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Anticipating Semilinear SPDEs (International Conference Stochastic Analysis and Stochastic Geometry)

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Anticipating Semilinear SPDEs [a]

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Swansea: April 19, 2007
Wales

[a] Results to appear in JFA [M-Z]

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Acknowledgment

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Does the following anticipating stochastic evolution equation (see):

$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt + F_0\big(v(t)\big)\,dt \\ &\qquad\qquad + Bv(t)\circ dW(t),\ t>0, \\ v(0) &= Y \end{aligned}\right\} \qquad (1)$$

admit a solution with a random initial condition $Y:\Omega\longrightarrow H$ in a Hilbert space H?

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Answer:

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admit a solution with a random initial condition $Y:\Omega\longrightarrow H$ in a Hilbert space H?

Answer:

YES! (provided Y is sufficiently regular).

Strategy

- Replace Y in see (1) by a deterministic initial condition x in H and get the corresponding (equivalent) Itô see:

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad + Bu(t,x)\,dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \tag{2}$$

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  with F a suitably modified non-linear drift.
- View the solution of the see (2) as a function (cocycle) $U(t,x,\omega)$ of three variables (t,x,ω) with Fréchet and Malliavin regularity in x and ω (resp.)

Strategy-Contd

- Consider the Stratonovich version of the Itô see (2):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F_0\big(u(t,x)\big)\,dt \\ &\qquad + Bu(t,x)\circ dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \tag{2'}$$

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- *In the above semilinear see, is it justified to replace the deterministic initial condition x by an arbitrary random variable Y* (substitution theorem)?

Strategy-Contd

- If YES, then get back the anticipating Stratonovich see (1) again:

$$\left.\begin{aligned} dU(t,Y) &= -AU(t,Y)\,dt + F_0\big(U(t,Y)\big)\,dt \\ &\qquad + BU(t,Y)\circ dW(t), \quad t>0 \\ U(0,Y) &= Y \end{aligned}\right\} \qquad (1)$$

by taking $v(t) := U(t,Y),\ t \geq 0.$

Difficulties

- Affirmative answer for the above question is known for a wide class of finite-dimensional sde's via substitution theorems ([Nu.1-2], [M-S.2]).

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- Known substitution theorems require a level of regularity of the cocycle $U(t,x,\omega)$ in t that is inconsistent with infinite-dimensionality of the stochastic dynamics (Cf. Theorem 3.2.6 [Nu.1], Theorem 5.3.4 [Nu.2]).
- Existing substitution theorems work under restrictive finite-dimensional or compactness constraints ([G-Nu-M]).

Difficulties-Contd

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- Failure of Sobolev inequalities in infinite dimensions.

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- Develop global spatial estimates on the semiflow, its Malliavin and Fréchet derivatives.
- Use of Malliavin calculus techniques is necessary because the initial condition and the underlying stochastic dynamics are infinite-dimensional.

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Substitution theorem provides a dynamic characterization of stable/unstable manifolds for semilinear see's near hyperbolic stationary states.

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Expect techniques developed in this analysis to yield similar substitution theorems for semiflows induced by sfde's.
Global moment estimates on the cocycle and its derivatives are interesting in their own right.

Expect results in this talk to lead to regularity in distribution of the invariant manifolds for semilinear spde's and sfde's.

The Set-up

- $(\Omega, \mathcal{F}, P) :=$ Wiener space of all continuous paths $\omega : \mathbf{R} \to E$, $\omega(0) = 0$, where E is a real separable Hilbert space.

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- Wiener shifts $\theta : \mathbf{R} \times \Omega \to \Omega$: Group of P-preserving ergodic transformations on $(\Omega, \mathcal{F}, P)$:

$$\theta(t, \omega)(s) := \omega(t+s) - \omega(t), \quad t, s \in \mathbf{R}, \ \omega \in \Omega.$$

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- $H :=$ real (separable) Hilbert space, norm $|\cdot|_H$.
- $\mathcal{B}(H) :=$ Borel σ-algebra of H.
- $L(H) :=$ Banach space of all bounded linear operators $H \to H$ given the uniform operator norm $\|\cdot\|_{L(H)}$.

Set-up: Brownian Motion

- $W :=$ E-valued Brownian motion $W : \mathbf{R} \times \Omega \to E$ with separable covariance Hilbert space $K \subset E$, Hilbert-Schmidt embedding.

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  $\{f_k : k \geq 1\} :=$ complete orthonormal basis of K; $W^k, k \geq 1$, standard independent one-dimensional Wiener processes ([D-Z.1], Chapter 4).

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- (W, θ) is a *helix*:
  $W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega))$

Set-up-contd

- $L_2(K,H) :=$ Hilbert space of all Hilbert-Schmidt operators $S : K \to H$, with norm

$$\|S\|_2 := \left[\sum_{k=1}^{\infty} |S(f_k)|_H^2\right]^{1/2}$$

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- $F_0 : H \to H$ is C_b^1.
- $F := F_0 + \frac{1}{2}\sum_{k=1}^{\infty} B_k^2$, where $B_k \in L(H)$ are given by

$$B_k(x) := B(x)(f_k),\ x \in H,\ k \geq 1; \text{ and } \sum_{k=1}^{\infty} \|B_k\|^2$$

converges.

Set-up: The Semilinear SEE

Consider the semilinear Itô stochastic evolution equation (see):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad + Bu(t,x)\,dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \qquad (2)$$

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$A : D(A) \subset H \to H$ is a closed linear operator on H.

Assume A has a complete orthonormal system of eigenvectors $\{e_n : n \geq 1\}$ with corresponding positive eigenvalues $\{\mu_n, n \geq 1\}$; i.e., $Ae_n = \mu_n e_n,\ n \geq 1$.

The Set-up-contd

Suppose $-A$ generates a strongly continuous semigroup of bounded linear operators $T_t : H \to H$, $t \geq 0$.

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Suppose $B : H \to L_2(K, H)$ is a bounded linear operator. The Itô integral in the see (2) is defined in the following sense ([D-Z.1], Chapter 4):

Set-up: The Itô Integral

Let $\psi : [0, a] \times \Omega \to L_2(K, H)$ be jointly measurable, $(\mathcal{F}_t)_{t \geq 0}$-adapted and

$$\int_0^a E\|\psi(t)\|^2_{L_2(K,H)}\, dt < \infty.$$

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Set

$$\int_0^a \psi(t)\, dW(t) := \sum_{k=1}^{\infty} \int_0^a \psi(t)(f_k)\, dW^k(t)$$

where the H-valued Itô integrals on the right hand side are with respect to the one-dimensional Wiener processes $W^k,\ k \geq 1$.

The Itô Integral-contd

Series converges in $L^2(\Omega, H)$ because

$$\sum_{k=1}^{\infty} E\left| \int_0^a \psi(t)(f_k)\, dW^k(t) \right|^2 = \int_0^a E\|\psi(t)\|^2_{L_2(K,H)}\, dt$$
$$< \infty.$$

Standing Hypotheses

- *Hypothesis* (A_1): $\sum_{n=1}^{\infty} \mu_n^{-1} \|B(e_n)\|^2_{L_2(K,H)} < \infty.$

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- *Hypothesis (B)*: $B : H \to L_2(K, H)$ extends to a bounded linear operator $B \in L(H, L(E, H))$; $\displaystyle\sum_{k=1}^{\infty} \|B_k\|^2 < \infty$, where $B_k \in L(H)$ is defined by

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- Requirement (b) above is satisfied if $A = -\Delta$, where Δ is the Laplacian on a compact smooth d-dimensional Riemannian manifold M with boundary, under Dirichlet boundary conditions.

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  - (a) The operator $B : H \to L_2(K, H)$ is Hilbert-Schmidt.
  - (b) $\liminf_{n \to \infty} \mu_n > 0$.
- Requirement (b) above is satisfied if $A = -\Delta$, where Δ is the Laplacian on a compact smooth d-dimensional Riemannian manifold M with boundary, under Dirichlet boundary conditions.
- No restriction on $\dim M$ under (A_1) for spdes.

Mild Solutions

A mild solution of the semilinear see (2) is a family of $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H,\ x \in H$, satisfying the following stochastic integral equation:

$$
\begin{aligned}
u(t,x,\cdot) = T_t x &+ \int_0^t T_{t-s} F(u(s,x,\cdot))\, ds \\
&+ \int_0^t T_{t-s} B u(s,x,\cdot)\, dW(s), \qquad t \geq 0,
\end{aligned}
\tag{3}
$$

([D-Z.1-2]).

Stratonovich Form

The Itô see (2) has the equivalent Stratonovich form

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad - \frac{1}{2}\sum_{k=1}^{\infty} B_k^2 u(t,x)\,dt + Bu(t,x)\circ dW(t) \\ u(0,x) &= x \in H \end{aligned}\right\} \qquad (2')$$

where $B_k \in L(H)$ are given by $B_k(x) := B(x)(f_k)$, $x \in H$, $k \geq 1$.

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- For each $\omega \in \Omega$, the map $U(t, x, \omega)$ is continuous in $(t, x) \in \mathbf{R}^+ \times H$; for fixed $(t, \omega) \in \mathbf{R}^+ \times \Omega$, $U(t, x, \omega)$ is C^k in $x \in H$.

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- $U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega)$ for all $t_1, t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.

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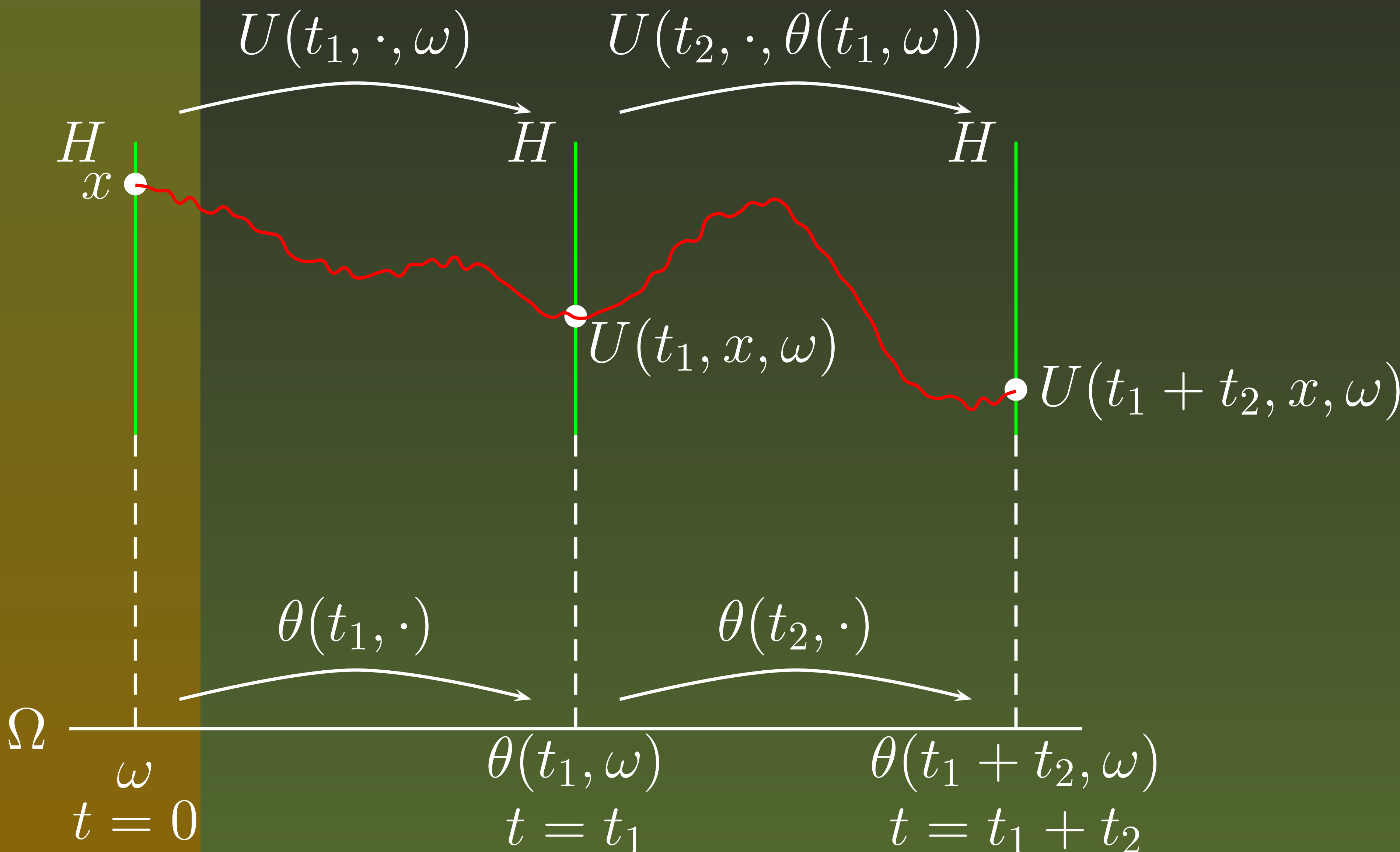
- $U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega)$ for all $t_1, t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.
- $U(0, x, \omega) = x$ for all $x \in H, \omega \in \Omega$.

The Cocycle Property



Existence of the Cocycle

Theorem 1:

Under Hypotheses (B) and (A_1), the Itô see (2) (or its Stratonovich version ($2'$)) admits a perfect jointly measurable C^1 cocycle (U, θ) where

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Proof of Theorem 1:

([M-Z-Z], Theorem 1.2.6). $\square$

Stationary Points

An $\mathcal{F}$-measurable random variable $Y : \Omega \to H$ is said be a stationary point for the cocycle (U, θ) if

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

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for all $(t, \omega) \in \mathbf{R}^+ \times \Omega$.

A stationary point of the see (2) corresponds to a stationary solution to the anticipating Stratonovich see (1).

Malliavin Regularity

For any integer $p \geq 2$, denote by $\mathbb{D}^{1,p}(\Omega, H)$ the Sobolev space of all $\mathcal{F}$-measurable random variables $Y : \Omega \to H$ which are p-integrable together with their Malliavin derivatives $\mathcal{D}Y$ ([Nu.1-2]).

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We now state the main substitution theorem in this talk.

Substitution

Theorem 2: (The Substitution Theorem)

Assume Hypotheses (B) and (A_1). Let $U : \mathbf{R}^+ \times H \times \Omega \to H$ be the C^1 cocycle generated by the see (2). Let $Y \in \mathbb{D}^{1,4}(\Omega, H)$ be a random variable. Then $v(t) := U(t, Y)$, $t \geq 0$, is a mild solution of the (anticipating) Stratonovich see

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$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt + F_0\big(v(t)\big)\,dt \\ &\qquad\qquad + Bv(t) \circ dW(t),\ t > 0, \\ v(0) &= Y. \end{aligned}\right\} \quad (1)$$

Substitution Theorem-contd

In particular, if $Y \in \mathbb{D}^{1,4}(\Omega, H)$ *is a stationary point of the see (2) (or* $(2')$*), then* $U(t,Y) = Y\big(\theta(t)\big),\ t \geq 0$*, is a stationary solution of the (anticipating) Stratonovich see (1):*

Substitution Theorem-contd

In particular, if $Y \in \mathbb{D}^{1,4}(\Omega, H)$ *is a stationary point of the see (2) (or (*$2'$*)), then* $U(t, Y) = Y\big(\theta(t)\big),\ t \geq 0$*, is a stationary solution of the (anticipating) Stratonovich see (1):*

$$\left.\begin{aligned} dY(\theta(t)) &= -AY(\theta(t))\,dt + F_0\big(Y(\theta(t))\big)\,dt \\ &\quad + BY(\theta(t)) \circ dW(t), t > 0, \\ Y(\theta(0)) &= Y. \end{aligned}\right\} \qquad (4)$$

Substitution Theorem-contd

Furthermore, assume that F_0 *is* C_b^2. *Then the linearized cocycle* $DU(t,Y)$ *is a mild solution of the linearized anticipating see*

Substitution Theorem-contd

Furthermore, assume that F_0 is C_b^2. Then the linearized cocycle $DU(t,Y)$ is a mild solution of the linearized anticipating see

$$\left.\begin{aligned} dDU(t,Y) &= -ADU(t,Y)\,dt \\ &\quad + DF_0\big(U(t,Y)\big)DU(t,Y)\,dt \\ &\quad + \big\{B \circ DU(t,Y)\big\} \circ dW(t),\ t>0, \\ DU(0,Y) &= \mathrm{id}_{L(H)}. \end{aligned}\right\} \tag{5}$$

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- Use the linear cocycle to get a pathwise variational integral equation equivalent to the semilinear see.
- Derive moment estimates for the nonlinear cocycle, its Fréchet and Malliavin derivatives.

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- Prove the substitution theorem when Y is replaced by its finite-dimensional projections Y_n: Use finite-dimensional projections to smooth out the semigroup T_t in t, and apply finite-dimensional substitution techniques.
- Rewrite each finite-dimensional anticipating Stratonovich integral in terms of a Skorohod integral plus a Lebesgue integral correction term.
- Let $n \longrightarrow \infty$ using dominated convergence and the moment estimates on the cocycle, its Fréchet and Malliavin derivatives.

Linear SEE

Existence of semiflows for mild solutions of linear see:

$$
\begin{aligned}
du(t,x,\cdot) &= -Au(t,x,\cdot)\,dt \\
&\qquad + Bu(t,x,\cdot)\,dW(t), \quad t>0 \\
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e.g. $A = -\Delta$ on compact smooth Riemannian manifold.

Mild Solutions: Linear Case

A *mild solution* of the linear see is a family of jointly measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes

$$u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H, \ \ x \in H$$

such that

$$u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} B u(s, x, \cdot)\, dW(s), \qquad t \geq 0.$$

Integral equation holds x-almost surely, $x \in H$.

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Integral equation holds x-almost surely, $x \in H$.

Is $u(t, x, \cdot)$ pathwise continuous linear in x?

Kolmogorov Fails!

Kolmogorov's continuity theorem fails for random field $I : L^2([0,1],\mathbf{R}) \to L^2(\Omega,\mathbf{R})$

$$I(x) := \int_0^1 x(t)\,dW(t), \quad x \in L^2([0,1],\mathbf{R}).$$

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No continuous (or even measurable linear!) selection

$$L^2([0,1],\mathbf{R}) \times \Omega \to \mathbf{R}$$

$$(x,\omega) \mapsto I(x,\omega)$$

of I ([Mo.1], pp. 144-148).

Lifting

- Lift semigroup $T_t, t \geq 0$, to a strongly continuous semigroup of bounded linear operators
$\tilde{T}_t : L_2(K, H) \to L_2(K, H), t \geq 0$, via composition
$\tilde{T}_t(C) := T_t \circ C,\ C \in L_2(K, H), t \geq 0.$

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- Lift stochastic integral

$$\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s),\ \ x \in H,\, t \geq 0,$$

to $L_2(H)$ for adapted square-integrable $v : \mathbf{R}^+ \times \Omega \to L_2(H)$. Denote lifting by $\displaystyle\int_0^t T_{t-s} B v(s)\, dW(s) \in L_2(H)$.

Lifting-contd

That is:

$$\left[\int_0^t T_{t-s} Bv(s)\, dW(s)\right](x) = \int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s)$$

for all $t \geq 0$, x-a.s..

The Linear Flow

Theorem 3:
Assume hypothesis (B) and (A_1)*. Then the mild solution of the linear see has a Borel (strongly) measurable* $(\mathcal{F}_t)_{t\geq 0}$*-adapted version* $\Phi : \mathbf{R}^+ \times \Omega \to L(H)$ *with the following properties:*

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- $E \sup_{0\leq t\leq a} \|\Phi(t,\cdot)\|_{L(H)}^{2p} < \infty$, *whenever* $p \geq 1$.

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- $E \sup_{0\leq t\leq a} \|\Phi(t,\cdot)\|_{L(H)}^{2p} < \infty$, *whenever* $p \geq 1$.
- (Φ, θ) *is a perfect* $L(H)$*-valued cocycle:*

$$\Phi(t+s,\omega) = \Phi(t,\theta(s,\omega)) \circ \Phi(s,\omega)$$

for all $s, t \geq 0$ *and all* $\omega \in \Omega$*;*

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  for all $s, t \geq 0$ *and all* $\omega \in \Omega$;
- $\sup\limits_{0\leq s\leq t\leq a} \|\Phi(t-s,\theta(s,\omega))\|_{L(H)} < \infty$, *for all* $\omega \in \Omega$.

Linear Flow-Contd: “Chaos"!

- *For each* $t > 0$ *and almost all* $\omega \in \Omega$, $\Phi(t,\omega) \in L_2(H)$ *has "chaos-type" representation*

$$\Phi(t,\cdot) = T_t + \sum_{n=1}^{\infty} \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \cdots \int_0^{s_{n-1}} T_{s_{n-1}-s_n} B T_{s_n}\, dW(s_n) \cdots dW(s_2)\, dW(s_1).$$

Iterated Itô stochastic integrals are lifted integrals in $L_2(H)$, *and series converges absolutely in* $L_2(H)$.

Semilinear SEE

Consider the semilinear Itô see:

$$\left.\begin{aligned} du(t) &= -Au(t)dt + F(u(t))dt \\ &\qquad + Bu(t)\,dW(t), \quad t > 0, \\ u(0) &= x \in H \end{aligned}\right\} \tag{2}$$

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Operators A, B satisfy hypothesis (B) and (A_1).
$F : H \to H$ is (Fréchet) C_b^1, with linear growth:

$$|F(v)| \leq C(1 + |v|), \quad v \in H$$

for some positive constant C.

Mild Solution: Semilinear SEE

Recall a *mild solution* of semilinear Itô see (2) is a family of jointly measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H,\ \ x \in H$, satisfying:

$$u(t,x,\cdot) = T_t(x) + \int_0^t T_{t-s}(F(u(s,x,\cdot)))\,ds \\ + \int_0^t T_{t-s}Bu(s,x,\cdot)\,dW(s),$$

for all $t \geq 0$, x-a.s. ([D–Z], Chapter 7, p. 182).

Random Integral Equation

Obtain a C^k perfect cocycle (U, θ) for mild solutions of the semilinear see, via the random integral equation on H:

$$U(t, x, \omega) = \Phi(t, \omega)(x) + \int_0^t \Phi(t - s, \theta(s, \omega))(F(U(s, x, \omega)))\, ds$$

for each $\omega \in \Omega$, $t \geq 0$, $x \in H$.

Estimates of the Cocycle

Get new global estimates on the non-linear cocycle $U : \mathbf{R}^+ \times H \times \Omega \to H$, its spatial Fréchet derivative $DU(t,x,\cdot)$ and its Malliavin derivatives $\mathcal{D}_u U(t,x,\cdot)$ for $u,t \in [0,a]$ and $x \in H$.

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Derivations are based on results in [M.Z.Z], Gronwall's lemma and the fact that W has stationary independent increments.

Estimates of Cocycle-Contd

Theorem 4:

Assume Hypotheses (B), (A_1) *and let* F *be* C_b^1. *Let* $U : \mathbf{R}^+ \times H \times \Omega \to H$ *be the cocycle generated by the mild solutions of the see (2). Fix any* $a \in (0, \infty)$. *Then:*

$$E \sup_{\substack{0 \le t \le a \\ x \in H}} \frac{|U(t, x, \cdot)|^{2p}}{(1 + |x|_H^{2p})} < \infty, \quad p \ge 1$$

$$E \sup_{\substack{0 \le t \le a \\ x \in H}} \|DU(t, x, \cdot)\|^{2p} < \infty, \quad p \ge 1$$

$DU :=$ *Fréchet derivative of* U *in the spatial variable* x.

More Estimates

Theorem 4$'$:

In the see (2), assume Hypotheses (B) and (A_1).

More Estimates

Theorem 4′:

In the see (2), assume Hypotheses (B) and (A_1).

(i) *Let* $u, t \in [0, a]$. *Define*

$$V(t, \cdot) := \Phi(t, \cdot) - T_t, \quad t \in [0, a].$$

Then $V(t, \cdot) \in \mathbb{D}^{1,2p}(\Omega, L_2(H))$ *and*

$$E\left[\sup_{u \le t \le a} \|\mathcal{D}_u V(t, \cdot)\|_{L_2(H)}^{2p}\right] < \infty$$

for all $p \geq 1$.

More Estimates-contd

(ii) *Suppose* F *is* C_b^1. *Then*

$$E\left[\sup_{\substack{0\le t\le a\\ x\in H}} \frac{|\mathcal{D}U(t,x,\cdot)|_H^{2p}}{(1+|x|_H^{2p})}\right] < \infty,$$

for all $p \geq 1$. $\mathcal{D}$:= *Malliavin derivative.*

More Estimates-contd

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for all $p \geq 1$. $\mathcal{D} :=$ *Malliavin derivative.*

(iii) *Let* F *be* C_b^2. *Then*

$$E\left[\sup_{\substack{0\le u,t\le a\\ x\in H}} \frac{\|\mathcal{D}_u DU(t,x,\cdot)\|^{2p}}{(1+|x|_H^{2p})}\right] < \infty$$

for all $p \geq 1$.

Finite-dimensional Projections

Objective:

To prove the substitution theorem when the random variable $Y \in \mathbb{D}^{1,4}(\Omega, H)$ is replaced by its finite-dimensional projections on H.

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$\{e_n : n \geq 1\}$:= complete orthonormal system of eigenvectors of A.

$H_n := L\{e_i : 1 \leq i \leq n\}$, the n-dimensional linear subspace of H spanned by $\{e_i : 1 \leq i \leq n\}$, for each $n \geq 1$.

Projections-contd

Define the projections $P_n : H \to H_n$, $n \geq 1$, by

$$P_n(x) := \sum_{k=1}^{n} < x, e_k, > e_k, \quad x \in H.$$

Projections-contd

Define the projections $P_n : H \longrightarrow H_n$, $n \geq 1$, by

$$P_n(x) := \sum_{k=1}^{n} < x, e_k, > e_k, \quad x \in H.$$

Define $Y_n : \Omega \longrightarrow H_n$ by

$$Y_n := P_n \circ Y, \quad n \geq 1.$$

Then $Y_n \longrightarrow Y$ as $n \longrightarrow \infty$ a.s.

Finite-dimensional Substitution

Theorem 5:

Assume (B) and (A_1) and suppose $Y \in \mathbb{D}^{1,4}(\Omega, H)$. Then

$$\left.\begin{aligned} dU(t, Y_n) &= -AU(t, Y_n)\,dt + F_0\big(U(t, Y_n)\big)\,dt \\ &\quad + BU(t, Y_n) \circ dW(t), t > 0, \\ U(0, Y_n) &= Y_n. \end{aligned}\right\} \quad (6)$$

for each $n \geq 1$.

Proof of Theorem 5

- Proof still requires Malliavin calculus techniques, largely due to the underlying strongly continuous semi-group dynamics in $\{T_t\}_{t\geq 0}$.

$\square$

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- Use global estimates on U to represent the Stratonovich integrals in terms of Skorohod integrals.
- Project the semigroup $\{T_t\}_{t\geq 0}$ onto H_m and use finite-dimensional substitutions.
- Then pass to the limit as $m \longrightarrow \infty$ using global estimates on U and dominated convergence.

$\square$

Proof of Substitution Theorem 2

Step 1:

Suppose $Y \in \mathbb{D}^{1,4}(\Omega, H)$, and assume Hypothesis (B) and (A_1).

Sufficient to show

$$\left.\begin{aligned} U(t,Y) = T_t(Y) &+ \int_0^t T_{t-s}F_0\big(U(s,Y)\big)\,ds \\ &+ \int_0^t T_{t-s}BU(s,Y)\circ dW(s). \end{aligned}\right\} \tag{10}$$

Proof of Theorem 2-contd

Step 2:

Pass to the limit as $n \to \infty$ in the finite-dimensional result:

$$\left.\begin{aligned} U(t, Y_n) = T_t(Y_n) + \int_0^t T_{t-s} F_0\big(U(s, Y_n)\big)\, ds \\ + \int_0^t T_{t-s} B U(s, Y_n) \circ dW(s), \\ t > 0,\ n \geq 1. \end{aligned}\right\} \qquad (7)$$

Localization

Denote by $\mathbb{L}^{1,2}$ the class of all processes $v : [0,t] \times \Omega \to H$ such that $v \in L^2([0,t] \times \Omega, H)$, $v(s,\cdot) \in \mathbb{D}^{1,2}(\Omega, H)$ for almost all $s \in [0,t]$ and $E[\int_0^t \int_0^t \|\mathcal{D}_u v(s,\cdot)\|_H^2 \, du \, ds] < \infty$.

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We say that v belongs to $\mathbb{L}^{1,2}_{loc}$ if there exists a sequence $(\Omega_m, v^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$ with the following properties:

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We say that v belongs to $\mathbb{L}^{1,2}_{loc}$ if there exists a sequence $(\Omega_m, v^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$ with the following properties:

(i) $\Omega_m \uparrow \Omega$ as $m \to \infty$,

(ii) $v = v^m$ on Ω_m.

Proof of Theorem 2

Step 3:

The Stratonovich integral

$$\int_0^t T_{t-s} BU(s, Y) \circ dW(s)$$

in (10) is well-defined:

Proof of Theorem 2

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The Stratonovich integral

$$\int_0^t T_{t-s} BU(s, Y) \circ dW(s)$$

in (10) is well-defined:
Sufficient to show that the process

$$v(s) := T_{t-s} BU(s, Y), s \leq t$$

is in $\mathbb{L}^{1,2}_{loc}$: *Localize* v using a bump function $\phi_m \in C_b^2(\mathbf{R}, \mathbf{R})$ such that $\phi_m(z) = 1$ for $|z| \leq m$ and $\phi_m(z) = 0$ for $|z| > m + 1$. ([Nu.2], Theorem 5.2.3).

Easy Limits

Step 4:

Pass to the limit a.s. as $n \to \infty$ in (7). Get easy a.s. limits:

$$\lim_{n\to\infty} U(t, Y_n) = U(t, Y)$$

$$\lim_{n\to\infty} T_t(Y_n) = T_t(Y)$$

$$\lim_{n\to\infty} \int_0^t T_{t-s} F_0\big(U(s, Y_n)\big)\, ds$$

$$= \int_0^t T_{t-s} F_0\big(U(s, Y)\big)\, ds$$

A Not-So-Easy Limit

Step 5:

A Not-So-Easy Limit

Step 5:

But following limit is non-trivial:

$$\left.\begin{aligned} \lim_{n\to\infty} \int_0^t T_{t-s} BU(s, Y_n) \circ dW(s) \\ = \int_0^t T_{t-s} BU(s, Y) \circ dW(s) \end{aligned}\right\} \qquad (11)$$

in probability.

Proof of Theorem 2-contd

Step 6:

Proof of Theorem 2-contd

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To prove (11), use localization:

Proof of Theorem 2-contd

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To prove (11), use localization:

$$\int_0^t T_{t-s} BU(s, Y_n) \circ dW(s)$$
$$= \int_0^t T_{t-s} BU(s, Y_n) \phi_m(|Y|_H) \circ dW(s),$$

on $\Omega_m := \{\omega : |Y(\omega)|_H \leq m\}$;

Proof of Theorem 2-contd

and

$$\int_0^t T_{t-s} BU(s,Y) \circ dW(s)$$

$$= \int_0^t T_{t-s} BU(s, Y)\phi_m(|Y|_H) \circ dW(s)$$

on Ω_m for any fixed integer $m \geq 1$.

Proof of Theorem 2-contd

Step 7:

(11) will follow from

$$\lim_{n\to\infty}\int_0^t T_{t-s}BU(s,Y_n)\phi_m(|Y|_H)\circ dW(s)$$
$$=\int_0^t T_{t-s}BU(s,Y)\phi_m(|Y|_H)\circ dW(s) \tag{12}$$

in probability for each $m \geq 1$.

Proof of Theorem 2-contd

To prove (12), fix $m \geq 1$ and let

$$g_n(s) := T_{t-s} BU(s, Y_n)\phi_m(|Y|_H),$$

$$g(s) := T_{t-s} BU(s, Y)\phi_m(|Y|_H)$$

for all $s \in [0, t]$. Then

$$\lim_{n\to\infty} E\left[\int_0^T \|g_n(s) - g(s)\|^2_{L_2(K,H)}\, ds\right] = 0 \qquad (13)$$

$$\lim_{n\to\infty} E\left[\int_0^T \int_0^T \|\mathcal{D}_u g_n(s) - \mathcal{D}_u g(s)\|^2_{L_2(K,H)}\, du\, ds\right] = 0. \qquad (14)$$

Proof of Theorem 2-contd

Compute:

$$(\mathcal{D}_+ g)_u := \lim_{s \to u+} \mathcal{D}_u g(s)$$

$$(\mathcal{D}_- g)_u := \lim_{s \to u-} \mathcal{D}_u g(s)$$

$$(\nabla g)_u := (\mathcal{D}_+ g)_u + (\mathcal{D}_- g)_u$$

Proof of Theorem 2-contd

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$$(\nabla g)_u := (\mathcal{D}_+ g)_u + (\mathcal{D}_- g)_u$$

and use path continuity to get

$$\lim_{n \to \infty} (\nabla g_n)_u = (\nabla g)_u, \quad a.s.$$

Proof of Theorem 2-contd

Step 8:

Proof of substitution theorem will be complete if:

$$\int_0^t g_n(s) \circ dW(s) = \int_0^t g_n(s)dW(s) + \frac{1}{2}\int_0^t (\nabla g_n)_s \, ds, \tag{15}$$

for $n \geq 1$; and

Proof of Theorem 2-contd

Step 8:

Proof of substitution theorem will be complete if:

$$\int_0^t g_n(s) \circ dW(s) = \int_0^t g_n(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g_n)_s \, ds, \tag{15}$$

for $n \geq 1$; and

$$\int_0^t g(s) \circ dW(s) = \int_0^t g(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g)_s \, ds \tag{16}$$

a.s.. Skorohod integrals on RHS.

Proof of Theorem 2-contd

Prove (15) and (16) from first principles, using approximations by Riemann sums: Lengthy computation.

Proof of Theorem 2-contd

Prove (15) and (16) from first principles, using approximations by Riemann sums: Lengthy computation.

Step 9:

Take $n \to \infty$ in RHS of (15) and get:

$$\lim_{n\to\infty} \int_0^t T_{t-s} BU(s, Y_n) \phi_m(|Y|_H) \circ dW(s) = \int_0^t T_{t-s} BU(s, Y) \phi_m(|Y|_H) \circ dW(s) \tag{12}$$

in probability for each $m \geq 1$. $\square$

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