12-13-2007

Numerics of Stochastic Systems with Memory (Mittag-Leffler Institute Seminar)

Salah-Eldin A. Mohammed
Southern Illinois University Carbondale, salah@sfde.math.siu.edu

Invited Talk; Mittag-Leffler Institute Seminar; Royal Swedish Academy of Sciences; Stockholm, Sweden; December 13, 2007

NUMERICS

OF STOCHASTIC SYSTEMS WITH MEMORY

Salah Mohammed [a]

http://sfde.math.siu.edu/

Institut Mittag-Leffler

Royal Swedish Academy of Sciences

Sweden: December 13, 2007

Acknowledgment

- Joint work with E. Buckwar, R. Kuske and T. Shardlow.

Acknowledgment

- Joint work with E. Buckwar, R. Kuske and T. Shardlow.
- Research supported by NSF Grants DMS-9703852, DMS-9975462, DMS-0203368, DMS-0705970 and Canadian PIMS.

Outline

- Objective: To approximate functions of solutions of stochastic delay differential equations (sdde's) arising in applications.

Outline

- Objective: To approximate functions of solutions of stochastic delay differential equations (sdde's) arising in applications.
- Formulation of the Euler scheme for sdde's. Convergence rate of order 1.

Outline

- Objective: To approximate functions of solutions of stochastic delay differential equations (sdde's) arising in applications.
- Formulation of the Euler scheme for sdde's. Convergence rate of order 1.
- For a semimartingale x, must study increments of $\psi(x(t-\tau), x(t))$ across partition points.

Outline

- Objective: To approximate functions of solutions of stochastic delay differential equations (sdde's) arising in applications.
- Formulation of the Euler scheme for sdde's. Convergence rate of order 1.
- For a semimartingale x, must study increments of $\psi(x(t-\tau), x(t))$ across partition points.
- *No semimartingale properties* for processes of the form $(x(t-\tau), x(t))$. But need an Itô formula!

Outline

- Objective: To approximate functions of solutions of stochastic delay differential equations (sdde's) arising in applications.
- Formulation of the Euler scheme for sdde's. Convergence rate of order 1.
- For a semimartingale x, must study increments of $\psi(x(t-\tau), x(t))$ across partition points.
- *No semimartingale properties* for processes of the form $(x(t-\tau), x(t))$. But need an Itô formula!
- Get a "tame" Itô formula for $\psi(x(t-\tau), x(t))$.

Outline-contd

- Proof of the Euler scheme: Uses tame Itô formula, tame character and Fréchet differentiability of the Euler approximation in the initial path, estimates on the Malliavin derivatives of the solution, Malliavin and Fréchet derivatives of the Euler approximation.

Outline-contd

- Proof of the Euler scheme: Uses tame Itô formula, tame character and Fréchet differentiability of the Euler approximation in the initial path, estimates on the Malliavin derivatives of the solution, Malliavin and Fréchet derivatives of the Euler approximation.
- Set-up is *non-anticipating*, but proof of convergence requires *anticipating* stochastic calculus techniques.

Outline-contd

- Proof of the Euler scheme: Uses tame Itô formula, tame character and Fréchet differentiability of the Euler approximation in the initial path, estimates on the Malliavin derivatives of the solution, Malliavin and Fréchet derivatives of the Euler approximation.
- Set-up is *non-anticipating*, but proof of convergence requires *anticipating* stochastic calculus techniques.
- Implementation of the scheme does not require knowledge of the Malliavin calculus.

Outline-contd

- Proof of the Euler scheme: Uses tame Itô formula, tame character and Fréchet differentiability of the Euler approximation in the initial path, estimates on the Malliavin derivatives of the solution, Malliavin and Fréchet derivatives of the Euler approximation.
- Set-up is *non-anticipating*, but proof of convergence requires *anticipating* stochastic calculus techniques.
- Implementation of the scheme does not require knowledge of the Malliavin calculus.
- Unlike (sode's), sdde's do not correspond to diffusions on Euclidean space. Thus techniques from deterministic pde's do not apply.

Motivation

Sdde's model noisy physical processes with memory:

- Laser dynamics with delayed feedback (Buldú, etal (2001), and Masoller (2002, 2003)).

Motivation

Sdde's model noisy physical processes with memory:

- Laser dynamics with delayed feedback (Buldú, etal (2001), and Masoller (2002, 2003)).
- Stochastic oscillator ensembles with delayed coupling (Goldobin, Rosenblum, Pikovsky (2003)).

Motivation

Sdde's model noisy physical processes with memory:

- Laser dynamics with delayed feedback (Buldú, etal (2001), and Masoller (2002, 2003)).
- Stochastic oscillator ensembles with delayed coupling (Goldobin, Rosenblum, Pikovsky (2003)).
- Models of delayed visual feedback systems (Beuter and Vasilakos (1993)), and (Longtin, Milton, Bos and Mackey (1990)).

Motivation

Sdde's model noisy physical processes with memory:

- Laser dynamics with delayed feedback (Buldú, etal (2001)), and Masoller (2002, 2003)).
- Stochastic oscillator ensembles with delayed coupling (Goldobin, Rosenblum, Pikovsky (2003)).
- Models of delayed visual feedback systems (Beuter and Vasilakos (1993)), and (Longtin, Milton, Bos and Mackey (1990)).
- Option-pricing models with memory (Arriojas, Hu, Mohammed and Pap (2007)).

Motivation-Contd

Model equations are non-linear and do not allow for explicit solutions.

Motivation-Contd

Model equations are non-linear and do not allow for explicit solutions.

Hence need numerical approximation methods of solution:

Sfde approximations

Sfde approximations

Strong (or almost sure) Euler scheme (order $1/2$) and strong Milstein scheme (order 1) for sdde's were developed by Ahmed, Elsanousi and Mohammed [A], Mohammed [Mo.1], Hu, Mohammed and Yan [H.M.Y] and Baker and Buckwar [B.B], Küchler and Platen [Kü.P].

Sfde approximations

Strong (or almost sure) Euler scheme (order $1/2$) and strong Milstein scheme (order 1) for sdde's were developed by Ahmed, Elsanousi and Mohammed [A], Mohammed [Mo.1], Hu, Mohammed and Yan [H.M.Y] and Baker and Buckwar [B.B], Küchler and Platen [Kü.P].

Weak approximations for sode's (without memory) are well-developed (Bally and Talay [B.T], Kloeden and Platen [K.P], Kohatsu-Higa [K]).

Sfde approximations

Strong (or almost sure) Euler scheme (order $1/2$) and strong Milstein scheme (order 1) for sdde's were developed by Ahmed, Elsanousi and Mohammed [A], Mohammed [Mo.1], Hu, Mohammed and Yan [H.M.Y] and Baker and Buckwar [B.B], Küchler and Platen [Kü.P].

Weak approximations for sode's (without memory) are well-developed (Bally and Talay [B.T], Kloeden and Platen [K.P], Kohatsu-Higa [K]).

Weak Euler scheme of order 1 for semilinear sfde's: Buckwar and Shardlow (2005); *linear* smooth memory drift term; memoryless diffusion term; 1-dim'l noise:

Approximations – cont’d

$$x(t) = \begin{cases} v + \int_0^t \int_{-\tau}^0 x(u+s)\,\mu(s)\,ds\,du \\ + \int_0^t f(x(u))\,du + \int_0^t g(x(u))\,dW(u), \;\; t \geq 0 \\ \eta(t), \quad -\tau < t < 0. \end{cases}$$

Initial condition $(v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-\tau, 0], \mathbf{R}^d)$.

Approximations – cont'd

$$x(t) = \begin{cases} v + \int_0^t \int_{-\tau}^0 x(u+s)\,\mu(s)\,ds\,du \\ + \int_0^t f(x(u))\,du + \int_0^t g(x(u))\,dW(u), \;\; t \geq 0 \\ \eta(t), \quad -\tau < t < 0. \end{cases}$$

Initial condition $(v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-\tau, 0], \mathbf{R}^d)$. Embed sfde as a semilinear see (without memory) in Hilbert space M_2. Weak approximation in M_2.

Approximations – cont'd

$$x(t) = \begin{cases} v + \int_0^t \int_{-\tau}^0 x(u+s)\,\mu(s)\,ds\,du \\ + \int_0^t f(x(u))\,du + \int_0^t g(x(u))\,dW(u), \ \ t \geq 0 \\ \eta(t), \quad -\tau < t < 0. \end{cases}$$

Initial condition $(v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-\tau, 0], \mathbf{R}^d)$.
Embed sfde as a semilinear see (without memory) in Hilbert space M_2. Weak approximation in M_2.

Duality methods for weak Euler scheme-independently by Clément, Kohatsu-Higa and Lamberton [CK-HL].

Approximations– cont'd

In this talk, we prove weak convergence of order 1 of the Euler scheme for *fully non-linear* sdde's.

Approximations– cont'd

In this talk, we prove weak convergence of order 1 of the Euler scheme for *fully non-linear* sdde's.

Allow for:

Approximations– cont'd

In this talk, we prove weak convergence of order 1 of the Euler scheme for *fully non-linear* sdde's.

Allow for:

multiple discrete delays

Approximations– cont'd

In this talk, we prove weak convergence of order 1 of the Euler scheme for *fully non-linear* sdde's.

Allow for:

multiple discrete delays

smooth memory

Approximations– cont'd

In this talk, we prove weak convergence of order 1 of the Euler scheme for *fully non-linear* sdde's.

Allow for:

multiple discrete delays

smooth memory

multidimensional Brownian noise

The weak Euler scheme

The weak Euler scheme

The SDDE:

The weak Euler scheme

The SDDE:

$$x(t) = \begin{cases} \eta(0) + \displaystyle\int_\sigma^t f\big(x(u-\tau_1), x(u)\big)\, du \\ \qquad + \displaystyle\int_\sigma^t g\big(x(u-\tau_2), x(u)\big)\, dW(u), \ t \geq \sigma, \\ \eta(t-\sigma), \qquad \sigma - \tau \leq t < \sigma, \quad \tau := \tau_1 \vee \tau_2. \end{cases} \tag{I}$$

The weak Euler scheme

The SDDE:

$$x(t) = \begin{cases} \eta(0) + \int_\sigma^t f\big(x(u-\tau_1), x(u)\big)\, du \\ \qquad + \int_\sigma^t g\big(x(u-\tau_2), x(u)\big)\, dW(u), \; t \geq \sigma, \\ \eta(t-\sigma), \qquad \sigma - \tau \leq t < \sigma, \quad \tau := \tau_1 \vee \tau_2. \end{cases} \tag{I}$$

Coefficients $f, g : \mathbf{R}^2 \to \mathbf{R}$ are of class C_b^3.

The weak Euler scheme

The SDDE:

$$x(t) = \begin{cases} \eta(0) + \int_\sigma^t f\big(x(u-\tau_1), x(u)\big)\, du \\ \qquad + \int_\sigma^t g\big(x(u-\tau_2), x(u)\big)\, dW(u),\ t \geq \sigma, \\ \eta(t-\sigma), \qquad \sigma - \tau \leq t < \sigma, \quad \tau := \tau_1 \vee \tau_2. \end{cases} \quad \textbf{(I)}$$

Coefficients $f, g : \mathbf{R}^2 \to \mathbf{R}$ are of class C_b^3.

Initial (random) path $\eta \in C^1([-\tau, 0], \mathbf{R})$.

The weak Euler scheme

The SDDE:

$$x(t) = \begin{cases} \eta(0) + \int_\sigma^t f\big(x(u-\tau_1), x(u)\big)\, du \\ \qquad + \int_\sigma^t g\big(x(u-\tau_2), x(u)\big)\, dW(u),\ t \geq \sigma, \\ \eta(t-\sigma), \qquad \sigma - \tau \leq t < \sigma, \quad \tau := \tau_1 \vee \tau_2. \end{cases} \tag{I}$$

Coefficients $f, g : \mathbf{R}^2 \longrightarrow \mathbf{R}$ are of class C_b^3.

Initial (random) path $\eta \in C^1([-\tau, 0], \mathbf{R})$.

Initial instant $\sigma \geq 0$. (<–)

The weak Euler scheme – cont'd

Noise: One-dimensional Brownian motion W on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

The weak Euler scheme – cont'd

Noise: One-dimensional Brownian motion W on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

Unique solution $x := x(\cdot; \sigma, \eta) : [\sigma - \tau, a] \times \Omega \to \mathbf{R}$ of (I), fixed $a > \sigma$.

The weak Euler scheme – cont'd

Noise: One-dimensional Brownian motion W on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

Unique solution $x := x(\cdot; \sigma, \eta) : [\sigma - \tau, a] \times \Omega \to \mathbf{R}$ of (I), fixed $a > \sigma$.

Partition $\pi := \{-\tau = t_{-m} < t_{-m+1} < \cdots < t_{-1} < 0 = t_0 < t_1 < t_2 \cdots < t_{n-1} < t_n = a\}$ of $[-\tau, a]$, with mesh:

$$|\pi| := \max\{(t_i - t_{i-1}) : -m + 1 \leq i \leq n\}.$$

The weak Euler scheme – cont'd

Noise: One-dimensional Brownian motion W on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

Unique solution $x := x(\cdot; \sigma, \eta) : [\sigma - \tau, a] \times \Omega \to$ **R** of (I), fixed $a > \sigma$.

Partition $\pi := \{-\tau = t_{-m} < t_{-m+1} < \cdots < t_{-1} < 0 = t_0 < t_1 < t_2 \cdots < t_{n-1} < t_n = a\}$ of $[-\tau, a]$, with mesh:

$$|\pi| := \max\{(t_i - t_{i-1}) : -m + 1 \leq i \leq n\}.$$

For any $u \in [\sigma, a]$, define $\lfloor u \rfloor := t_{i-1} \vee \sigma$ whenever $u \in [t_{i-1}, t_i] \cap [\sigma, a]$.

The weak Euler scheme – cont'd

The weak Euler scheme – cont'd

Euler approximation:

$y := y(\cdot;\sigma,\eta) : [\sigma-\tau, a] \times \Omega \to \mathbf{R}$ of $x(\cdot;\sigma,\eta)$ is the solution of the sdde:

$$y(t) = \begin{cases} \eta(0) + \int_\sigma^t f\big(y(\lfloor u \rfloor - \tau_1), y(\lfloor u \rfloor)\big)\, du \\ + \int_\sigma^t g\big(y(\lfloor u \rfloor - \tau_2), y(\lfloor u \rfloor)\big)\, dW(u), t \geq \sigma, \\ \eta(t-\sigma), \qquad \sigma - \tau \leq t < \sigma, \;\; \tau := \tau_1 \vee \tau_2. \end{cases} \quad \text{(II)}$$

$$\eta^\pi(s) := \Big(\frac{t_i - s}{t_i - t_{i-1}}\Big)\eta(t_{i-1}) + \Big(\frac{s - t_{i-1}}{t_i - t_{i-1}}\Big)\eta(t_i),$$

$s \in [t_{i-1}, t_i), -m+1 \leq i \leq 0.$

The weak Euler scheme – cont'd

Main result:

Weak convergence of order 1
for the Euler scheme of the sdde (I).

Theorem 1-Weak Convergence

Let π *be a partition of* $[-\tau, a]$ *with mesh* $|\pi|$*, and* $\phi : \boldsymbol{R} \to \boldsymbol{R}$ *be of class* C_b^3*. In the sdde (I), assume that the coefficients* f, g *are* C_b^3*. Let* $x(\cdot; \sigma, \eta)$ *be the unique solution of (I) starting at* $\sigma \in (0, a]$ *with initial path* $\eta \in C^1\big([-\tau, 0], \boldsymbol{R}\big)$*.*

Theorem 1-Weak Convergence

Let π *be a partition of* $[-\tau, a]$ *with mesh* $|\pi|$*, and* $\phi : \boldsymbol{R} \to \boldsymbol{R}$ *be of class* C_b^3*. In the sdde (I), assume that the coefficients* f, g *are* C_b^3*. Let* $x(\cdot;\sigma,\eta)$ *be the unique solution of (I) starting at* $\sigma \in (0, a]$ *with initial path* $\eta \in C^1\big([-\tau, 0], \boldsymbol{R}\big)$*. Denote by* $y(\cdot;\sigma,\eta)$ *the Euler approximation to* $x(\cdot;\sigma,\eta)$ *defined by (II).*

Theorem 1-Weak Convergence

Let π *be a partition of* $[-\tau, a]$ *with mesh* $|\pi|$, *and* $\phi : \boldsymbol{R} \to \boldsymbol{R}$ *be of class* C_b^3. *In the sdde (I), assume that the coefficients* f, g *are* C_b^3. *Let* $x(\cdot;\sigma,\eta)$ *be the unique solution of (I) starting at* $\sigma \in (0, a]$ *with initial path* $\eta \in C^1\big([-\tau, 0], \boldsymbol{R}\big)$. *Denote by* $y(\cdot;\sigma,\eta)$ *the Euler approximation to* $x(\cdot;\sigma,\eta)$ *defined by (II).*

Then there is a positive constant C *and a positive integer* q *such that*

Theorem 1 – cont'd

$$|E\phi(x(t;\sigma,\eta)) - E\phi(y(t;\sigma,\eta^\pi))| \leq C(1 + \|\eta\|_{C^1}^q)|\pi|$$

for all $\eta \in C^1\big([-\tau,0],\boldsymbol{R}\big)$ *and all* $t \in [\sigma-\tau,a]$.

Theorem 1 – cont'd

$$|E\phi(x(t;\sigma,\eta)) - E\phi(y(t;\sigma,\eta^\pi))| \le C(1 + \|\eta\|_{C^1}^q)|\pi|$$

for all $\eta \in C^1\big([-\tau,0],\boldsymbol{R}\big)$ *and all* $t \in [\sigma-\tau,a]$*. The constant* C *may depend on* a,q *and the test function* ϕ*, but is independent of* $\pi,\eta,\ \sigma\in[0,a]$ *and* $t\in[\sigma-\tau,a]$*.*

Markov Property

For the solution

$$x : [-\tau, a] \times \Omega \to \mathbf{R}$$

of sdde (I), denote its *segment* $x_t \in C([-\tau, 0], \mathbf{R})$, $t \in [0, a]$, by

$$x_t(s) := x(t+s), \quad s \in [-\tau, 0], \quad t \in [0, a].$$

Markov Property

For the solution

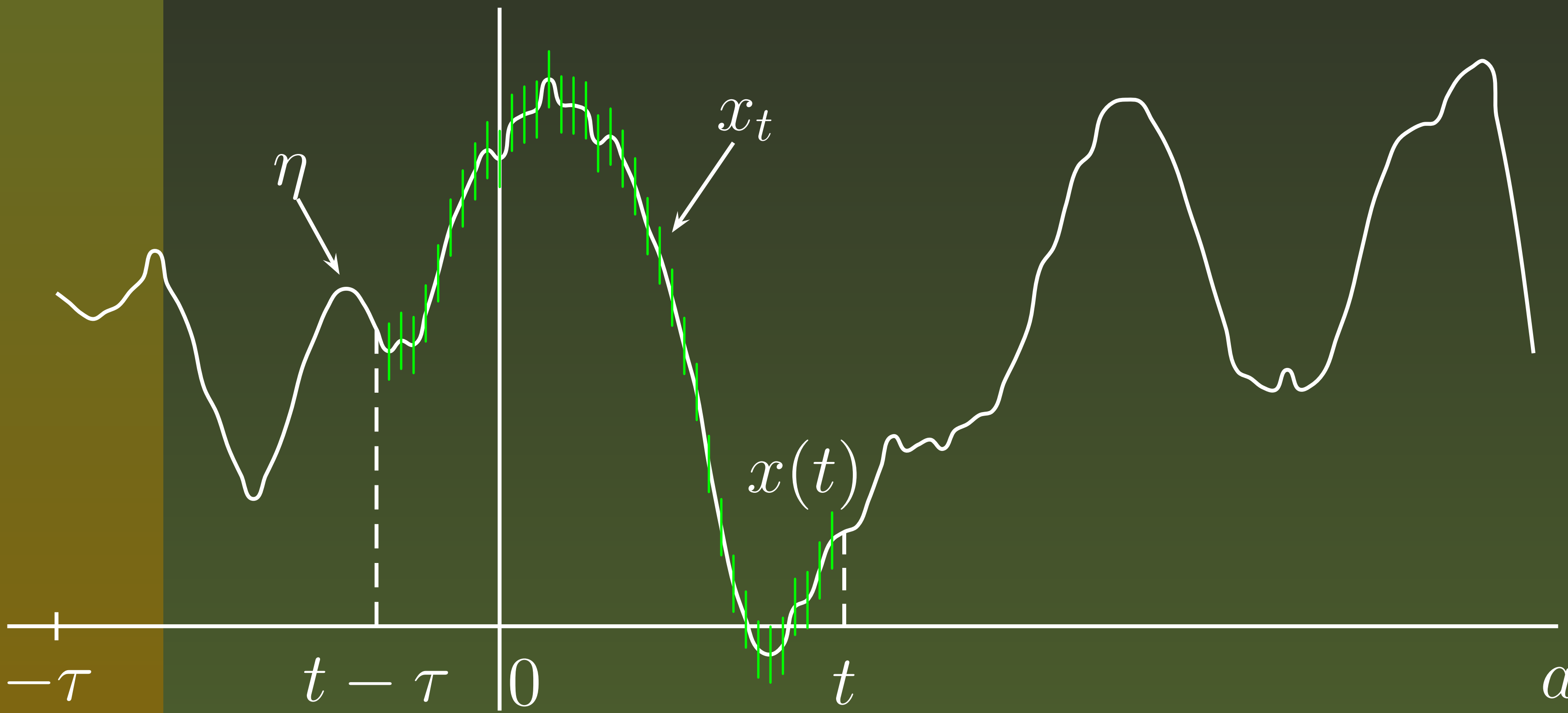
$$x : [-\tau, a] \times \Omega \to \mathbf{R}$$

of sdde (I), denote its *segment* $x_t \in C([-\tau, 0], \mathbf{R})$, $t \in [0, a]$, by

$$x_t(s) := x(t+s), \quad s \in [-\tau, 0], \quad t \in [0, a].$$

$x_t \in C([-\tau, 0], \boldsymbol{R}),\ t \geq 0,$ *is Markov.*

The segment process



Outline of Proof

Step 1:

Let $t \in [\sigma, a]$ and $\pi := \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[0, a]$. W.l.o.g, assume that $\sigma = t_0 = 0, t = t_n$.

Outline of Proof

Step 1:

Let $t \in [\sigma, a]$ and $\pi := \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[0, a]$. W.l.o.g, assume that $\sigma = t_0 = 0, t = t_n$.

Using telescoping, the Markov property for x_t and y_t, and Fréchet differentiability of $y(t_n; t_i, \eta)$ in η:

Outline of Proof

Step 1:

Let $t \in [\sigma, a]$ and $\pi := \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[0, a]$. W.l.o.g, assume that $\sigma = t_0 = 0, t = t_n$.

Using telescoping, the Markov property for x_t and y_t, and Fréchet differentiability of $y(t_n; t_i, \eta)$ in η:

$$
\begin{aligned}
&E\phi\big(x(t_n;0,\eta)\big) - E\phi\big(y(t_n;0,\eta)\big) \\
&= E\phi\big(y(t_n;t_n,x_{t_n}(\,\cdot\,;0,\eta))\big) - E\phi\big(y(t_n;0,\eta)\big) \\
&= \sum_{i=1}^{n}\{E\phi\big(y(t_n;t_i,x_{t_i}(\cdot;0,\eta))\big) \\
&\qquad - E\phi\big(y(t_n;t_{i-1},x_{t_{i-1}}(\,\cdot\,;0,\eta))\big)\}
\end{aligned}
$$

Outline of Proof – cont'd

$$= \sum_{i=1}^{n} \Big\{ E\phi\Big(y(t_n; t_i, x_{t_i}(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,; 0, \eta)))\Big)$$

$$- E\phi\Big(y(t_n; t_i, y_{t_i}(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,; 0, \eta)))\Big) \Big\}$$

$$= \sum_{i=1}^{n} E \int_0^1 D(\phi \circ y)\Big(t_n; t_i, \lambda x_{t_i}(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,; 0, \eta))$$

$$+ (1-\lambda) y_{t_i}\Big(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,; 0, \eta)\Big)\Big)\, d\lambda$$

$$\cdot \Big[x_{t_i}\Big(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,; 0, \eta)\Big)$$

$$- y_{t_i}\Big(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,; 0, \eta)\Big)\Big].$$

(M.V. Theorem)

Outline of Proof– cont'd

Step 2:

Main task is to show that each of the terms in the above sum is $O((t_i - t_{i-1})^2)$: Use the tame Itô formula. Get multiple Skorohod integrals of the form

Outline of Proof– cont'd

Step 2:

Main task is to show that each of the terms in the above sum is $O((t_i - t_{i-1})^2)$: Use the tame Itô formula. Get multiple Skorohod integrals of the form

$$J_1^i := \int_{t_{i-1}-t_i}^{0} Y(ds) \int_{t_{i-1}}^{t_i+s} \int_{t_{i-1}}^{u} \Sigma_1(v)\, dv\, dW(u),$$

$$J_2^i := \int_{t_{i-1}-t_i}^{0} Y(ds) \int_{t_{i-1}}^{t_i+s} \int_{t_{i-1}}^{u} \Sigma_2(v)\, dW(v-\tau_2)\, dW(u),$$

$$J_3^i := \int_{t_{i-1}-t_i}^{0} Y(ds) \int_{t_{i-1}}^{t_i+s} \int_{t_{i-1}}^{u} \Sigma_3(v)\, dW(v-\tau_1)\, du.$$

Outline of Proof– cont'd

The discrete random measure Y and the processes $\Sigma_j, j = 1, 2, 3,$ are Malliavin smooth and possibly anticipate the lagged Brownian motions $W(\cdot - \tau_i)$, $i = 1, 2.$

Outline of Proof– cont’d

The discrete random measure Y and the processes $\Sigma_j, j = 1, 2, 3$, are Malliavin smooth and possibly anticipate the lagged Brownian motions $W(\cdot - \tau_i)$, $i = 1, 2$.

Step 3:

To estimate the expectations EJ_j^i in Step 2, use the definition of the Skorohod integral as adjoint of the weak differentiation operator, coupled with estimates on higher-order moments of Malliavin derivatives of Σ_j’s, $j = 1, 2, 3$.

Outline of Proof– cont'd

The discrete random measure Y and the processes $\Sigma_j, j = 1, 2, 3,$ are Malliavin smooth and possibly anticipate the lagged Brownian motions $W(\cdot - \tau_i)$, $i = 1, 2$.

Step 3:

To estimate the expectations EJ_j^i in Step 2, use the definition of the Skorohod integral as adjoint of the weak differentiation operator, coupled with estimates on higher-order moments of Malliavin derivatives of Σ_j's, $j = 1, 2, 3.$ These estimates follow from higher moments of the solution x, its Euler approximations y and Malliavin derivatives of linearizations of y.

Outline of Proof– cont'd

This gives

$$|EJ_j^i| = O((t_i - t_{i-1})^2), \quad j = 1, 2, 3.$$

Summing over $i = 1, \cdots, n$, we get the required order of convergence 1 for the weak Euler scheme.

Step 4:

Replace η in $y(t; \sigma, \eta)$ by its P-L approx η^π via the estimates

$$\begin{aligned} |E\phi(x(t; \sigma, \eta)) - E\phi(x(t; \sigma, \eta^\pi))| \\ \leq C\|\eta - \eta^\pi\|_C \leq C\|\eta'\|_\infty |\pi|. \end{aligned}$$

□

THE PROOF

Example

Example

One-dimensional sdde:

$$\begin{aligned} dX(t) &= g(X(t-1), X(t))\, dW(t), \quad t > 0, \\ X(t) &= W(t), \quad -1 \leq t \leq 0. \end{aligned}$$

$g : \mathbf{R}^2 \to \mathbf{R}$ smooth function. For Euler scheme of order 1, seek a stochastic differential of $g(X(t-1), X(t))$ on RHS of sdde.

Example – Cont’d

For $t \in (0, 1]$, formally expect something like:

$$
\begin{aligned}
& dg(X(t-1), X(t)) \\
& = \frac{\partial g}{\partial x_2}(W(t-1), X(t))\, g(W(t-1), X(t))\, dW(t) \\
& + \frac{\partial g}{\partial x_1}(W(t-1), X(t))\, dW(t-1) \quad (\textit{anticipating!}) \\
& + \text{second-order terms} \cdots
\end{aligned}
$$

Example – Cont'd

- *LHS is adapted but anticipating integral(s) on RHS.*

Example – Cont'd

- *LHS is adapted but anticipating integral(s) on RHS.*
- $(\mathcal{F}_t)_{0\leq t\leq 1}$-adapted process $(X(t-1), X(t)) \in \mathbf{R}^2$ is not a semimartingale with respect to any natural filtration.

Example – Cont'd

- *LHS is adapted but anticipating integral(s) on RHS.*
- $(\mathcal{F}_t)_{0\leq t\leq 1}$-adapted process $(X(t-1), X(t)) \in \mathbf{R}^2$ is not a semimartingale with respect to any natural filtration.
- Still need an Itô formula for tame functions:

$$g(X(t-1), X(t)) = g(X_t(-1), X_t(0)).$$

where $X_t(s) := X(t+s), \; s \in [-1, 0], t \geq 0$.

The tame Itô formula

The tame Itô formula

Objective is to obtain an Itô formula for tame functionals on the Banach $C([-\tau, 0], \mathbf{R}^d)$, acting on segments of sample-continuous random processes $[-\tau, \infty] \times \Omega \to \mathbf{R}^d$: *tame Itô formula* ([H.M.Y])

The tame Itô formula

Objective is to obtain an Itô formula for tame functionals on the Banach $C([-\tau, 0], \mathbf{R}^d)$, acting on segments of sample-continuous random processes $[-\tau, \infty] \times \Omega \to \mathbf{R}^d$: *tame Itô formula* ([H.M.Y])

First, some notation:

The tame Itô formula

Objective is to obtain an Itô formula for tame functionals on the Banach $C([-\tau,0],\mathbf{R}^d)$, acting on segments of sample-continuous random processes $[-\tau,\infty]\times\Omega\to\mathbf{R}^d$: *tame Itô formula* ([H.M.Y])

First, some notation:

$W(t)$, $t\geq 0$, one-dimensional standard Brownian motion on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$.

The tame Itô formula

Objective is to obtain an Itô formula for tame functionals on the Banach $C([-\tau, 0], \mathbf{R}^d)$, acting on segments of sample-continuous random processes $[-\tau, \infty] \times \Omega \to \mathbf{R}^d$: *tame Itô formula* ([H.M.Y])

First, some notation:

$W(t)$, $t \geq 0$, one-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

For simplicity, set $W(t) := 0$ if $t \leq 0$.

The tame Itô formula

Objective is to obtain an Itô formula for tame functionals on the Banach $C([-\tau, 0], \mathbf{R}^d)$, acting on segments of sample-continuous random processes $[-\tau, \infty] \times \Omega \to \mathbf{R}^d$: *tame Itô formula* ([H.M.Y])

First, some notation:

$W(t)$, $t \geq 0$, one-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

For simplicity, set $W(t) := 0$ if $t \leq 0$.

$\mathcal{D} :=$ the weak (Malliavin) differentiation operator associated with $\{W(t) : t \geq 0\}$.

The tame Itô formula – Cont'd

Let $p > 1$, k a positive integer; $\mathbb{L}^{k,p} := L^p([0,a], \mathbb{D}^{k,p})$, where $\mathbb{D}^{k,p}$ is the closure of all random variables Y with k-th weak derivatives in $L^p(\Omega, H^{\otimes^k})$ under the norm

$$\|Y\|_{k,p} := (E|Y|^p)^{1/p} + \left(\sum_{j=1}^{k} E\|\mathcal{D}^j Y\|^p_{H^{\otimes^j}} \right)^{1/p}.$$

The tame Itô formula – Cont'd

Let $p > 1$, k a positive integer; $\mathbb{L}^{k,p} := L^p([0,a], \mathbb{D}^{k,p})$, where $\mathbb{D}^{k,p}$ is the closure of all random variables Y with k-th weak derivatives in $L^p(\Omega, H^{\otimes^k})$ under the norm

$$\|Y\|_{k,p} := (E|Y|^p)^{1/p} + \left(\sum_{j=1}^{k} E\|\mathcal{D}^j Y\|^p_{H^{\otimes j}} \right)^{1/p}.$$

In above formula, $H := L^2([0,a], \mathbf{R})$. The spaces $\mathbb{L}^{k,p}_{loc}, p > 4$, are defined to be the set of all processes X such that there is an increasing sequence of $\mathcal{F}$-measurable sets $A_n, n \geq 1$, and processes $X_n \in \mathbb{L}^{k,p}$,

The tame Itô formula – Cont'd

$n \geq 1$, such that $X = X_n$ a.s. on A_n for each $n \geq 1$, and $\bigcup_{n=1}^{\infty} A_n = \Omega$. Weak differentiation operator $\mathcal{D}$ is local and hence extends unambiguously to the spaces $\mathbb{L}^{k,p}_{loc}$, $p > 4$.

The tame Itô formula – Cont'd

$n \geq 1$, such that $X = X_n$ a.s. on A_n for each $n \geq 1$, and

$$\bigcup_{n=1}^{\infty} A_n = \Omega.$$ Weak differentiation operator $\mathcal{D}$ is local and

hence extends unambiguously to the spaces $\mathbb{L}^{k,p}_{loc}$, $p > 4$.

See ([Nu.1], pp. 61, 151, 161) for further properties of weak derivatives and the spaces $\mathbb{L}^{k,p}$.

The tame Itô formula – Cont'd

$n \geq 1$, such that $X = X_n$ a.s. on A_n for each $n \geq 1$, and $\bigcup_{n=1}^{\infty} A_n = \Omega$. Weak differentiation operator $\mathcal{D}$ is local and hence extends unambiguously to the spaces $\mathbb{L}^{k,p}_{loc}$, $p > 4$.

See ([Nu.1], pp. 61, 151, 161) for further properties of weak derivatives and the spaces $\mathbb{L}^{k,p}$.

$C^{1,2}([0,a] \times \mathbf{R}^k, \mathbf{R}) :=$ space of all functions $\phi : [0,a] \times \mathbf{R}^k \to \mathbf{R}$ which are C^1 in the time variable $[0,a]$ and C^2 in the space variables $\mathbf{R}^k$.

The tame Itô formula – Cont'd

Let $X : [-\tau, \infty) \times \Omega \to \mathbf{R}$ be a pathwise-continuous (not necessarily adapted) $\mathbf{R}$-valued process given by

$$X(t) = \begin{cases} \eta(0) + \int_0^t u(s)\,dW(s) + \int_0^t v(s)\,ds, \\ \qquad\qquad t > 0, \\ \eta(t), \quad -\tau \le t \le 0, \end{cases} \qquad (1)$$

where $\eta \in C := C([-\tau, 0], \mathbf{R})$ and is of bounded variation, $u \in \mathbb{L}^{2,p}_{loc}, p > 4$, and $v \in \mathbb{L}^{1,4}_{loc}$.

The tame Itô formula – Cont'd

Let $X : [-\tau, \infty) \times \Omega \to \mathbf{R}$ be a pathwise-continuous (not necessarily adapted) $\mathbf{R}$-valued process given by

$$X(t) = \begin{cases} \eta(0) + \int_0^t u(s)\, dW(s) + \int_0^t v(s)\, ds, \\ \qquad\qquad t > 0, \\ \eta(t), \quad -\tau \le t \le 0, \end{cases} \tag{1}$$

where $\eta \in C := C([-\tau, 0], \mathbf{R})$ and is of bounded variation, $u \in \mathbb{L}^{2,p}_{loc}, p > 4$, and $v \in \mathbb{L}^{1,4}_{loc}$. The stochastic integral is a Skorohod integral.

The tame Itô formula – Cont'd

Set $u(t) := 0$ for $t < 0$, and

$$v(t) := \eta'(t), \qquad -\tau \leq t \leq 0,$$

where η' is the (usual!) derivative of η.

The tame Itô formula – Cont'd

Set $u(t) := 0$ for $t < 0$, and

$$v(t) := \eta'(t), \qquad -\tau \leq t \leq 0,$$

where η' is the (usual!) derivative of η.

Let $\Pi : C([-\tau, 0], \mathbf{R}) \to \mathbf{R}^k$ be the *tame projection* associated with $s_1, \cdots, s_k \in [-\tau, 0]$; that is

$$\Pi(\eta) := (\eta(s_1), \cdots, \eta(s_k))$$

for all $\eta \in C := C([-\tau, 0], \mathbf{R})$.

The tame Itô formula – Cont'd

For any sample-continuous process

$$X : [-\tau, a] \times \Omega \to \mathbf{R}$$

recall its *segment* $X_t \in C([-\tau, 0], \mathbf{R})$, $t \in [0, a]$:

$$X_t(s) := X(t+s), \quad s \in [-\tau, 0], \quad t \in [0, a].$$

Get the *tame Itô formula*:

Theorem 2 (Tame Itô Formula)

Assume that X is a continuous process defined by (1), where $\eta : [-\tau, 0] \to \boldsymbol{R}$ is of bounded variation, $u \in \mathbb{L}^{2,4}_{loc}$, and $v \in \mathbb{L}^{1,4}_{loc}$. Suppose $\phi \in C^{1,2}([0,a] \times \boldsymbol{R}^k, \boldsymbol{R})$. Then for all $t \in [0,a]$ we have a.s.

Theorem 2 (Tame Itô Formula)

Assume that X is a continuous process defined by (1), where $\eta : [-\tau, 0] \to \boldsymbol{R}$ is of bounded variation, $u \in \mathbb{L}^{2,4}_{loc}$, and $v \in \mathbb{L}^{1,4}_{loc}$. Suppose $\phi \in C^{1,2}([0,a] \times \boldsymbol{R}^k, \boldsymbol{R})$. Then for all $t \in [0,a]$ we have a.s.

$$\phi(t, \Pi(X_t)) - \phi(0, \Pi(X_0)) = \int_0^t \frac{\partial \phi}{\partial s}(s, \Pi(X_s))\, ds$$

$$+ \sum_{i=1}^k \int_0^t \frac{\partial \phi}{\partial x_i}(s, \Pi(X_s))\, dX(s + s_i)$$

$$+ \frac{1}{2} \sum_{i,j=1}^k \int_0^t \frac{\partial^2 \phi}{\partial x_i \partial x_j}(s, \Pi(X_s)) u(s + s_i) \bigtriangledown_{s_i, s_j} X(s)\, ds$$

Theorem 2 – Cont'd

where

$$\nabla_{s_i,s_j} X(s) := \mathcal{D}^{+}_{s+s_i} X(s+s_j) + \mathcal{D}^{-}_{s+s_i} X(s+s_j)$$

and

Theorem 2 – Cont'd

where

$$\nabla_{s_i,s_j} X(s) := \mathcal{D}^+_{s+s_i} X(s+s_j) + \mathcal{D}^-_{s+s_i} X(s+s_j)$$

and

$$\mathcal{D}^+_{s+s_i} X(s+s_j) := \lim_{\epsilon \to 0+} \mathcal{D}_{s+s_i} X(s+s_j+\epsilon),$$

$$\mathcal{D}^-_{s+s_i} X(s+s_j) := \lim_{\epsilon \to 0+} \mathcal{D}_{s+s_i} X(s+s_j-\epsilon).$$

Theorem 2 – Cont'd

where

$$\nabla_{s_i,s_j} X(s) := \mathcal{D}^+_{s+s_i} X(s+s_j) + \mathcal{D}^-_{s+s_i} X(s+s_j)$$

and

$$\mathcal{D}^+_{s+s_i} X(s+s_j) := \lim_{\epsilon \to 0+} \mathcal{D}_{s+s_i} X(s+s_j+\epsilon),$$

$$\mathcal{D}^-_{s+s_i} X(s+s_j) := \lim_{\epsilon \to 0+} \mathcal{D}_{s+s_i} X(s+s_j-\epsilon).$$

Proof. Hu, Mohammed and Yan [H.M.Y], Theorem 2.3.

□

Corollary 3

Let $\psi : \boldsymbol{R}^2 \to \boldsymbol{R}$ *be of class* C^2*, and suppose* x *solves the sdde*

$$x(t) = \begin{cases} \eta(0) + \int_0^t f\big(x(u-\tau_1), x(u)\big)\, du \\ \qquad + \int_0^t g\big(x(u-\tau_2), x(u)\big)\, dW(u),\ t > 0 \\ \eta(t), \quad -\tau < t < 0,\ \tau := \tau_1 \vee \tau_2, \end{cases} \tag{I}$$

where the coefficients $f, g : \boldsymbol{R}^2 \to \boldsymbol{R}$ *are of class* C_b^2*, and* $\eta \in C\big([-\tau, 0], \boldsymbol{R}\big)$ *is of bounded variation.*

Corollary 3 – cont'd

Suppose $\delta > 0$*. Then a.s.*

Corollary 3 – cont'd

Suppose $\delta > 0$. *Then a.s.*

$$
\begin{aligned}
d\psi\big(x(t-\delta), x(t)\big)
&= \frac{\partial\psi}{\partial x_1}(x(t-\delta), x(t))\, 1_{[0,\delta)}(t)\, d\eta(t-\delta) \\
&+ \frac{\partial\psi}{\partial x_1}(x(t-\delta), x(t)) 1_{[\delta,\infty)}(t) \\
&\qquad \times \Big[f\big(x(t-\delta-\tau_1), x(t-\delta)\big)\, dt \\
&\qquad\qquad + g\big(x(t-\delta-\tau_2), x(t-\delta)\big)\, dW(t-\delta)\Big] \\
&+ \frac{\partial\psi}{\partial x_2}\big(x(t-\delta), x(t)\big) f\big(x(t-\tau_1), x(t)\big)\, dt
\end{aligned}
$$

Corollary 3 – cont'd

$$
\begin{aligned}
&+ \frac{\partial \psi}{\partial x_2}\big(x(t-\delta), x(t)\big) g\big(x(t-\tau_2), x(t)\big)\, dW(t) \\
&+ \frac{\partial^2 \psi}{\partial x_1 \partial x_2}\big(x(t-\delta), x(t)\big) g\big(x(t-\delta-\tau_2), x(t-\delta)\big) \times \\
&\qquad\qquad\qquad \times 1_{[\delta,\infty)}(t) \mathcal{D}_{t-\delta} x(t)\, dt \\
&+ \frac{1}{2}\frac{\partial^2 \psi}{\partial x_1^2}\big(x(t-\delta), x(t)\big) g\big(x(t-\delta-\tau_2), x(t-\delta)\big)^2 1_{[\delta,\infty)}(t)\, dt \\
&+ \frac{1}{2}\frac{\partial^2 \psi}{\partial x_2^2}\big(x(t-\delta), x(t)\big) g\big(x(t-\tau_2), x(t)\big)^2\, dt, \qquad t > 0.
\end{aligned}
\tag{2}
$$

Proof of Corollary 3

Suppose $t > \delta$. Apply Theorem 2 with $\phi := \psi(x_1, x_2)$, $X = x$, $s_1 = -\delta$, $s_2 = 0$, where x solves the sdde (I).

For $0 < t < \delta$, result follows from classical Itô formula because η is BV. $\square$

Remark

In the second term on the right hand side of (2), the $(\mathcal{F}_t)_{t\geq 0}$*-adapted factor* $\dfrac{\partial \psi}{\partial x_1}(x(t-\delta), x(t))$ *anticipates the differential* $dW(t-\delta)$.

Lemma 1

Fix a partition point t_i in π. Then for a.a. $\omega \in \Omega$, the function

$$[t_i, a] \times C\big([-\tau, 0], \boldsymbol{R}\big) \ni (t, \eta) \mapsto y(t, \omega; t_i, \eta) \in \boldsymbol{R}$$

is tame: That is, there exists a deterministic function $F : \boldsymbol{R}^+ \times \boldsymbol{R}^k \times \boldsymbol{R}^h \times \boldsymbol{R}^l \to \boldsymbol{R}$ which is continuous in the time variable $\boldsymbol{R}^+$, of class C_b^2 in all space variables $\boldsymbol{R}^k, \boldsymbol{R}^h, \boldsymbol{R}^l$, and fixed numbers $t_1, t_2, \ldots, t_k \leq t$, $s_1, s_2, \ldots, s_h \leq t$, $\mu_1, \mu_2, \ldots, \mu_l \in [-\tau, 0]$ such that a.s.

Lemma 1 – cont'd

$$y(t;t_i,\eta) = F\Big(t, W(t), W(t_1), W(t_2), \ldots, W(t_k), s_1, s_2, \ldots, s_h, \eta(\mu_1), \eta(\mu_2), \ldots, \eta(\mu_l)\Big)$$

for all $\eta \in C\big([-\tau,0],\boldsymbol{R}\big)$*. In particular, for a.a.* $\omega \in \Omega$ *and each* $t \in [t_i, a]$*, the map*

$$C\big([-\tau,0],\boldsymbol{R}\big) \ni \eta \mapsto y(t,\omega;t_i,\eta) \in \boldsymbol{R}$$

is C^1 *(in the Fréchet sense), and*

Lemma 1 – cont'd

$$Dy(t,\omega;t_i,\eta)(\xi) = \sum_{m=1}^{l} \partial_m F\Big(t, W(t,\omega), W(t_1,\omega), \ldots,$$
$$W(t_k,\omega), s_1, \ldots, s_h, \eta(\mu_1),$$
$$\cdots, \eta(\mu_m), \ldots, \eta(\mu_l)\Big)\xi(\mu_m)$$

for all $\eta, \xi \in C\big([-\tau, 0], \boldsymbol{R}\big)$. $\partial_m F$ *is the partial derivative of* F *with respect to the variable* $\eta(\mu_m)$.

Lemma 1 – cont'd

$$Dy(t,\omega;t_i,\eta)(\xi) = \sum_{m=1}^{l} \partial_m F\Big(t, W(t,\omega), W(t_1,\omega), \ldots, W(t_k,\omega), s_1, \ldots, s_h, \eta(\mu_1), \cdots, \eta(\mu_m), \ldots, \eta(\mu_l)\Big)\xi(\mu_m)$$

for all $\eta, \xi \in C\Big([-\tau, 0], \boldsymbol{R}\Big)$. $\partial_m F$ *is the partial derivative of* F *with respect to the variable* $\eta(\mu_m)$.

Proof of Lemma 1: By induction, forward steps along partition intervals $[0, t_1], (t_1, t_2], \cdots$. $\square$

Warning!

Warning!

Lemma 1 is *false* if the Euler approximation y is replaced by the exact solution x of the sdde (I):

Warning!

Lemma 1 is *false* if the Euler approximation y is replaced by the exact solution x of the sdde (I):

$$x(t, \omega; 0, \eta) \text{ is highly irregular in } \eta!$$

Warning!

Lemma 1 is *false* if the Euler approximation y is replaced by the exact solution x of the sdde (I):

$x(t, \omega; 0, \eta)$ *is highly irregular in* η*!*

This dictates telescoping argument is wrt Euler approximation y *and* *not* *the solution* x *of (I)*

Lemma 2

Assume that f, g *are* C_b^2 *and let* $\pi := \{t_0, t_1, \cdots, t_n\}$ *be a partition of* $[0, a]$. *For each* $1 \leq i \leq n$, *define the process* $\Lambda^i : [-\tau, 0] \times \Omega \to \boldsymbol{R}$ *by*

$$\Lambda^i := x_{t_i}\big(\cdot\,; t_{i-1}, x_{t_{i-1}}(\cdot\,; 0, \eta)\big) - y_{t_i}\big(\cdot\,; t_{i-1}, x_{t_{i-1}}(\cdot\,; 0, \eta)\big).$$

Denote

$$x(u) := x(u; 0, \eta), \;\; y(u) := y(u; 0, \eta)$$

for $u \in [-\tau, a]$. *Then*

Lemma 2 – cont'd

$$\Lambda^i(s) = \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \Big[f\Big(x(u-\tau_1), x(u)\Big) - f\Big(x\Big(\lfloor u \rfloor - \tau_1\Big), x\Big(\lfloor u \rfloor\Big)\Big)\Big]\, du$$

$$+ \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \Big[g\Big(x(u-\tau_2), x(u)\Big) - g\Big(x\Big(\lfloor u \rfloor - \tau_2\Big), x\Big(\lfloor u \rfloor\Big)\Big)\Big]\, dW(u)$$

$$:= \sum_{j=1}^{10} \Lambda_j^i(s), \qquad s \in [-\tau, 0],$$

where

Lemma 2 – cont'd

$$\Lambda_1^i(s) := \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial f}{\partial x_1}\big(x(v-\tau_1), x(v)\big) \times f\big(x(v-2\tau_1), x(v-\tau_1)\big) 1_{[\tau_1,\infty)}(v)\, dv\, du$$

$$+ \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial f}{\partial x_1}\big(x(v-\tau_1), x(v)\big) \times g\big(x(v-\tau_1-\tau_2), x(v-\tau_1)\big) \times 1_{[\tau_1,\infty)}(v)\, dW(v-\tau_1)\, du$$

$$+ \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial f}{\partial x_1}\big(x(v-\tau_1), x(v)\big) \times 1_{[0,\tau_1)}(v)\, d\eta(v-\tau_1)\, du$$

Lemma 2 – cont’d

$$\Lambda_2^i(s) := \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial f}{\partial x_2}\big(x(v-\tau_1), x(v)\big) \times f\Big(x(v-\tau_1), x(v)\Big)\, dv\, du$$

$$+ \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial f}{\partial x_2}\big(x(v-\tau_1), x(v)\big) \times g\Big(x(v-\tau_2), x(v)\Big)\, dW(v)\, du$$

Lemma 2 – cont'd

$$\Lambda_3^i(s) := \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial^2 f}{\partial x_1 \partial x_2}\big(x(v-\tau_1), x(v)\big) \times g\big(x(v-\tau_1-\tau_2), x(v-\tau_1)\big) \times 1_{[\tau_1,\infty)}(v)\mathcal{D}_{v-\tau_1}x(v)\, dv\, du$$

Lemma 2 – cont'd

$$\Lambda_4^i(s) := \frac{1}{2}\int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial^2 f}{\partial x_1^2}\big(x(v-\tau_1), x(v)\big)$$
$$\times\, g\Big(x(v-\tau_1-\tau_2), x(v-\tau_1)\Big)^2$$
$$\times\, 1_{[\tau_1,\infty)}(v)\, dv\, du$$

Lemma 2 – cont'd

$$\Lambda_4^i(s) := \frac{1}{2}\int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial^2 f}{\partial x_1^2}\big(x(v-\tau_1), x(v)\big) \times g\Big(x(v-\tau_1-\tau_2), x(v-\tau_1)\Big)^2 \times 1_{[\tau_1,\infty)}(v)\, dv\, du$$

$$\Lambda_5^i(s) := \frac{1}{2}\int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial^2 f}{\partial x_2^2}\big(x(v-\tau_1), x(v)\big) \times g\Big(x(v-\tau_2), x(v)\Big)^2\, dv\, du$$

Lemma 2 – cont’d

$$\Lambda_6^i(s) := \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial g}{\partial x_1}\big(x(v-\tau_2), x(v)\big) \times f\Big(x(v-\tau_1-\tau_2), x(v-\tau_2)\Big) \times 1_{[\tau_2,\infty)}(v)\, dv\, dW(u)$$

$$+ \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial g}{\partial x_1}\big(x(v-\tau_2), x(v)\big) \times 1_{[0,\tau_2)}(v)\, d\eta(v-\tau_2)\, dW(u)$$

Lemma 2 – cont'd

$$\Lambda_7^i(s) := \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial g}{\partial x_1}\Big(x(v-\tau_2), x(v)\Big) \times g\Big(x(v-2\tau_2), x(v-\tau_2)\Big) \times 1_{[\tau_2,\infty)}(v)\, dW(v-\tau_2)\, dW(u)$$

Lemma 2 – cont'd

$$\Lambda_7^i(s) := \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial g}{\partial x_1}\big(x(v-\tau_2), x(v)\big) \times g\big(x(v-2\tau_2), x(v-\tau_2)\big) \times \times 1_{[\tau_2,\infty)}(v)\, dW(v-\tau_2)\, dW(u)$$

$$\Lambda_8^i(s) := \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u \rfloor}^{u} \frac{\partial^2 g}{\partial x_1 \partial x_2}\big(x(v-\tau_2), x(v)\big) \times g\big(x(v-2\tau_2), x(v-\tau_2)\big) \times \times 1_{[\tau_2,\infty)}(v) \mathcal{D}_{v-\tau_2} x(v)\, dv\, dW(u)$$

Lemma 2 – cont'd

$$\Lambda_9^i(s) := \frac{1}{2}\int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u\rfloor \vee \tau_1}^{u} \frac{\partial^2 g}{\partial x_1^2}\big(x(v-\tau_2), x(v)\big)$$
$$\times\, g\Big(x(v-2\tau_2), x(v-\tau_2)\Big)^2 \times$$
$$\times\, 1_{[\tau_2,\infty)}(v)\, dv\, dW(u)$$

Lemma 2 – cont'd

$$\Lambda_9^i(s) := \frac{1}{2}\int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u\rfloor \vee \tau_1}^{u} \frac{\partial^2 g}{\partial x_1^2}\big(x(v-\tau_2), x(v)\big)$$
$$\times\, g\Big(x(v-2\tau_2), x(v-\tau_2)\Big)^2 \times$$
$$\times\, 1_{[\tau_2,\infty)}(v)\, dv\, dW(u)$$

$$\Lambda_{10}^i(s) := \frac{1}{2}\int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{\lfloor u\rfloor}^{u} \frac{\partial^2 g}{\partial x_2^2}\big(x(v-\tau_2), x(v)\big)$$
$$\times\, g\Big(x(v-\tau_2), x(v)\Big)^2 dv\, dW(u)$$

for all $s \in [-\tau, 0]$.

Convention

From now on *all* positive constants

will be denoted by the *same* letter C

e.g.

$$C = 2C = \frac{1}{2}C = \cdots$$

etc..

Lemma 3

Suppose $f, g \in C_b^2$*. Then for any* $p \geq 1$ *there is a positive constant* $C := C(p, a, f, g)$ *such that*

$$\sup_{\sigma-\tau\leq u,t\leq a} E|\mathcal{D}_u y(t;\sigma,\eta)|^{2p}$$

$$< C\Big(1 + E\|\eta\|_C^{2p} + \sup_{\sigma-\tau\leq s\leq\sigma} E\|\mathcal{D}_s\eta\|_\infty^{2p}\Big);$$

Lemma 3

Suppose $f, g \in C_b^2$*. Then for any* $p \geq 1$ *there is a positive constant* $C := C(p, a, f, g)$ *such that*

$$\sup_{\sigma-\tau \leq u, t \leq a} E|\mathcal{D}_u y(t; \sigma, \eta)|^{2p}$$

$$< C\Big(1 + E\|\eta\|_C^{2p} + \sup_{\sigma-\tau \leq s \leq \sigma} E\|\mathcal{D}_s \eta\|_\infty^{2p}\Big);$$

$$\sup_{\substack{\sigma-\tau \leq u, t \leq a \\ \|\xi\|_\infty \leq 1}} E|\mathcal{D}_u Dy(t; \sigma, \eta)(\xi)|^{2p}$$

$$< C\Big(1 + E\|\eta\|_C^{4p} + \sup_{\sigma-\tau \leq s \leq \sigma} E\|\mathcal{D}_s \eta\|_\infty^{4p}\Big)^{1/2}$$

for all $\eta \in L^{4p}\Big(\Omega, C\Big([-\tau, 0], \boldsymbol{R}\Big); \mathcal{F}_\sigma\Big)$ *with finite RHS.*

Lemma 4

Suppose $f, g \in C_b^3$. *Then for any* $p \geq 1, \sigma \in [0, a]$,

$$\sup_{\sigma-\tau\leq u,w,t\leq a} E|\mathcal{D}_w\mathcal{D}_u y(t;\sigma,\eta)|^{2p}$$

$$< C\Big(1 + E\|\eta\|_C^{4p} + \sup_{\sigma-\tau\leq s\leq\sigma} E\|\mathcal{D}_s\eta\|_\infty^{4p}$$

$$+ \sup_{\sigma-\tau\leq s_1,s_2\leq\sigma} E\|\mathcal{D}_{s_1}\mathcal{D}_{s_2}\eta\|_\infty^{4p}\Big)$$

for all $\eta \in L^{4p}\big(\Omega, C\big([-\tau, 0], \boldsymbol{R}\big); \mathcal{F}_\sigma\big)$ *with RHS finite.*
$C := C(p, a, f, g) > 0$ *independent of* $t \in [\sigma - \tau, a], \sigma, \eta$.

Lemma 4

Suppose $f, g \in C_b^3$. *Then for any* $p \geq 1, \sigma \in [0, a]$,

$$\sup_{\sigma-\tau\leq u,w,t\leq a} E|\mathcal{D}_w\mathcal{D}_u y(t;\sigma,\eta)|^{2p}$$

$$< C\Big(1 + E\|\eta\|_C^{4p} + \sup_{\sigma-\tau\leq s\leq\sigma} E\|\mathcal{D}_s\eta\|_\infty^{4p}$$

$$+ \sup_{\sigma-\tau\leq s_1,s_2\leq\sigma} E\|\mathcal{D}_{s_1}\mathcal{D}_{s_2}\eta\|_\infty^{4p}\Big)$$

for all $\eta \in L^{4p}\big(\Omega, C\big([-\tau, 0], \boldsymbol{R}\big); \mathcal{F}_\sigma\big)$ *with RHS finite.*
$C := C(p, a, f, g) > 0$ *independent of* $t \in [\sigma - \tau, a], \sigma, \eta$.
Similar estimate for $E|\mathcal{D}_w\mathcal{D}_u Dy(t;\sigma,\eta)(\xi)|^{2p}$.

Proof of Theorem 1

Fix $t \in [\sigma, a]$. Let $\pi := \{0 = t_0, t_1, t_2, \cdots, t_n = a\}$ be a partition of $[0, a]$. W.l.o.g, assume that $\sigma = 0$, $t = t_n$.

Proof of Theorem 1

Fix $t \in [\sigma, a]$. Let $\pi := \{0 = t_0, t_1, t_2, \cdots, t_n = a\}$ be a partition of $[0, a]$. W.l.o.g, assume that $\sigma = 0$, $t = t_n$.

By telescoping and the Markov property for the segments x_t and y_t ([Mo.1], [Mo.2]), write:

Proof of Theorem 1

Fix $t \in [\sigma, a]$. Let $\pi := \{0 = t_0, t_1, t_2, \cdots, t_n = a\}$ be a partition of $[0, a]$. W.l.o.g, assume that $\sigma = 0$, $t = t_n$.

By telescoping and the Markov property for the segments x_t and y_t ([Mo.1], [Mo.2]), write:

$$
\begin{aligned}
&E\phi\big(x(t_n;0,\eta)\big) - E\phi\big(y(t_n;0,\eta)\big) \\
&\quad = E\phi\big(y(t_n;t_n,x_{t_n}(\,\cdot\,;0,\eta))\big) - E\phi\big(y(t_n;0,\eta)\big) \\
&\quad = \sum_{i=1}^{n}\Big\{E\phi\big(y(t_n;t_i,x_{t_i}(\,\cdot\,;t_{i-1},x_{t_{i-1}}(\,\cdot\,;0,\eta)))\big) \\
&\qquad\qquad - E\phi\big(y(t_n;t_i,y_{t_i}(\,\cdot\,;t_{i-1},x_{t_{i-1}}(\,\cdot\,;0,\eta)))\big)\Big\}
\end{aligned}
$$

Proof of Theorem 1 – cont'd

$$= \sum_{i=1}^{n} E \int_0^1 D(\phi \circ y)\Big(t_n; t_i, \lambda x_{t_i}(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,; 0, \eta))$$

$$+ (1-\lambda) y_{t_i}\big(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,; 0, \eta)\big)\Big)\, d\lambda$$

$$\cdot \Big[x_{t_i}\big(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,; 0, \eta)\big)$$

$$- y_{t_i}\big(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,; 0, \eta)\big)\Big]\,.$$

Proof of Theorem 1 – cont'd

For simplicity, denote each random measure

$$\Big\{D(\phi\circ y)\big(t_n; t_i, \lambda x_{t_i}\big(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,;0,\eta)\big)\\ +(1-\lambda)y_{t_i}\big(\,\cdot\,; t_{i-1}, x_{t_{i-1}}(\,\cdot\,;0,\eta)\big)\big)\Big\}\,(ds)$$

by

$$D(\phi\circ y)_i(\lambda, ds)$$

for each $\lambda\in[0,1]$.

Proof of Theorem 1 – cont'd

Thus, by Fubini's theorem,

$$E\Big(\phi\Big(x(t_n;0,\eta)\Big)\Big) - E\phi\Big(y(t_n;0,\eta)\Big)$$

$$= \sum_{j=1}^{10}\sum_{i=1}^{n}\int_0^1\int_{-\tau}^0 E\,D(\phi\circ y)_i(\lambda,ds)\Lambda_j^i(s)\,d\lambda. \tag{3}$$

Proof of Theorem 1 – cont’d

Thus, by Fubini’s theorem,

$$E\Big(\phi\Big(x(t_n;0,\eta)\Big)\Big) - E\phi\Big(y(t_n;0,\eta)\Big)$$

$$= \sum_{j=1}^{10}\sum_{i=1}^{n}\int_0^1\int_{-\tau}^0 E\,D(\phi\circ y)_i(\lambda,ds)\Lambda_j^i(s)\,d\lambda. \tag{3}$$

Estimate each of the 10 terms

$$\sum_{i=1}^{n}\int_0^1\int_{-\tau}^0 E\Big\{D(\phi\circ y)_i(\lambda,ds)\Lambda_j^i(s)\Big\}\,d\lambda,\, j=1,2,\ldots,10$$

on RHS of (3), for any fixed $\lambda\in[0,1]$.

Proof of Theorem 1 – cont'd

Let $j = 10$. Fix $\lambda \in [0,1]$. Since the Skorohod integral is the adjoint of the Malliavin derivative, a computation via Lemma 2 (Dy tame) gives:

$$\begin{aligned} I_{10}^i &:= \int_{-\tau}^0 E\, D(\phi \circ y)_i(\lambda, ds)\Lambda_{10}^i(s) \\ &= \int_{-\tau}^0 E\, D\phi\Big(y\big(t_n; t_i, \lambda x_{t_i} + (1-\lambda)y_{t_i}\big) \\ &\qquad \times Dy\big(t_n; t_i, \lambda x_{t_i} + (1-\lambda)y_{t_i}\big)(ds)\Lambda_{10}^i(s) \\ &= Y_1^i + Y_2^i, \end{aligned} \tag{4}$$

Proof of Theorem 1 – cont'd

$$Y_1^i := \int_{t_{i-1}}^{t_i} EX(u)Dy(t_n; t_i, \lambda x_{t_i} + (1-\lambda)y_{t_i})(\xi^u)\,du \tag{5}$$

Proof of Theorem 1 – cont'd

$$Y_1^i := \int_{t_{i-1}}^{t_i} EX(u)Dy(t_n; t_i, \lambda x_{t_i} + (1-\lambda)y_{t_i})(\xi^u)\, du \tag{5}$$

where

$$X(u) := \frac{1}{2}\mathcal{D}_u D\phi\Big(y\Big(t_n; t_i, \lambda x_{t_i} + (1-\lambda)y_{t_i}\Big)$$
$$\times \int_{\lfloor u \rfloor}^{u} \frac{\partial^2 g}{\partial x_2^2}\Big(x(v-\tau_2), x(v)\Big) g\Big(x(v-\tau_2), x(v)\Big)^2\, dv,$$

Proof of Theorem 1 – cont'd

$$Y_1^i := \int_{t_{i-1}}^{t_i} EX(u) Dy(t_n; t_i, \lambda x_{t_i} + (1-\lambda) y_{t_i})(\xi^u)\, du \tag{5}$$

where

$$X(u) := \frac{1}{2}\mathcal{D}_u D\phi\Big(y\Big(t_n; t_i, \lambda x_{t_i} + (1-\lambda) y_{t_i}\Big)$$

$$\times \int_{\lfloor u \rfloor}^{u} \frac{\partial^2 g}{\partial x_2^2}\big(x(v-\tau_2), x(v)\big) g\big(x(v-\tau_2), x(v)\big)^2\, dv,$$

and $\xi^u \in L^\infty([-\tau, 0], \mathbf{R})$ is given by

$$\xi^u(s) := 1_{[t_{i-1},(t_i+s)\vee t_{i-1})}(u),\ s \in [-\tau, 0],\ u \in [0, a];$$

Proof of Theorem 1 – cont’d

Proof of Theorem 1 – cont'd

$$Y_2^i := \int_{t_{i-1}}^{t_i} EZ(u)\mathcal{D}_u Dy(t_n; t_i, \lambda x_{t_i} + (1-\lambda)y_{t_i})(\xi^u)\, du \tag{5}$$

where

$$Z(u) := \frac{1}{2} D\phi\Big(y\Big(t_n; t_i, \lambda x_{t_i} + (1-\lambda)y_{t_i}\Big)$$

$$\times \int_{\lfloor u \rfloor}^{u} \frac{\partial^2 g}{\partial x_2^2}\Big(x(v-\tau_2), x(v)\Big) g\Big(x(v-\tau_2), x(v)\Big)^2 \, dv.$$

Proof of Theorem 1 –cont'd

By the linearization of (II) and Gronwall's lemma, we get

$$\sup_{\substack{\xi \in L^\infty([-\tau,0],\mathbf{R}) \\ \|\xi\|_\infty \leq 1}} \sup_{\sigma-\tau\leq t\leq a} E|Dy(t;\sigma,\eta)(\xi)|^{2p} \leq C \qquad (6)$$

for every $p \geq 1$.

Proof of Theorem 1 – cont'd

Using (similar) moment estimates on the solution, the Euler approximation and its Fréchet and Malliavin derivatives:

$$|Y_1^i| \leq C(1 + E\|\eta\|_C^3)(t_i - t_{i-1})^2 \tag{7}$$

$$|Y_2^i| \leq C(1 + E\|\eta\|_C^4)(t_i - t_{i-1})^2 \tag{8}$$

Positive constants C are independent of η and the partition $\{t_1, t_2, \cdots, t_n\}$.

Proof of Theorem 1 – cont'd

Putting things together:

$$\left|\sum_{i=1}^{n} I_{10}^{i}\right| = \left|\sum_{i=1}^{n}\int_0^1\int_{-\tau}^{0} E\Big\{D(\phi\circ y)_i(\lambda, ds)\Lambda_{10}^{i}(s)\Big\}\,d\lambda\right|$$

$$\le C(1+E\|\eta\|_C^3)\sum_{i=1}^{n}(t_i - t_{i-1})^2$$

$$+\,C(1+E\|\eta\|_C^4)\sum_{i=1}^{n}(t_i - t_{i-1})^2$$

$$\le C(1+E\|\eta\|_C^4)|\pi|. \qquad (9)$$

Proof of Theorem 1 – cont'd

Develop estimates similar to (9) for the 9 cases
$j = 1, 2, 3, 4, 5, 6, 7, 8, 9.$ $\square$

Extensions

Extensions cover:

- Nonlinear $\mathbf{R}^d$-valued sdde's with several delays.

Extensions

Extensions cover:

- Nonlinear $\mathbf{R}^d$-valued sdde's with several delays.
- Quasitame memory dependence.

Extensions

Extensions cover:

- Nonlinear $\mathbf{R}^d$-valued sdde's with several delays.
- Quasitame memory dependence.
- Mixed history dependence: discrete and quasitame.

Extensions

Extensions cover:

- Nonlinear $\mathbf{R}^d$-valued sdde's with several delays.
- Quasitame memory dependence.
- Mixed history dependence: discrete and quasitame.
- Multidimensional Brownian motion.

REFERENCES

[A] Ahmed, T. A. *Stochastic Functional Differential Equations with Discontinuous Initial Data*, M.Sc. Thesis, University of Khartoum, Sudan (1983).(<–)

[AHMP] Arriojas, M., Hu, Y., Mohammed, S.-E.A. and Pap, G., *A Delayed Black and Scholes Formula*, Stochastic Analysis and Applications, Vol. 25, 2, (2007), 471 - 492.(¡–)

[B.B] Baker, C. T. H. and Buckwar, E., *Numerical analysis of explicit one-step methods for stochastic delay differential equations.*, LMS J. Comput. Math. **3** (2000), 315–335 (electronic).(¡–)

REFERENCES – cont'd

[B.T] Bally, V., and Talay, D., *The law of the Euler scheme for stochastic differential equations. I. Convergence rate of the distribution function*, Probab. Theory Related Fields, 104 (1996), 43–60.(<–)

[B.M] Bell, D. and Mohammed, S.-E. A., *The Malliavin calculus and stochastic delay equations*, Journal of Functional Analysis **99 No. 1** (1991), 75–99.

[B.S] Buckwar, E. and Shardlow, T., *Weak approximation of stochastic differential delay equations*, IMA J. Numer. Anal. **25** (2005), 57–86.(<–)

REFERENCES – cont’d

[B.V] Beuter, A. and Vasilakos, K., *Effects of noise on a delayed visual feedback system.* J. Theor. Biology **165** (1993), 389–407.(<–)

[BGMTS] Buldú, J.M., Garcia-Ojalvo, J., Mirasso, C.R., Torrent, M.C., and Sancho, J.M., *Effect of external noise correlation in optical coherence resonance.* Phys. Rev. E **64** (2001), 051109-1–051109-4. (<–)

[CK-HL] E. Clément, A. Kohatsu-Higa and D. Lamberton, *A duality approach for weak approximation of stochastic differential equations*. (<–)

REFERENCES – cont'd

[GRP] Goldobin, D., Rosenblum, M., and Pikovsky, A., *Coherence of noisy oscillators with delayed feedback* Physica A **327** (2003), 124–128. (<–)

[Hu] Hu, Y., *Strong and weak order of time discretization schemes of stochastic differential equations*, In Séminaire de Probabilités XXX, ed. by J. Azema, P.A. Meyer and M. Yor, Lecture Notes in Mathematics **1626,** Springer-Verlag, 1996, 218-227.

[H.M.Y] Hu, Y., Mohammed, S.-E. A., and Yan, F., *Discrete-time approximations of stochastic delay equations: The Milstein scheme*, The Annals of Probability, Vol. 32, No. 1A, (2004), 265-314. (<–)

REFERENCES – cont'd

[K] Kohatsu-Higa, Y., *Weak approximations. A Malliavin calculus approach.* Math. Comput. **70**(233) (2001), 135–172.(<–)

[LMBM] Longtin, A., Milton, J.G., Bos, J. and Mackey, M.C., *Noise and critical behavior of the pupil light reflex at oscillation onset.* Phys. Rev. A **41** (1990), 6992–7005.(<–)

[K.P] Kloeden, P. E. and Platen E., *Numerical Solutions of Stochastic Differential Equations*, Springer-Verlag, New York-Berlin (1995).(<–)

REFERENCES – cont'd

[Kü.P] Küchler, U. and Platen, E. *Strong Discrete Time Approximation of Stochastic Differential Equations with Time Delay.* Mathematics and Computer Simulation 54 (2000), 189-205.(<–)

[Ma.1] Masoller, C., *Numerical investigation of noise-induced resonance in a semiconductor laser with optical feedback.*, Physica D *168-169* (2002), 171–176.

[Ma.2] Masoller, C., *Distribution of residence times of time-delayed bistable systems driven by noise.* Phys. Rev. Lett. **90**(2) (2003), 020601-1–020601-4. (<–)

REFERENCES – cont'd

[Mo.1] Mohammed, S.-E. A., *Stochastic Functional Differential Equations*, Research Notes in Mathematics, no. 99, Pitman Advanced Publishing Program, Boston-London-Melbourne (1984).(<–)

[Mo.2] Mohammed, S.-E. A., Stochastic differential systems with memory: Theory, examples and applications. In "Stochastic Analysis", Decreusefond L. Gjerde J., Øksendal B., Ustunel A.S., edit., *Progress in Probability* 42, Birkhauser (1998), 1-77.(<–)

[N.P] Nualart, D. and Pardoux, E., *Stochastic calculus with anticipating integrands*, Probability Theory and Related fields **78** (1988), 535–581.

REFERENCES – cont'd

[Nu.1] Nualart, D., *The Malliavin Calculus and Related Topics*, Springer-Verlag, 1995.(<–)

[Nu.2] Nualart, D., *Analysis on Wiener Space and Anticipating Stochastic Calculus*, Ecole d'Été de Probabilités, Saint Flour 1995.

[Y] Yan, F., *Topics on Stochastic Delay Equations*, Ph.D. Dissertation, Southern Illinois University at Carbondale, August, 1999.

THE END

THANK YOU!

Extensions

Extensions cover:

- Nonlinear $\mathbf{R}^d$-valued sdde's with several delays.

Extensions

Extensions cover:

- Nonlinear $\mathbf{R}^d$-valued sdde's with several delays.
- Quasitame memory dependence.

Extensions

Extensions cover:

- Nonlinear $\mathbf{R}^d$-valued sdde's with several delays.
- Quasitame memory dependence.
- Mixed history dependence: discrete and quasitame.

Extensions

Extensions cover:

- Nonlinear $\mathbf{R}^d$-valued sdde's with several delays.
- Quasitame memory dependence.
- Mixed history dependence: discrete and quasitame.
- Multidimensional Brownian motion.

Extensions—Notation

Let $W(t) := (W_1(t), W_2(t), \cdots, W_m(t))\, t \geq 0$, be m-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

Extensions—Notation

Let $W(t) := (W_1(t), W_2(t), \cdots, W_m(t))\, t \geq 0$, be m-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

Consider a finite number of delays $\{\tau_1^i : 1 \leq i \leq k_1\}$, $\{\tau_2^{j,l} : 1 \leq j \leq k_{2,l}, 1 \leq l \leq m\}$, with maximum delay $\tau := \max\{\tau_1^i, \tau_2^{j,l} : 1 \leq i \leq k_1, 1 \leq j \leq k_{2,l}, 1 \leq l \leq m\}$. We will designate the memory in our sfde by a collection of *tame projections*

Extensions—-cont'd

$$\Pi^1 : C := C([-\tau, \mathbf{R}^d) \rightarrow \mathbf{R}^{d_1}, \quad \Pi^{2,l} : C \rightarrow \mathbf{R}^{d_{2,l}}$$

$$\Pi^1(\eta) := (\eta(\tau_1^1), \eta(\tau_1^2), \cdots, \eta(\tau_1^{k_1})),$$
$$\Pi^{2,l}(\eta) := (\eta(\tau_2^{1,l}), \eta(\tau_2^{2,l}), \cdots, \eta(\tau_2^{k_{2,l},l}))$$

for all $\eta \in C$, and *quasitame projections*

$$\Pi_q^1 : C \rightarrow \mathbf{R}^{d_1^q}, \quad \Pi_q^{2,l} : C \rightarrow \mathbf{R}^{d_{2,l}^q}$$

where $d_1 = k_1 d, d_1^q = k_2 d, d_{2,l} = k_{2,l} d, d_{2,l}^q = k_{2,l} d$ are integer multiples of d, for $1 \leq l \leq m$.

Extensions—-cont'd

The quasitame projections are of the form:

$$\Pi_q^1(\eta) := \Big(\int_{-\tau}^0 \sigma_1^1(\eta(s))\mu_1^1(s)\,ds, \int_{-\tau}^0 \sigma_2^1(\eta(s))\mu_2^2(s)\,ds, \\ \cdots, \int_{-\tau}^0 \sigma_{k_2}^1(\eta(s))\mu_{k_2}^1(s)\,ds \Big)$$

Extensions—cont'd

$$\Pi_q^{2,l}(\eta) := \Big(\int_{-\tau}^{0} \sigma_1^2(\eta(s))\mu_1^2(s)\,ds, \int_{-\tau}^{0} \sigma_2^2(\eta(s))\mu_2^2(s)\,ds, \cdots, \int_{-\tau}^{0} \sigma_{k_{2,l}}^2(\eta(s))\mu_{k_{2,l}}^2(s)\,ds\Big)$$

for all $\eta \in C$. The functions $\sigma_i^1, \sigma_j^2, \mu_i^1, \mu_j^2$ are smooth.

Extensions—cont'd

Let

$$f : \mathbf{R}^+ \times \mathbf{R}^{d_1} \times \mathbf{R}^{d_1^q} \longrightarrow \mathbf{R}^d, \qquad g_l : \mathbf{R}^+ \times \mathbf{R}^{d_{2,l}} \times \mathbf{R}^{d_{2,l}^q} \longrightarrow \mathbf{R}^d$$

be functions of class C^1 in the first variable and C_b^3 in all space variables.

Extensions—cont'd

Consider the sfde

$$dx(t) = f(t, \Pi^1(x_t), \Pi_q^1(x_t))\,dt$$
$$+ \sum_{l=1}^{m} g_l(t, \Pi^{2,l}(x_t), \Pi_q^{2,l}(x_t))\,dW_l(t), \ \ \sigma < t < a, \qquad \text{(III)}$$

with initial path

$$x_\sigma = \eta \in H^{1,\infty}([-\tau, 0], \mathbf{R}^d).$$

Extensions—-cont'd

Let $\pi := \{t_{-m}, \cdots, t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[-\tau, a]$ with mesh $|\pi|$. The Euler approximations y of x are given by

$$\begin{aligned} dy(t) = f(\lfloor t \rfloor, \Pi^1(y_{\lfloor t \rfloor}), \Pi_q^1(y_{\lfloor t \rfloor}))\, dt \\ + \sum_{l=1}^{m} g_l(\lfloor t \rfloor, \Pi^{2,l}(y_{\lfloor t \rfloor}), \Pi_q^{2,l}(y_{\lfloor t \rfloor}))\, dW_l(t), \\ \sigma < t < a, \end{aligned} \qquad \text{(IV)}$$

with initial path

$$y_\sigma = \eta \in H^{1,\infty}([-\tau, 0], \mathbf{R}^d).$$

Extensions—cont'd

Under sufficient regularity hypotheses on the coefficients of (III), one gets weak convergence of order 1 for the Euler approximations y in (IV) to the exact solution x.

Theorem 4

Let $\phi : \boldsymbol{R} \to \boldsymbol{R}$ be of class C_b^3. In the sfde (III), let $f,\ g_{\ l}$, $1 \leq l \leq m$, be C^1 in the time variable and C_b^3 in all space variables. Let $x(\cdot;\sigma,\eta)$ be the unique solution of (III) with initial process $\eta \in H^{1,\infty}\big([-\tau,0],\boldsymbol{R}^d\big)$.

Theorem 4

Let $\phi : \boldsymbol{R} \to \boldsymbol{R}$ be of class C_b^3. In the sfde (III), let $f,\ g_l$, $1 \le l \le m$, be C^1 in the time variable and C_b^3 in all space variables. Let $x(\cdot;\sigma,\eta)$ be the unique solution of (III) with initial process $\eta \in H^{1,\infty}\big([-\tau,0],\boldsymbol{R}^d\big)$. Denote by $y(\cdot;\sigma,\eta)$ the Euler approximation to $x(\cdot;\sigma,\eta)$ defined by (IV). Let η^π be the piecewise-linear approximation of η.

Theorem 4

Let $\phi : \boldsymbol{R} \to \boldsymbol{R}$ be of class C_b^3. In the sfde (III), let f, g_l, $1 \leq l \leq m$, be C^1 in the time variable and C_b^3 in all space variables. Let $x(\cdot;\sigma,\eta)$ be the unique solution of (III) with initial process $\eta \in H^{1,\infty}\big([-\tau,0],\boldsymbol{R}^d\big)$. Denote by $y(\cdot;\sigma,\eta)$ the Euler approximation to $x(\cdot;\sigma,\eta)$ defined by (IV). Let η^π be the piecewise-linear approximation of η. Then there is a positive constant C and a positive integer q such that

$$|E\phi(x(t;\sigma,\eta)) - E\phi(y(t;\sigma,\eta^\pi))| \leq C(1 + E\|\eta\|_{1,\infty}^q)|\pi|$$

for all $\eta \in H^{1,\infty}\big([-\tau,0],\boldsymbol{R}^d\big)$, all $t \in [\sigma-\tau, a]$, and all $\sigma \in [0,a]$.

Theorem 4 – cont'd

The constant C *may depend on* a, q *and the test function* ϕ*, but is independent of* π, η*,* $t \in [\sigma - \tau, a]$ *and* $\sigma \in [0, a]$.

Theorem 4 – cont'd

The constant C *may depend on* a, q *and the test function* ϕ*, but is independent of* π, η*,* $t \in [\sigma - \tau, a]$ *and* $\sigma \in [0, a]$*.*

Proof:

Very similar to that of Theorem 1: Main difference is a straightforward application of the classical Itô formula combined with the tame Itô formula. □

Lemma 5

Let $\psi : \boldsymbol{R}^{+} \times \boldsymbol{R}^{3} \to \boldsymbol{R}$ *be of class* C^1 *in the time-variable and* C_b^2 *in the three space variables* x_1, x_2, x_3*. Suppose* x *solves the sfde (III) (for* $d = 1$*) with coeffi cients satisfying the hypotheses of Theorem 4. Assume that* h, μ *are smooth functions. Let* $\delta > 0$*. Then:*

Lemma 5

Let $\psi : \boldsymbol{R}^{+} \times \boldsymbol{R}^{3} \to \boldsymbol{R}$ *be of class* C^1 *in the time-variable and* C_b^2 *in the three space variables* x_1, x_2, x_3. *Suppose* x *solves the sfde (III) (for* $d = 1$*) with coeffi cients satisfying the hypotheses of Theorem 4. Assume that* h, μ *are smooth functions. Let* $\delta > 0$. *Then:*

$$d\psi\Big(t, x(t-\delta), x(t), \int_{-\delta}^{0} h(x(t+s))\mu(s)\,ds\Big)$$

$$= \frac{\partial\psi}{\partial t}(t, x(t-\delta), x(t), \int_{-\delta}^{0} h(x(t+s))\mu(s)\,ds)\,dt$$

Lemma 5—contd

$$+\frac{\partial\psi}{\partial x_1}(t, x(t-\delta), x(t), \int_{-\delta}^{0} h(x(t+s))\mu(s)\,ds)\,1_{[0,\delta)}(t)\cdot$$
$$\cdot\, d\eta(t-\delta)$$
$$+\frac{\partial\psi}{\partial x_1}(t, x(t-\delta), x(t), \int_{-\delta}^{0} h(x(t+s))\mu(s)\,ds)1_{[\delta,\infty)}(t)\times$$
$$\times\,[f(t-\delta, \Pi^1(x_{t-\delta}), \Pi_q^1(x_{t-\delta}))\,dt$$
$$+\sum_{l=1}^{m} g_l(t-\delta, \Pi^{2,l}(x_{t-\delta}), \Pi_q^{2,l}(x_{t-\delta}))\,dW_l(t-\delta)]$$
$$+\frac{\partial\psi}{\partial x_2}\Big(t, x(t-\delta), x(t), \int_{-\delta}^{0} h(x(t+s))\mu(s)\,ds\Big)\times$$

Lemma 5 – cont'd

$$\times \left[f(t, \Pi^1(x_t), \Pi^1_q(x_t))\, dt + \sum_{l=1}^{m} g_l(t, \Pi^{2,l}(x_t), \Pi^{2,l}_q(x_t))\, dW_l(t)\right]$$

$$+ \frac{\partial \psi}{\partial x_3}\Big(t, x(t-\delta), x(t), \int_{-\delta}^{0} h(x(t+s))\mu(s)\, ds\Big) \times$$

$$\times \left[h(x(t)\mu(0) - h(x(t-\delta)\mu(-\delta) - \int_{t-\delta}^{t} h(x(u))\mu'(u-t)\, du\right] dt$$

$$+ \frac{\partial^2 \psi}{\partial x_1 \partial x_2}\Big(t, x(t-\delta), x(t), \int_{-\delta}^{0} h(x(t+s))\mu(s)\, ds\Big) \times$$

$$\times \sum_{l=1}^{m} g_l(t-\delta, \Pi^{2,l}(x_{t-\delta}), \Pi^{2,l}_q(x_{t-\delta})) 1_{[\delta,\infty)}(t) \mathcal{D}_{t-\delta} x(t)\, dt$$

Lemma 5 – cont'd

$$+\frac{1}{2}\frac{\partial^2\psi}{\partial x_1^2}\Big(t, x(t-\delta), x(t), \int_{-\delta}^{0} h(x(t+s))\mu(s)\,ds\Big)$$

$$\times\sum_{l=1}^{m} g_l(t-\delta, \Pi^{2,l}(x_{t-\delta}), \Pi_q^{2,l}(x_{t-\delta}))^2 1_{[\delta,\infty)}(t)\,dt$$

$$+\frac{1}{2}\frac{\partial^2\psi}{\partial x_2^2}\Big(t, x(t-\delta), x(t), \int_{-\delta}^{0} h(x(t+s))\mu(s)\,ds\Big)$$

$$\times\sum_{l=1}^{m} g_l(t, \Pi^{2,l}(x_t), \Pi_q^{2,l}(x_t))^2\,dt,$$

for all $t > 0$.

Appropriate generalizations of Lemma 5 hold for higher dimensional versions of the sfde (III) ($d > 1$).

Duality Methods

Weak convergence of the Euler scheme for a class of SFDE's with smooth coefficients:

$$b\left(\int_{-r}^{0} x(u+s)\,d\nu(s)\right)$$

and

$$\sigma\left(\int_{-r}^{0} x(u+s)\,d\nu(s)\right)$$

ν a finite measure on $[-r,0]$ and $b,\sigma:\mathbf{R}\to\mathbf{R}$ sufficiently smooth real-valued functions- due independently to Clément, Kohatsu-Higa and Lamberton [CK-HL]. Uses duality techniques.

THE VERY END