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An Application of Differential Equations in the Study of Elastic Columns

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AN APPLICATION OF DIFFERENTIAL EQUATIONS

IN THE STUDY OF ELASTIC COLUMNS

by

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CHAPTER 1

INTRODUCTION

The area of differential equations is a very broad field of study. The versatility of differential equations allows the area to be applied to a variety of topics from physics to population growth to the stock market. They are a useful tool for modeling and studying naturally occurring phenomena such as determining when beams may break as well as predicting future outcomes such as the spread of disease or the changes in populations of different species over time. Anytime an unknown phenomena is changing with respect to time or space, a differential equation is involved.

In more general terms, a differential equation is simply an equation involving an unknown function and its derivatives. To be more technical, a differential equation is a "mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders"to a particular phenomena [2]. Differential equations generally fall into two categories: ordinary differential equations (ODE) or partial differential equations (PDE), the distinction being that ODEs involve unknown functions of one independent variable while PDEs involve unknown functions of more than one independent variable. In this paper we will focus on ordinary differential equations.

Some defining characteristics of a differential equation are its order and if it is linear. The order of the equation refers to the highest order derivative present. A differential equation is said to be linear if it is linear in its dependent variable and its derivatives, i.e. in the case of an ODE, if it can be written in the form

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y' + a_0(x)y = Q(x) \text{ [1].}$$

In addition, we say that linear differential equations are homogeneous when $Q(x) = 0$. A very important property of homogeneous linear ordinary differential equations says that there are n linearly independent solutions for an n^{th} order equation and that all solutions can be written as a linear combination of these solutions.

Initial conditions are when y and its derivatives are evaluated at a single point. Typically an n^{th} order ODE will have $y(x_0), y'(x_0), y''(x_0), \ldots, y^{(n-1)}(x_0)$ given. A differential equation together with these initial conditions is called an initial value problem (IVP).

If y and/or its derivatives are evaluated at two different points we say that we have a boundary condition. A boundary value problem (BVP) is a differential equation with boundary conditions. An example of a BVP is the equation

$$y'' + p(x)y' + q(x)y = g(x)$$

with the boundary condition

$$y(\alpha) = a, \quad y(\beta) = b.$$

When $g(x) = 0$ and $a = b = 0$, the BVP is said to be homogeneous.

The following are examples of common differential equations. The first two examples are IVPs and the last two are examples of IBVPs.

Example 1.0.1. The Mass-Spring Equation. The motion of a mass on the end of a spring can be given by

$$my'' + cy' + ky = f(t),$$

$$y(0) = a, y'(0) = b,$$

where y is the position of the mass from equilibrium, m is the mass, c is a damping constant, k is the spring constant, and $f(t)$ is an outside forcing function. Usually $y(0)$ and $y'(0)$ are given and represent the initial position and velocity, respectively.

Example 1.0.2. The population of mosquitoes in a certain area increases at a rate proportional to the current population, and, in the absence of other factors, the population doubles each week. If there are 200,000 mosquitoes in the area initially, and predators eat 20,000 mosquitoes each day. What is the population of mosquitoes in the area at any time [1]?

$$\begin{aligned} p'(t) &= r(t) - q(t), \\ p(0) &= 200,000, \end{aligned}$$

where $r(t)$ is the rate of growth for the population and $q(t)$ is the death rate for the population. The solution for this problem is

$$p(t) = 201,977.31 - 1977.31e^{t\ln(2)},$$

where $t \geq 0$ is time measured in weeks.

This type of problem requires us to model a differential equation to fit the stipulations of the growth rate of the population as well as the death rate. For more applied situations, such as developing a disease model, there will typically be a system of differential equations to solve as opposed to a single ODE.

Example 1.0.3. The Heat Equation. Consider a one-dimensional bar of length L. The transfer of heat in this bar is given by

$$\alpha^2 u_{xx} = u_t, \quad \text{for } 0 < x < L,\ t \geq 0$$

with boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \text{for } t > 0$$

and initial condition

$$u(x,0) = f(x), \quad \text{for } 0 \leq x \leq L$$

where u represents the heat in the bar for every $x \in [0, L]$ and $t \geq 0$ and f is the initial temperature distribution [1]. It should be noted that the boundary conditions can be interpreted as holding the temperature at the ends of the bar at zero degrees.

Example 1.0.4. The Wave Equation. The motion of a string of length L can be described by

$$\alpha^2 u_{xx} = u_{tt}, \quad \text{for } 0 < x < L,\ t \geq 0$$

with boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \text{for } t \geq 0$$

and initial conditions

$$u(x,0) = f(x), \quad \text{for } 0 \leq x \leq L$$

$$u_t(x,0) = g(x), \quad \text{for } 0 \leq x \leq L,$$

where u represents the position of the string from equilibrium at any $x \in [0, L]$ and $t \geq 0$, f is the initial position and g is the initial velocity [1]. Here the boundary conditions represent the string being held at the equilibrium position.

This paper examines the application of boundary value problems in determining the buckling load of an elastic column. The boundary conditions are determined by the length of the column and how it is supported (i.e. clamped end, hinged end, etc.). The following sections will discuss some of the theory behind these boundary value problems, some typical problems and their solutions, and how to interpret these results in the context of our problem.

CHAPTER 2

DEFINITIONS AND THEOREMS

In this chapter we introduce some basic theory of IVPs and BVPs as applicable to our area of study. For simplicity, we are considering first and second order differential equations.

2.1 INITIAL VALUE PROBLEMS

Initial value problems, and linear problems in particular, can be separated from boundary value problems. There is a rich literature involving linear IVPs. The following theorem concerns the existence of solutions to IVPs.

Theorem 2.1.1. [1] *Consider the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t),$$

$$y(t_0) = a, y'(t_0) = b,$$

where p, q, and g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

In general, for nonlinear ODEs the existence of a solution will be given in some interval which contains the initial value. The above theorem not only talks about a solution existing but also tells us the interval in which it does exist. The next theorem highlights another difference between linear and nonlinear problems.

Theorem 2.1.2. (Principle of Superposition) [1] *If y_1 and y_2 are two solutions of the differential equation*

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

This theorem does not hold for nonlinear problems and highlights one of the main differences between linear and nonlinear problems. It can be illustrated by many examples in ODE books. The following definition is extremely important when dealing with solutions to linear functions.

Definition. [1] *Suppose y_1 and y_2 are solutions of a differential equation. We define the Wronskian, W, as*

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2.$$

This definition along with Theorem 2.1.2 gives us the following.

Theorem 2.1.3. [1] *Suppose that y_1 and y_2 are two solutions of the differential equation*

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the Wronskian

$$W = y_1y_2' - y_1'y_2$$

is not zero at the point t_0 where the initial conditions,

$$y(t_0) = a, y'(t_0) = b$$

are assigned. Then there is a choice of the constants c_1, c_2 *for which*

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and its initial conditions.

It is this choice of constants that will aid in simpifying the process of solving the problems of this paper. Finally, we have

Theorem 2.1.4. [1] *If* y_1 *and* y_2 *are two solutions of the differential equation*

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and if there is a point t_0 *where the Wronskian of* y_1 *and* y_2 *is nonzero, then the family of solutions*

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients c_1 *and* c_2 *includes every solution of the differential equation.*

Notice that the last theorem deals only with a linear ODE and does not involve an IVP. In particular, it is extremely important in that it tells us first, that every second order ODE has two solutions, second, that these solutions are linearly independent, third, that these are the only solutions to the ODE, and finally, that all solutions can be written as a linear combination of these two solutions.

2.2 BOUNDARY VALUE PROBLEMS

Suppose we have the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

with boundary conditions

$$y(\alpha) = a, \quad y(\beta) = b.$$

In order to solve this BVP, we need to find a function $y = \varphi$ such that φ satisfies the differential equation on the interval $\alpha < x < \beta$ and takes on the values of a and b at the endpoints of the interval [1]. To find φ, we first examine the general solution to the ODE and then use the boundary conditions to determine if there are constants to solve the problem.

Although the idea of finding solutions to linear IVPs and BVPs is fairly straight forward, the results can be vastly different. As we saw in the previous section, linear IVPs have the existence of a unique solution in an interval which is well-defined. Boundary value problems, on the other hand, may have a unique solution, no solution, or infinitely many solutions depending on the conditions of the problem. Consider the problem

$$y'' + y = 0,$$

subject to the boundary conditions

$$y(0) = a, \quad y\left(\frac{\pi}{2}\right) = b.$$

Here, one solution exists. If we change the BC to

$$y(0) = a, \quad y(2\pi) = b$$

then we have no solution if $a \neq b$. However, if $a = b$ we have an infinite number of solutions.

In this sense, we may relate BVPs to systems of linear algebraic equations. Consider the linear system

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{x}$ is an $n \times 1$ vector to be determined, and $\mathbf{b}$ is a given $n \times 1$ vector. The solution to the system is dependent on the matrix $\mathbf{A}$. If $\mathbf{A}$ is nonsingular, the system will have a unique solution. On the other hand, if $\mathbf{A}$ is singular, the system may have no solution or an infinite number of solutions. In the case of the homogeneous linear system

$$\mathbf{Ax} = \mathbf{0},$$

the trivial solution $\mathbf{x}= \mathbf{0}$ always exists. Moreover, if $\mathbf{A}$ is nonsingular, the trivial solution is the only solution. However, if $\mathbf{A}$ is singular, there are infinitely many non-trivial solutions. This homogeneous linear system is similar to the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

with boundary conditions

$$y(\alpha) = 0 \text{ and } y(\beta) = 0,$$

where α and β are the endpoints of our interval. Thus we need to solve

$$\begin{pmatrix} y_1(\alpha) & y_2(\alpha) \\ y_1(\beta) & y_2(\beta) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where y_1 and y_2 are solutions to the ODE. This corresponds exactly to the algebraic problem mentioned above. It is important to note that, from the previous section, we know that there are exactly two linearly independent solutions to the ODE.

We may take the relation between BVPs and linear algebraic systems further by considering the linear system

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

This sytem has the solution $\mathbf{x} = \mathbf{0}$ for all values of λ, but for certain values of λ, the solution has non-trivial solutions. We call these values of λ eigenvalues and their respective solutions, $\mathbf{x}$, the corresponding eigenfunctions. The problem as a whole is called an eigenvalue problem. An example of an eigenvalue problem in ODE's would be

$$y'' + \lambda y = 0$$

with boundary conditions

$$y(0) = 0 \text{ and } y(\beta) = 0 \text{ [1].}$$

The relationship between boundary value problems and systems of linear equations provides a very useful tool in determining the type of solution for a BVP as well as solving for the constant values of the general homogeneous solution to a linear ODE. The idea is the same as it is for an algebraic system. We want to determine all values of λ for which the nontrivial solution y exists. We will use this relationship between BVPs and systems of linear equations to solve the eigenvalue problems in the next chapter.

CHAPTER 3

APPLICATION OF BOUNDARY VALUE PROBLEMS

3.1 A SIMPLY SUPPORTED COLUMN

Consider the differential equation

$$y^{(4)} + \lambda y'' = 0,$$

where the solution $y = \varphi$ is the eigenfunction corresponding to λ. The problem is to find, for each of the following boundary conditions, the smallest eigenvalue, which determines the buckling load, as well as the corresponding eigenfunction, which determines the shape of the buckled column:

Problem 1: $y(0) = y''(0) = 0$, $y(L) = y''(L) = 0$, a column with both ends hinged.

Problem 2: $y(0) = y''(0) = 0$, $y(L) = y'(L) = 0$, a column where one end is hinged and the other end is clamped.

Problem 3: $y(0) = y'(0) = 0$, $y(L) = y'(L) = 0$, a column where both ends are clamped.

Solutions:

We first note that the ODE is linear and from the previous chapter we can extend our results from second order to fourth order. Thus, this fourth order ODE has four linearly independent solutions independent of the boundary conditions. We also know that all solutions to the ODE can be written as a linear combination of these four solutions. We will first solve this 4^{th} order linear ODE. Notice that the character of the solution changes for $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. When we apply the BC's, we will consider each case separately and require that the solution be non-trivial in order to properly determine the buckling load and the shape of the buckled column.

Case 1:

Suppose $\lambda < 0$. Let $\lambda = -\mu^2$ for $\mu \neq 0$. Then the characteristic equation for the ODE is

$$r^4 - \mu^2 r^2 = 0.$$

Thus, the roots are $r = 0, 0, \mu, -\mu$ and the homogeneous solution is

$$y = c_1 + c_2 x + c_3 e^{\mu x} + c_4 e^{-\mu x}.$$

Case 2:

Suppose $\lambda = 0$. Then the characteristic equation is

$$r^4 = 0$$

and the homogeneous solution is

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3.$$

Case 3:

Suppose $\lambda > 0$. Let $\lambda = \mu^2$ for $\mu \neq 0$. Then the characteristic equation is

$$r^4 + \mu^2 r^2 = 0$$

and the homogeneous solution is

$$y = c_1 + c_2 x + c_3 \cos(\mu x) + c_4 \sin(\mu x).$$

Thus, we have three possible forms for the solution to the ODE. Now let us apply the different boundary conditions to find nontrivial solutions to the different situations. Recall that, in order for λ to be an eigenvalue, we need to find nonzero eigenfunctions. For each of the three problems, this amounts to finding at least one nonzero c_i. This is equivalent to determining for which values of λ a matrix will be singular.

3.1.1 Problem 1

Solve $y^{(4)} + \lambda y'' = 0$ subject to $y(0) = y''(0) = 0,\ \ y\,(L) = y''\,(L) = 0$.

Case 1: Let $\lambda < 0$ and $\lambda = -\mu^2$ for $\mu \neq 0$. Then the homogeneous solution is

$$y = c_1 + c_2 x + c_3 e^{\mu x} + c_4 e^{-\mu x}.$$

Applying the boundary conditions, we see that

$$\begin{aligned} y(0) &= c_1 + c_3 + c_4 = 0 \\ y''(0) &= c_3 \mu^2 + c_4 \mu^2 = 0 \\ y(L) &= c_1 + c_2 L + c_3 e^{\mu L} + c_4 e^{-\mu L} = 0 \\ y''(L) &= c_3 \mu^2 e^{\mu L} + c_4 \mu^2 e^{-\mu L} = 0. \end{aligned}$$

In order to determine if this case will yield nontrivial solutions, we can form a matrix from the LHS of the boundary equations and check its determinant. Thus, we have the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & \mu^2 & \mu^2 \\ 1 & L & e^{\mu L} & e^{-\mu L} \\ 0 & 0 & \mu^2 e^{\mu L} & \mu^2 e^{-\mu L} \end{pmatrix}$$

and

$$\det(\mathbf{A}) = 2\mu^4 L \sinh(\mu L).$$

Since μ and L are nonzero, this solution will also be nonzero. Therefore, $\mathbf{A}$ is nonsingular. Hence, by Section 2.2, the system and, thus, the ODE will have only the trivial solution, $y \equiv 0$.

Case 2: Let $\lambda = 0$. Then the homogeneous solution is

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3.$$

Applying the boundary conditions yields

$$\begin{aligned} y(0) &= c_1 = 0 \\ y''(0) &= 2c_3 = 0 \\ y(L) &= c_1 + c_2 L + c_3 L^2 + c_4 L^3 = 0 \\ y''(L) &= 2c_3 + 6c_4 L = 0. \end{aligned}$$

These boundary conditions can be related to the matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 0 & 2 & 6L \end{pmatrix}$$

with determinant

$$\det(\mathbf{B}) = -12L^2.$$

Thus, $\det \mathbf{B} \neq 0$ and so $\mathbf{B}$ is nonsingular. Therefore, the trivial solution is the only solution to this case.

Case 3: Let $\lambda > 0$ with $\lambda = \mu^2$ for $\mu \neq 0$. Then the homogeneous solution is

$$y = c_1 + c_2 x + c_3 \cos(\mu x) + c_4 \sin(\mu x).$$

Applying the boundary conditions yields

$$y(0) = c_1 + c_3 = 0 \tag{3.3a}$$

$$y''(0) = -c_3\mu^2 = 0 \tag{3.3b}$$

$$y(L) = c_1 + c_2 L + c_3 \cos(\mu L) + c_4 \sin(\mu L) = 0 \tag{3.3c}$$

$$y''(L) = -c_3\mu^2 \cos(\mu L) - c_4\mu^2 \sin(\mu L) = 0. \tag{3.3d}$$

The related matrix for these boundary conditions is

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -\mu^2 & 0 \\ 1 & L & \cos \mu L & \sin(\mu L) \\ 0 & 0 & -\mu^2 \cos(\mu L) & \mu^2 \sin \mu L \end{pmatrix}$$

with determinant

$$\det(\mathbf{C}) = \mu^4 L \sin \mu L.$$

Since we require a nontrivial solution, we want $\det(\mathbf{C}) = 0$. This will happen when $\mu L = n\pi$. Solving for μ gives $\mu_n = \frac{n\pi}{L}$. To determine φ, we must solve for the constants.

By solving equation (3.3b) for c_3 we have $c_3 = 0$ and so, by equation (3.3a), we also have $c_1 = 0$. Thus equations (3.3c) and (3.3d) are now

$$c_2 L + c_4 \sin(\mu L) = 0 \tag{3.4}$$

$$\text{and} \quad -c_4 \mu^2 \sin(\mu L) = 0. \tag{3.5}$$

From above we have that $\mu = \frac{n\pi}{L}$. Since, $c_4 \neq 0$, we have $c_2 = 0$. If $\mu_n = \frac{n\pi}{L}$ then $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $y_n = \sin\left(\frac{n\pi x}{L}\right)$. Recall that we want the smallest eigenvalue. Thus, $\lambda_1 = \frac{\pi^2}{L^2}$ is the buckling load and $y_1 = \varphi_1 = \sin\left(\frac{\pi x}{L}\right)$ is the shape of the buckled column.

3.1.2 Problem 2

Solve $y^{(4)} + \lambda y'' = 0$ subject to $y(0) = y''(0) = 0$, $y(L) = y'(L) = 0$. As shown previously there are three possible general solutions to the ODE to consider.

Case 1: Suppose $\lambda < 0$. Then let $\lambda = -\mu^2$ where $\mu \neq 0$. Then the general homogeneous solution is

$$y = c_1 + c_2 x + c_3 e^{\mu x} + c_4 e^{-\mu x}.$$

Applying our boundary conditions, we have

$$\begin{aligned}
y(0) &= c_1 + c_3 + c_4 = 0 \\
y''(0) &= c_3\mu^2 + c_4\mu^2 = 0 \\
y(L) &= c_1 + c_2 L + c_3 e^{\mu L} + c_4 e^{-\mu L} = 0 \\
y'(L) &= c_2 + c_3 \mu e^{\mu L} - c_4 \mu e^{-\mu L} = 0.
\end{aligned}$$

Using a matrix to determine the type of solution to this system, we have

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & \mu^2 & \mu^2 \\ 1 & L & e^{\mu L} & e^{-\mu L} \\ 0 & 1 & \mu e^{\mu L} & -\mu e^{-\mu L} \end{pmatrix}$$

which has determinant

$$\begin{aligned}
\det(\mathbf{D}) &= \mu^3 L e^{\mu L} + \mu^3 L e^{-\mu L} - \mu^2 e^{\mu L} + \mu^2 e^{-\mu L} \\
&= \mu^2 \left(2\mu L \cosh(\mu L) - 2\sinh(\mu L)\right).
\end{aligned}$$

This determinant will only be zero when $2\mu L \cosh(\mu L) = 2\sinh(\mu L)$ or when $\mu L = \tanh(\mu L)$. This will only happen when $\mu L = 0$. Since neither μ nor L is zero, this determinant is nonzero. Thus $\mathbf{D}$ is nonsingular and $y \equiv 0$ is the solution to this ODE.

Case 2: Suppose $\lambda = 0$. Then the homogeneous solution is

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3.$$

Applying our boundary conditions yields

$$\begin{aligned} y(0) &= c_1 = 0 \\ y''(0) &= 2c_3 = 0 \\ y(L) &= c_1 + c_2 L + c_3 L^2 + c_4 L^3 = 0 \\ y'(L) &= c_2 + 2c_3 L + 3c_4 L^2 = 0. \end{aligned}$$

The matrix for this system is

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{pmatrix}$$

and has determinant

$$\det(\mathbf{E}) = -4L^3.$$

Since L is nonzero, the matrix is nonsingular. Therefore, the trivial solution is once again the only solution to the ODE.

Case 3: Let $\lambda > 0$ with $\lambda = \mu^2$ and $\mu \neq 0$. Then our general solution is

$$y = c_1 + c_2 x + c_3 \cos(\mu x) + c_4 \sin(\mu x).$$

Applying the boundary condtions gives

$$y(0) = c_1 + c_3 = 0 \tag{3.8a}$$

$$y''(0) = -c_3 \mu^2 = 0 \tag{3.8b}$$

$$y(L) = c_1 + c_2 L + c_3 \cos(\mu L) + c_4 \sin(\mu L) = 0 \tag{3.8c}$$

$$y'(L) = c_2 - c_3 \mu \sin(\mu L) + c_4 \mu \cos(\mu L) = 0. \tag{3.8d}$$

The corresponding matrix for the boundary conditions is

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -\mu^2 & 0 \\ 1 & L & \cos(\mu L) & \sin(\mu L) \\ 0 & 1 & -\mu\sin(\mu L) & \mu\cos(\mu L) \end{pmatrix}$$

and has determinant

$$\det(\mathbf{F}) = (\mu L\cos(\mu L) - \sin(\mu L))\mu^2.$$

Since $\mu \neq 0$,$\det(\mathbf{F}) = 0$ when $\mu L\cos(\mu L) - \sin(\mu L) = 0$. Let $\gamma = \mu L$. Then solving for γ, we need to determine when

$$\gamma = \tan(\gamma).$$

This yields the solution set

$$\{0, 4.4934, 7.7253, 54.9597, \dots\}.$$

Since the buckling load must be a real number and we require the smallest buckling load, we choose $\gamma = 4.4934$. Thus, $\mu_1 = \frac{4.4934}{L}$ and $\lambda_1 = \frac{(4.4934)^2}{L^2}$. To find the buckling shape, we need to determine y.

By equation (3.8b) we have $c_3 = 0$. Then by equation (3.8a) $c_1 = 0$ as well. Thus, equation (3.8d) is now

$$c_2 + c_4\mu\cos(\mu L) = 0$$

$$\text{so} \qquad c_2 = -c_4\mu\cos(\mu L).$$

Thus, by substituting for $c_1, c_2,$ and c_3 our general solution is now

$$y = c_4 sin(\mu x) - c_4 \mu x \cos(\mu L).$$

Since we have chosen the smallest buckling load, λ_1, the solution to the ODE will be the shape of the buckled column. Therefore, the nontrivial solution to the ODE is

$$\varphi_1 = -\mu_1 x \cos(\mu_1 L) + \sin(\mu_1 x).$$

3.1.3 Problem 3

Solve $y^{(4)} + \lambda y'' = 0$ subject to $y(0) = y'(0) = 0$, $y(L) = y'(L) = 0$. Let us examine the three possible general solutions.

Case 1: Let $\lambda < 0$ and $\lambda = -\mu^2$ for $\mu \neq 0$. Then the homogeneous solution is

$$y = c_1 + c_2 x + c_3 e^{\mu x} + c_4 e^{-\mu x}.$$

Applying our boundary conditions, we have

$$\begin{aligned}
y(0) &= c_1 + c_3 + c_4 = 0 \\
y'(0) &= c_2 + c_3 \mu - c_4 \mu = 0 \\
y(L) &= c_1 + c_2 L + c_3 e^{\mu L} + c_4 e^{-\mu L} = 0 \\
y'(L) &= c_2 + c_3 \mu e^{\mu L} - c_4 \mu e^{-\mu L} = 0.
\end{aligned}$$

The corresponding matrix for the boundary conditions is

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & \mu & -\mu \\ 1 & L & e^{\mu L} & e^{-\mu L} \\ 0 & 1 & \mu e^{\mu L} & -\mu e^{-\mu L} \end{pmatrix}$$

with determinant

$$\begin{aligned} \det(\mathbf{G}) &= -\mu^2 L e^{\mu L} + \mu^2 L e^{-\mu L} + 2\mu e^{\mu L} + 2\mu e^{-\mu L} - 4\mu \\ &= -2\mu^2 L \sinh(\mu L) + 4\mu \cosh(\mu L) - 4\mu \\ &= -2\mu(\mu L \sinh(\mu L) - 2\cosh(\mu L) + 2). \end{aligned}$$

This determinant will equal zero when $\mu L \sinh(\mu L) - 2\cosh(\mu L) + 2 = 0$ or when $\mu L = \frac{2\cosh(\mu L) - 2}{\sinh(\mu L)}$. This will only happen if $\mu L = 0$. Therefore, by our stipulations that $\mu \neq 0$ and $L \neq 0$, we have that $\det(\mathbf{G}) \neq 0$. Thus, the matrix representation for the boundary conditions is nonsingular. Therefore, only the trivial solution, $y \equiv 0$, will satisfy this case.

Case 2: Suppose $\lambda = 0$. Then the general homogeneous solution is

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3.$$

Applying our boundary conditions yields

$$\begin{aligned} y(0) &= c_1 = 0 \\ y'(0) &= c_2 = 0 \\ y(L) &= c_1 + c_2 L + c_3 L^2 + c_4 L^3 = 0 \\ y'(L) &= c_2 + 2c_3 L + 3c_4 L^2 = 0 \end{aligned}$$

Then we may use the following matrix to determine the type of solution for the ODE

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{pmatrix}$$

which has determinant

$$\det(\mathbf{H}) = L^4.$$

Thus, $\mathbf{H}$ is nonsingular and so only the trivial solution will satisfy the general solution.

Case 3: Suppose $\lambda > 0$ and $\lambda = \mu^2$ for $\mu \neq 0$. Then we have the following homogeneous solution

$$y = c_1 + c_2 x + c_3 \cos(\mu x) + c_4 \sin(\mu x).$$

Applying the boundary condtions gives

$$y(0) = c_1 + c_3 = 0 \tag{3.11a}$$

$$y'(0) = c_2 + c_4\mu = 0 \tag{3.11b}$$

$$y(L) = c_1 + c_2 L + c_3 \cos(\mu L) + c_4 \sin(\mu L) = 0 \tag{3.11c}$$

$$y'(L) = c_2 - c_3\mu \sin(\mu L) + c_4\mu \cos(\mu L) = 0. \tag{3.11d}$$

Our corresponding matrix for the boundary conditions is

$$\mathbf{J} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \mu \\ 1 & L & \cos(\mu L) & \sin(\mu L) \\ 0 & 1 & -\mu\sin(\mu L) & \mu\cos(\mu L) \end{pmatrix}$$

and has determinant

$$\det(\mathbf{J}) = -\mu(\mu L\sin(\mu L) + 2(\cos(\mu L) - 1)).$$

In order for the system to have a nontrivial solution, we need $\det(\mathbf{J}) = 0$. This will happen when $\mu L\sin(\mu L) + 2(\cos(\mu L) - 1) = 0$. Let $\gamma = \mu L$. Then we have the following

$$-2(1 - \cos(\gamma)) = -\gamma\sin(\gamma). \tag{3.12}$$

To simplify equation (3.12) we can divide by a negative and use the half-angle formula on the LHS. Then

$$\begin{aligned} 2(2\sin^2\left(\frac{\gamma}{2}\right)) &= \gamma\sin(\gamma) \\ \text{so} \quad 4\sin^2\left(\frac{\gamma}{2}\right) &= \gamma\sin(\gamma). \end{aligned} \tag{3.13}$$

Solving equation (3.13) for γ yields the solution set

$$\{0, 2\pi, 4\pi, 8.98682, 15.4505, 37.69911, 40.74261, \dots\},$$

which can be verified graphically. Using the same reasoning as in Problem 2 Case 3, we must choose $\gamma = 2\pi$. Thus, $\mu_1 = \frac{2\pi}{L}$ so $\lambda_1 = \left(\frac{2\pi}{L}\right)^2$. To solve for φ_1 we may plug-in our value of μ_1 to the boundary conditions and solve accordingly. Thus, we have the system

$$c_1 + c_3 = 0 \tag{3.14a}$$

$$c_2 + c_4\frac{2\pi}{L} = 0 \tag{3.14b}$$

$$c_1 + c_2L + c_3\cos(\frac{2\pi}{L}L) + c_4\sin(\frac{2\pi}{L}L) = 0 \tag{3.14c}$$

$$c_2 - c_3\frac{2\pi}{L}\sin(\frac{2\pi}{L}L) + c_4\frac{2\pi}{L}\cos(\frac{2\pi}{L}L) = 0. \tag{3.14d}$$

We can solve for c_1 and c_2 in equations (3.14a) and (3.14b), respectively, and equations (3.14c) and (3.14d) will simplify. Hence, we may rewrite the system as

$$c_1 = -c_3 \tag{3.15a}$$

$$c_2 = -c_4\frac{2\pi}{L} \tag{3.15b}$$

$$c_1 + c_2L + c_3 = 0 \tag{3.15c}$$

$$c_2 + c_4\frac{2\pi}{L} = 0. \tag{3.15d}$$

From equations (3.15a) and (3.15c), we have $c_2L = 0$ and so $c_2 = 0$. Then by equation (3.15b) $c_4 = 0$. This leaves us with $c_1 = -c_3$. Substituting for c_1 in the general solution, we have

$$y = -c_3 + c_3\cos(\mu x).$$

Thus, we have $\varphi_1 = 1 - \cos\left(\frac{2\pi x}{L}\right)$ by choice of $c_3 = -1$.

CLAMPED END, FREE END

3.1.4 Problem 4

Since the boundary conditions for this particular BVP vary according to how a column is supported, some interesting cases will arise. When one end of a column is fixed and the other is free, the eigenvalue parameter also appears in the boundary conditions. In order to determine the buckling load and buckling shape for this type of column, we must solve $y^{(4)} + \lambda y'' = 0$ subject to the boundary conditions

$$y\left(0\right) = y'\left(0\right) = 0,\ y''\left(L\right) = 0, \text{ and } y'''\left(L\right) + \lambda y'\left(L\right) = 0.$$

We begin by looking at the case where $\lambda < 0$, $\lambda = -\mu^2$, and $\mu \neq 0$. Then our general solution is

$$y = c_1 + c_2 x + c_3 e^{\mu x} + c_4 e^{-\mu x}.$$

Applying our boundary conditions, we have

$$\begin{aligned}
y(0) &= c_1 + c_3 + c_4 = 0 \\
y'(0) &= c_2 + c_3\mu - c_4\mu = 0 \\
y''(L) &= c_3\mu^2 e^{\mu L} + c_4\mu^2 e^{-\mu L} = 0 \\
y'''(L) + \lambda y'(L) &= \\
\mu^3 c_3 e^{\mu L} - \mu^3 c_4 e^{-\mu L} &- \mu^2[c_2 + c_3\mu e^{\mu L} - c_4\mu e^{-\mu L}] = 0.
\end{aligned}$$

The matrix relating to these boundary conditions is

$$\mathbf{K} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & \mu & -\mu \\ 0 & 0 & \mu^2 e^{\mu L} & \mu^2 e^{-\mu L} \\ 0 & -\mu^2 & 0 & 0 \end{pmatrix}$$

with determinant

$$\det(\mathbf{K}) = -2\mu^5 \cosh(\mu L).$$

Since $\mu \neq 0$ and $L > 0$, $\det(\mathbf{K}) \neq 0$. Therefore, only the trivial solution will satisfy the system of equations for the boundary conditions and, thus, the BVP.

Case 2: Let $\lambda = 0$. Then the homogeneous solution is

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3.$$

Applying our boundary conditions yields

$$\begin{aligned} y(0) &= c_1 = 0 \\ y'(0) &= c_2 = 0 \\ y''(L) &= 2c_3 + 6c_4 L = 0 \\ y'''(L) + \lambda y'(L) &= 6c_4 = 0. \end{aligned}$$

Then we have the matrix

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 6L \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

which has determinant

$$\det(\mathbf{L}) = 12.$$

Since $\det(\mathbf{L}) \neq 0$, we have a nonsingular matrix and once again only the trvial solution exists.

Case 3: Let $\lambda > 0$ with $\lambda = \mu^2$ for $\mu \neq 0$. Then the general homogeneous solution is

$$y = c_1 + c_2 x + c_3 \cos(\mu x) + c_4 \sin(\mu x)$$

Applying the boundary conditions yields

$$\begin{aligned}
y(0) &= c_1 + c_3 = 0 \\
y'(0) &= c_2 + c_4 \mu = 0 \\
y''(L) &= -c_3 \mu^2 \cos(\mu L) - c_4 \mu^2 \sin(\mu L) = 0 \\
y'''(L) + \lambda y'(L) &= \\
\mu^3 c_3 \sin(\mu L) &- \mu^3 c_4 \cos(\mu L) + \mu^2 \left[c_2 - c_3 \mu \sin(\mu L) + c_4 \mu \cos(\mu L)\right] = 0.
\end{aligned}$$

After simplifying the RHS of the last boundary condition the corresponding matrix for the boundary conditions is

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \mu \\ 0 & 0 & -\mu^2 \cos(\mu L) & -\mu^2 \sin(\mu L) \\ 0 & \mu^2 & 0 & 0 \end{pmatrix}$$

and has determinant

$$\det(\mathbf{M}) = \mu^5 \cos(\mu L).$$

For a nontrivial solution we need $\det(\mathbf{M}) = 0$. This will occur only when $\cos(\mu L) = 0$. Thus, $\mu L = \frac{(2n-1)\pi}{2}$. Hence, $\mu_1 = \frac{\pi}{2L}$ and so $\lambda_1 = \frac{\pi^2}{4L^2}$. To determine φ_1, we can substitute for μ_1 in the boundary conditions and solve the system accordingly. Therefore, our boundary conditions form the system

$$c_1 + c_3 = 0 \tag{3.19a}$$

$$c_2 + c_4 \frac{\pi}{2L} = 0 \tag{3.19b}$$

$$-c_3 \left(\frac{\pi}{2L}\right)^2 \cos\left(\frac{\pi}{2L} L\right) - c_4 \left(\frac{\pi}{2L}\right)^2 \sin\left(\frac{\pi}{2L} L\right) = 0 \tag{3.19c}$$

$$c_2 \left(\frac{\pi}{2L}\right)^2 = 0. \tag{3.19d}$$

From equation (3.19d), $c_2 = 0$. Simplifying equation (3.19c), we have $-c_4 \left(\frac{\pi}{2L}\right)^2 = 0$ and so $c_4 = 0$. This leaves us with $c_1 + c_3 = 0$ or $c_1 = -c_3$. Substituting into our general solution we have

$$y = -c_3 + c_3 \cos\left(\frac{\pi}{2L} x\right)$$

Thus, $\varphi_1 = 1 - \cos\left(\frac{\pi}{2L} x\right)$ by choosing $c_3 = -1$.

CHAPTER 4

ANALYSIS AND APPLICATION

It is important to note that the context of the problems from Chapter 3 deal with ideal columns. An ideal column is "one that is perfectly straight before loading, is made of homogeneous material, and upon which the load is applied through the centroid of the cross section" [4]. Naturally, ideal columns are impossible to construct. However, applying the analysis of ideal columns is crucial for determining the stability and construction of columns for structures.

The results of Chapter 3 show that both the buckling load, λ_1, and buckling shape, φ_1, are dependent only on the length of the column. To be more specific, the buckling load "is independent of the strength of the material; rather it depends only on the column's dimensions . . . and the material's stiffness," which is also referred to as the material's modulus of elasticity [4]. The buckling load is also the maximum weight a column may support. If any amount of weight over the buckling load is applied, then buckling, as determined by φ_1, will occur. As for the shape of the buckled column, we must consider the buckling load, or critical load, in more applied terms.

The Euler Load is given by

$$P_{cr} = \tfrac{n^2\pi^2 EI}{L^2},$$

where E is the modulus of elasticity for the material of the column, I is the least moment of intertia of the column, and L is the unsupported length of the column.

This equation pertains to columns that are free to rotate or "pinned" at the ends. We can consider this equation to be the applied variation of our theoretical critical loads. This is due to the fact that our results only take into consideration how a particular column is supported while the Euler Load also takes into consideration other properties of the column such as the stiffness of its material and its least moment of interia. A key fact of the Euler Equation and our results is that the value of n represents the number of waves in the buckled column. Consider the Euler Load. If we let $n = 2$, then this new value will be four times the buckling load. This buckled, or deflected, shape is said to be unstable. Thus, more than one wave in a buckled column cannot exist [4]. This is why we choose $n = 1$, the least buckling load, for both the Euler Load and ideal columns as seen in Chapter 3.

Structurally speaking, "efficient columns are designed so that most of the column's cross-sectional area is located as far away as possible from the principal centroidal axes for the section." Thus, hollow sections such as tubes are a more preferable type of column as opposed to solid columns [4].

The following is a table of other boundary conditions, buckling loads, and the resulting function modeling the shape of the buckled column. Note that because of Theorem 2.1.3 we are free to choose constants for φ_1 so long as the constants satisfy the boundary conditions. Thus, there may be minor discrepancies in the buckled shape listed and our resulting buckled shape [3].

Table 4.1. Boundary Conditions and Corresponding Buckling Shapes [3]

Boundary Conditions	Theoretical Effective Length	Engineering Effective Length	Buckling Shape
Free-Free	L	$1.2L$	$\sin\left(\frac{\pi x}{L}\right)$
Hinged-Free	L	$1.2L$	$\sin\left(\frac{\pi x}{L}\right)$
Hinged-Hinged* (see Problem 1)	L	L	$\sin\left(\frac{\pi x}{L}\right)$
Guided-Free	$2L$	$2.1L$	$\sin\left(\frac{\pi x}{2L}\right)$
Guided-Hinged	$2L$	$2L$	$\cos\left(\frac{\pi x}{2L}\right)$
Guided-Guided	L	$1.2L$	$\cos\left(\frac{\pi x}{L}\right)$
Clamped-Free (see Problem 4)	$2L$	$2.1L$	$1-\cos\left(\frac{\pi x}{2L}\right)$
Clamped-Hinged** (see Problem 2)	$0.7L$	$0.8L$	$\sin(kx)-kL\cos(kx)+kL$
Clamped Guided	L	$1.2L$	$1-\cos\left(\frac{\pi x}{L}\right)$
Clamped-Clamped (see Problem 3)	$0.5L$	$0.65L$	$1-\cos\left(\frac{2\pi x}{L}\right)$

*Simply-supported column

**$k = 1.4318\frac{\pi}{L}$

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