Path

$$t \mapsto W(t,\omega)$$

Glossary-Contd

Each Brownian shift

$$\theta(t,\cdot) : \Omega \to \Omega, \quad t \in \mathbf{R}$$

$$\theta(t,\omega)(s) := W(t+s,\omega) - W(t,\omega), \quad s \in \mathbf{R},\ \omega \in \mathbf{\Omega}.$$

transforms the probability space Ω into itself (by moving the sample points ω around) while preserving the probabilities of all events.

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transforms the probability space Ω into itself (by moving the sample points ω around) while preserving the probabilities of all events.

Theorem:

The probability space Ω is perfectly mixed by the Brownian shift $\theta(t)$: The only events that are unchanged are either sure or impossible. (alias "ergodicity")

Examples: Noisy Feedback

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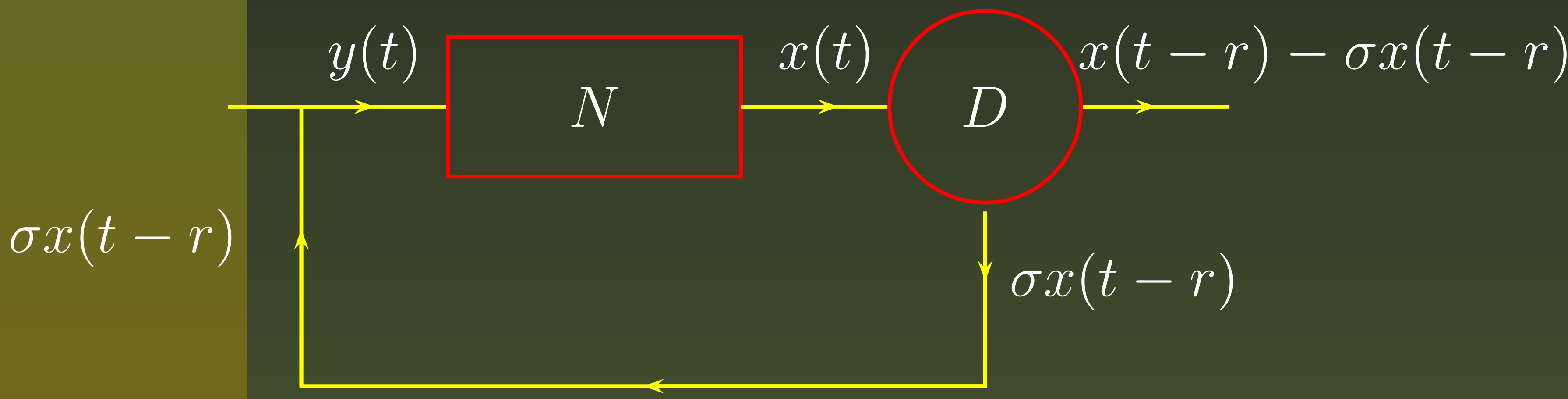

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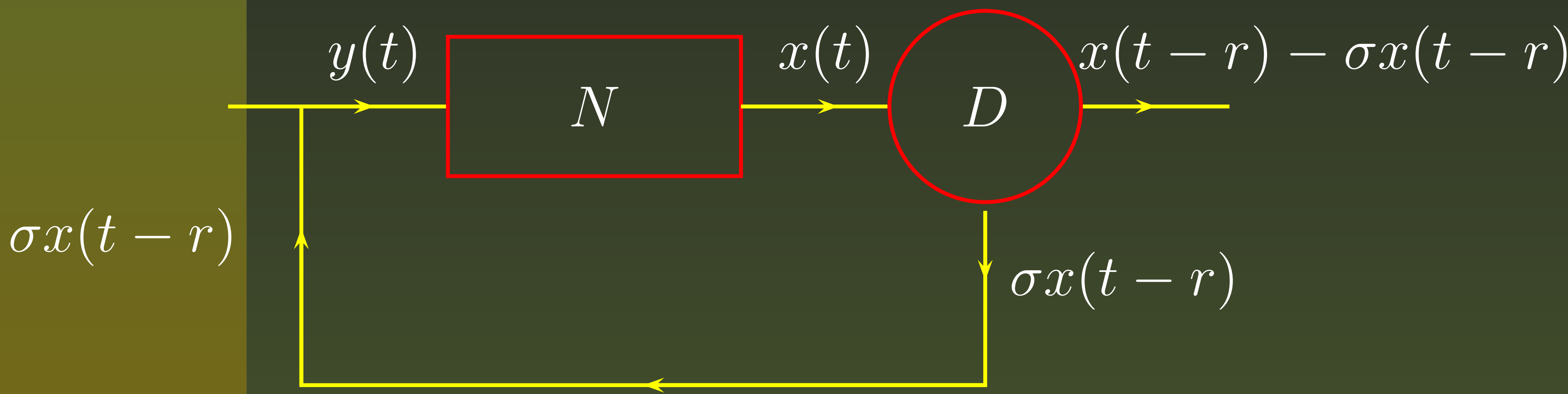

Box N: Input signal $= y(t)$, output $= x(t)$ at time $t > 0$ related by

$$\frac{dx(t)}{dt} = y(t)\,\frac{dW(t)}{dt}$$

where $W(t)$ is Brownian motion "white noise" in EE.

Noisy Feedback– Cont'd

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$$\frac{dx(t)}{dt} = \sigma x(t-r)\frac{dW(t)}{dt}, \qquad t > 0 \qquad \text{(I)}$$

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To solve (I), need an initial process $\eta(t)$, $-r \leq t \leq 0$:

$$x(t) = \eta(t) \qquad -r \leq t \leq 0$$

Noisy Feedback-Contd

View (I) as a stochastic integral

$$x(t) = \eta(0) + \int_0^t \sigma x(u - r)\, dW(u), \qquad t > 0$$

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$$x(t) = \eta(0) + \int_0^t \sigma x(u-r)\, dW(u), \qquad t > 0$$

Use idea of stochastic integration with respect to Brownian motion (K. Itô):

Partition time interval $[0, t]$ by points

$$0 = u_0 < u_1 < u_2 < \cdots u_i < u_{i+1} < \cdots u_n = t$$

which get closer and closer to each other as n gets infinitely large.

Partition of $[0, t]$

Noisy Feedback-Contd

The corresponding sums:

$$\sum_{i=0}^{n-1} \sigma x(u_i - r)[W(u_{i+1}) - W(u_i)]$$

will approach the Itô stochastic integral:

$$\int_0^t \sigma x(u - r)\, dW(u)$$

as the number of partition points n gets larger and larger.

Noisy Feedback-contd

To solve

$$dx(t) = \sigma x(t-r)\, dW(t), \qquad t > 0 \qquad \text{(I)}$$

proceed by successive forward (stochastic) integrations:

$$0 \le t \le r,\ \ r \le t \le 2r,\ \ 2r \le t \le 3r,\ \ \cdots\ ,$$

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The current value $x(t)$ of the solution x of (I) is non-Markov.

Segment Process

Segment Process

The segment x_t is a path $[-r, 0] \to \mathbf{R}$ defined by

$$x_t(s) := x(t+s), \qquad -r \leq s \leq 0$$

Segment Process-Contd

The solution $x(t)$ of the stochastic delay equation

$$dx(t) = \sigma x(t-r)\, dW(t), \qquad t > 0$$

is non-Markov, but the segment process x_t is Markov within the state space of all paths η.

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is non-Markov, but the segment process x_t is Markov within the state space of all paths η.

In order to capture the true dynamics of the stochastic delay equation, we observe the random evolution of the segment x_t rather than the current value $x(t)$

Feedback Without Delay

Conside the case $\mathbf{r} = \mathbf{0}$: (I) becomes a linear stochastic differential equation (without memory)

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$x(t)$ is Markov (no delay= no memory).

Simple Population Dynamics

- Consider a large population $x(t)$ at time t evolving with a constant birth rate $\beta > 0$ and a constant death rate α per capita.

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- Assume there is migration whose overall rate is distributed like white noise $\sigma\dot{W}$ (mean zero and variance $\sigma > 0$), where W is one-dimensional Brownian motion.

Simple Population – Cont'd

The change in population $\Delta x(t)$ over a small time interval $(t, t + \Delta t)$ is

$$\Delta x(t) = -\alpha x(t)\Delta t + \beta x(t-r)\Delta t + \sigma \dot{W}\Delta t$$

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Associate with the above stochastic delay equation the initial path η

$$x(s) = \eta(s), \quad -r \leq s \leq 0.$$

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A population $x(t)$ at time t evolving logistically with development (incubation) period $r > 0$ under Gaussian type noise (e.g. migration on a molecular level):

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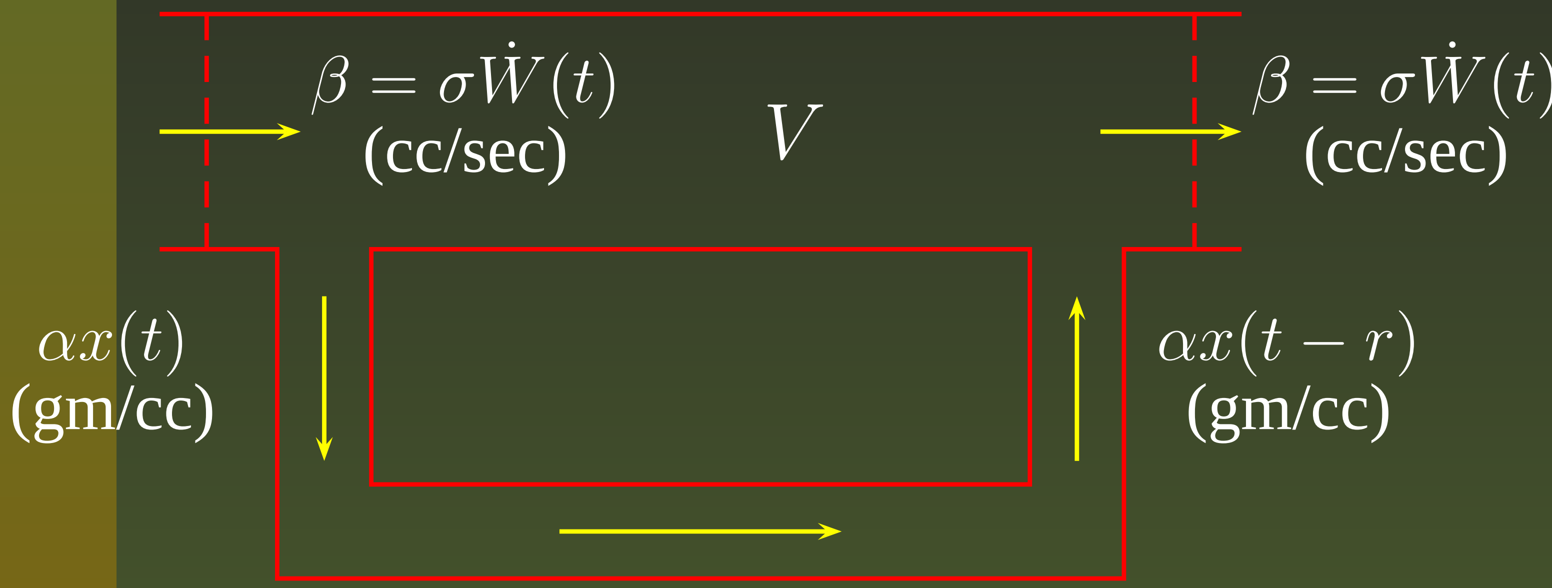
with initial condition

$$x(t) = \eta(t) \qquad -r \leq t \leq 0.$$

Fluid Flow



$\beta = \sigma \dot{W}(t)$
(cc/sec)
V
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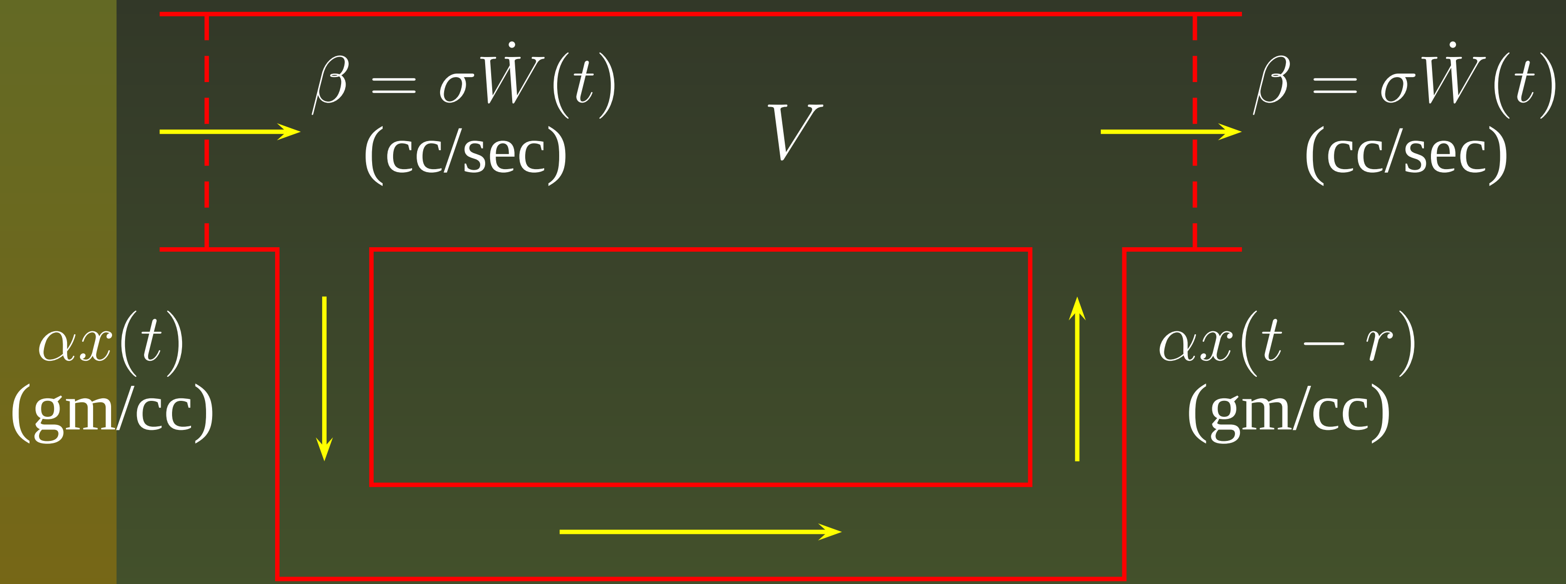

Main canal has dye (pollutant) with concentration $x(t)$ (gm/cc) at time t.

A fixed proportion of fluid in the main canal is pumped into the side canal(s).

Fluid Flow– Cont'd

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Then get the following stochastic delay equation:

$$\left.\begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r))\}\, dt + \sigma x(t)\, dW(t), \ \ t > 0 \\ x(s) &= \eta(s), \quad -r \le s \le 0 \end{aligned}\right\}$$

where η is a path $[-r, 0] \to \mathbf{R}$, ν and μ are real constants.

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Trading Strategy: $\pi_S(t)$ shares of stock $S(t)$ and $\pi_B(t)$ of bond $B(t)$.

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Constant volatility g and h corresponds to Black-Scholes model.

Stock Dynamics

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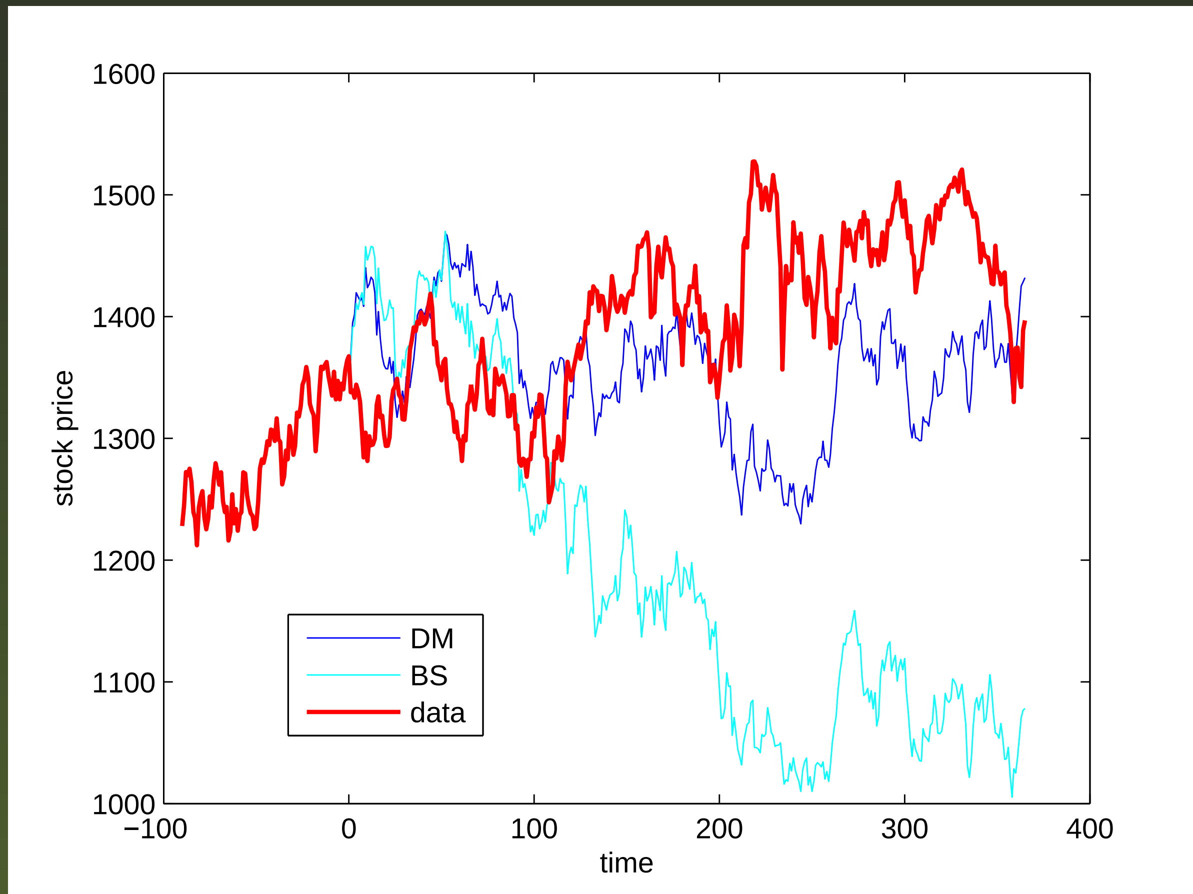

Stock prices when $h =$ constant, $b = 2$, $T = 365$, $L = 100$.

Stock data: DJX Index at CBOE.

Delayed BS Formula

(–>)

“Now let’s do the math”!

Stochastic Systems with Memory

Combine all dynamic models encountered so far in a single stochastic equation of the form

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W is Brownian motion; x_t is the segment process (encoding the memory of the solution process x); η is a given initial path $[-r, 0] \to \mathbf{R}$ (starting process for x).

State Space

Collect all possible initial states η in a state space, denoted by H, which contains all continuous paths $[-r, 0] \to \mathbf{R}$.
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distance between two paths η_1 and η_2:

$$\left(\int_{-r}^{0} [\eta_1(s) - \eta_2(s)]^2 \, ds \right)^{1/2}$$

State Space-contd

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angle between paths: η_1 and η_2 in H are perpendicular if

$$\int_{-r}^{0} \eta_1(s)\eta_2(s)\,ds = 0$$

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The state space is BIG: has infinite dimension. That is infinitely many mutually perpendicular paths:

$$\sin\left(\frac{\pi s}{r}\right),\ \sin\left(\frac{2\pi s}{r}\right),\ \sin\left(\frac{3\pi s}{r}\right), \cdots, \sin\left(\frac{n\pi s}{r}\right), \cdots$$

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$$\int_{-r}^{0} \sin\left(\frac{\pi s}{r}\right) \sin\left(\frac{2\pi s}{r}\right) ds = 0$$

Existence

A random dynamical system with memory is a relation between the current rate of change of the system and its past random states.

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Theorem:

Under appropriate (fairly general) conditions on the coefficients h, g, the stochastic equation with memory has a unique solution x for each choice of the initial state η in the state space H.

Random Dynamics with Memory

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- The expanding manifolds have fixed (non-random) finite dimension.
- The contracting manifolds have infinite dimension.

Theorem:

Under regularity conditions, for each sample point ω, we can observe the whole state space as it mixes under the random flow.

The Flow

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$$X(t, \eta, \omega)$$

of three variables: time t, state η and chance ω, changing continuously in (t, η) and satisfying:

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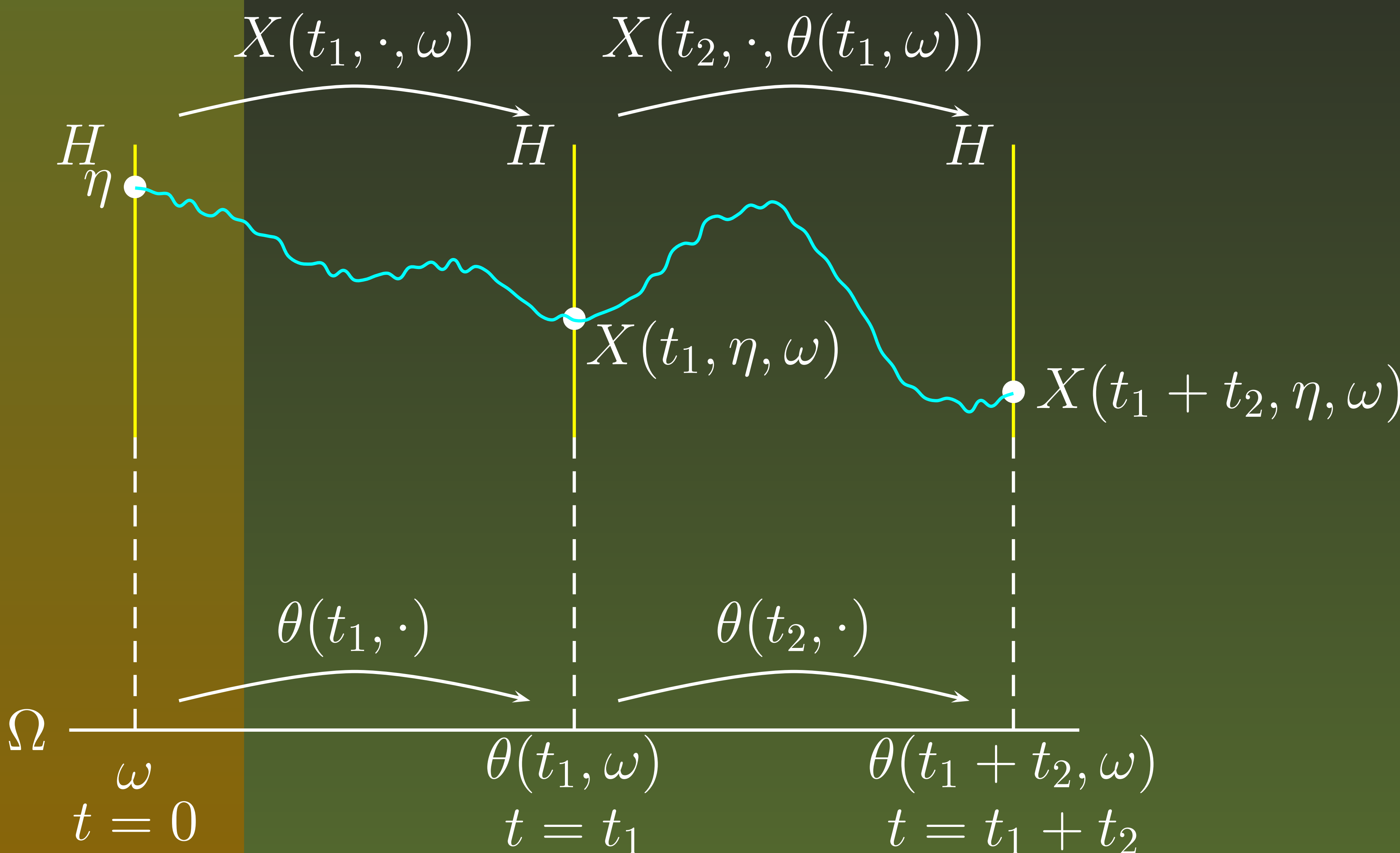
- $X(0, \eta, \omega) = \eta$ for all initial paths $\eta \in H$, and all $\omega \in \Omega$.

The Flow Property



Stationary Point-Equilibrium

A random variable $Y : \Omega \longrightarrow H$ is a *stationary point* for the flow (X, θ) if

$$X(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

for all $t \in \mathbf{R}^+$ and every $\omega \in \Omega$.

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Denote a stationary trajectory by

$$X(t, Y) = Y(\theta(t)).$$

Random Tubes

Theorem:

Within the state space H, *each stationary point* $Y(\omega)$ *has a ball* $B(Y(\omega), \rho(\omega))$ *center* $Y(\omega)$ *and radius* $\rho(\omega)$ *with the property that for any* $\eta \in B(Y(\omega), \rho(\omega))$ *the distance between* $X(t, \eta, \omega)$ *and* $Y(\omega)$ *grows like* $e^{\lambda_i t}$ *for large* t *where*

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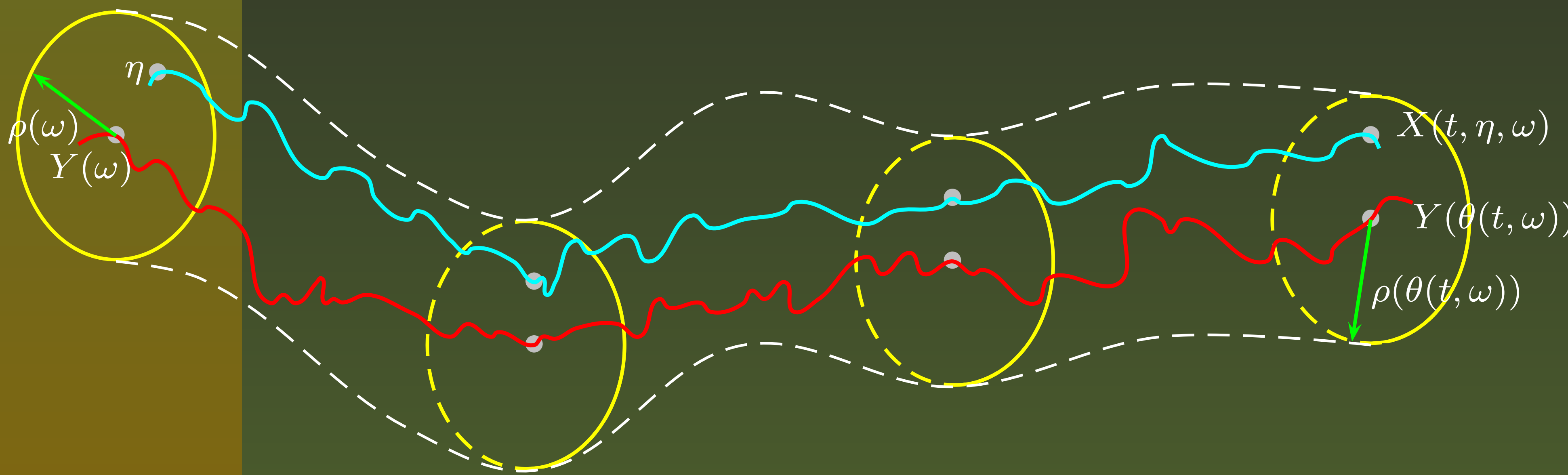
$$\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$$

are fixed countable and non-random. These represent exponential growth rates of the random flow near its equilibrium.

A Random Tube



Hyperbolicity

An equilibrium $Y(\omega)$ is hyperbolic if all exponential growth rates λ_i are non-zero:

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λ_{i_0} = largest negative growth rate.

λ_{i_0-1} = least positive growth rate.

Theorem

Let Y be a hyperbolic equilibrium of the stochastic delay equation. Then there is a random tube $B(Y(\omega), \rho(\omega))$ around Y, a smooth stable manifold $\mathcal{S}(\omega)$, and unstable one $\mathcal{U}(\omega)$ in $B(Y(\omega), \rho(\omega))$ with the following properties:

Theorem

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The stable manifold $\mathcal{S}(\omega)$ *is the set of all states* η *in* $B(Y(\omega), \rho(\omega))$ *such that the distance between* $X(t, \eta, \omega)$ *and* $Y(\theta(t, \omega))$ *decays like* $e^{\lambda_{i_0} t}$ *for large* t.

Theorem-contd

(Flow-invariance of the stable manifolds): *The stable manifold* $\mathcal{S}(\omega)$ *is eventually transported into* $\mathcal{S}(\theta(t,\omega))$*: That is*

$X(t,\cdot,\omega)(\mathcal{S}(\omega))$ *is a subset of* $\mathcal{S}(\theta(t,\omega))$ *for all large* t.

Theorem-contd

The unstable manifold $\mathcal{U}(\omega)$ *is the set of all states* η *in* $B(Y(\omega), \rho(\omega))$ *such that there is a unique continuous-time history process also denoted by* $y(\cdot, \omega) : (-\infty, 0] \to H$ *such that* $y(0, \omega) = \eta$, $X(t, y(s, \omega), \theta(s, \omega)) = y(t + s, \omega)$ *for all* $s \leq 0$, $0 \leq t \leq -s$, *and the distance between* $y(-t, \omega)$ *and* $Y(\theta(-t, \omega))$ *decays like* $e^{-\lambda_{i_0 - 1} t}$ *for large* t.

Theorem-contd

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The dimension of the unstable manifold $\mathcal{U}(\omega)$ *is finite and non-random.*

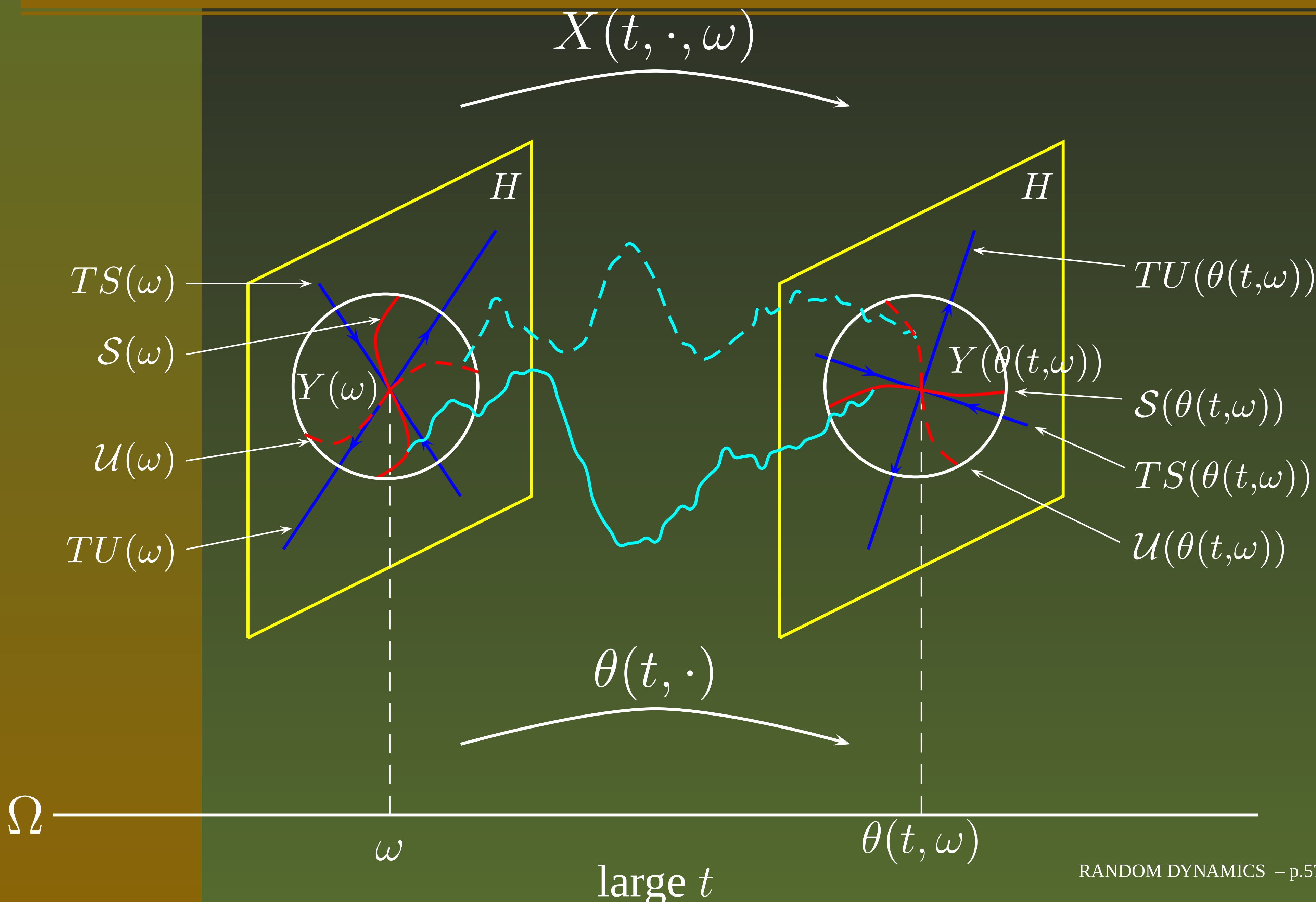
Theorem-contd

(Flow-invariance of the unstable manifolds):

The remote history of the unstable manifold $\mathcal{U}(\omega)$ *may be traced back to* $\mathcal{U}(\theta(-t,\omega))$*: That is* $\mathcal{U}(\omega)$ *is a subset of* $X(t,\cdot,\theta(-t,\omega))(\mathcal{U}(\theta(-t,\omega)))$ *for sufficiently large* t.

$$\mathcal{U}(\omega) \subseteq X(t,\cdot,\theta(-t,\omega))(\mathcal{U}(\theta(-t,\omega)))$$

Stable/Unstable Manifolds



Proof

[M.S]

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THE END!

THANK YOU!