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A Characterization of Power Method Transformations through The Method of Percentiles

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Abstract

This paper derives closed-form solutions for the fifth-ordered power method polynomial transformation based on the method of percentiles (MOP). A proposed MOP univariate procedure is described and compared with the method of moments (MOM) in the context of distribution fitting and estimating skew, kurtosis, fifth- and sixth-ordered functions. The MOP methodology is also extended from univariate to multivariate data generation. The MOP procedure has an advantage over the MOM because it does not require numerical integration to compute intermediate correlations. In addition, the MOP procedure can be applied to distributions where mean and(or) variance do(does) not exist. Simulation results demonstrate that the proposed MOP procedure is superior to the MOM in terms of estimation, relative bias, and relative error.

keywords: intermediate correlation; Monte Carlo; power method; percentile; multivariate; simulation

1. Introduction

The power method (PM, Fleishman, 1978; Headrick, 2010) is a traditional procedure used for simulating continuous non-normal distributions. Some applications of the power method have included such topics as ANOVA (e.g., Berkovits, Hancock, & Nevitt, 2000; Lix & Fouladi, 2007), asset pricing theories (Affleck-Graves & MacDonald, 1989), business-cycle features (Hess & Iwata, 1997), cluster analysis (Steinley & Henson, 2005), item parameter estimation (Kirisci, Hsu, & Yu, 2001), item response theory (Harwell, Stone, Hsu, & Kirisci, 1996; Stone, 2003), factor analysis (Benson & Fleishman, 1994; Flora & Curran, 2004), price risk (Mahul, 2003), structural equation models (Hau & Marsh, 2004; Henson, Reise, & Kim, 2007), and toxicology (Hothorn & Lehmacher, 2007).

The PM transformation can be generally expressed as (Headrick, 2010, p.12-13)

$$q(Z) = \sum_{i=1}^{m} c_i Z^{i-1}, \tag{1}$$

where $q(Z)$ is a polynomial used to perform the transformation on Z, c_i is a constant coefficient defining the nature of the transformation, and Z is a standard normal random variable with probability density function (PDF) $\phi(z)$ and cumulative distribution function (CDF) $\Phi(z)$. Setting $m = 4$ (or $m = 6$) in Equation (1) gives the third-order (or fifth-order) class of PM distributions.

The values of c_i associated with (1) can be determined from either the method of moments (MOM; e.g., Headrick, Kowalchuk, & Sheng, 2008; Kowalchuk & Headrick, 2010), or the method of percentiles (MOP; e.g., Hoaglin, 1983; Hoaglin, Mosteller, & Tukey, 1985). Specifically, the conventional MOM determines the values of c_i from the specified α_3 (skew), α_4 (kurtosis), α_5, and α_6. On the other hand, the MOP obtains the values of c_i given the specified γ_3 (left-right tail-weight ratios), γ_4 (tail-weight factors), γ_5, and γ_6.

Conventional moment-based PM have unfavorable attributes to the extent that the estimates of conventional skew and kurtosis associated with heavy tailed or skewed distributions can be substantially biased, have high variance, or can be influenced by outliers (e.g., Headrick, 2011; Headrick & Pant, 2012a, 2012c, 2013; Hodis, Headrick, & Sheng, 2012; Karian & Dudewicz, 2003). On the other hand, the MOP, which is based on the methodology described in Karian and Dudewicz (e.g., Karian & Dudewicz, 2003, 2011) in the context of the generalized lambda distribution (GLD), has demonstrated to be an attractive and computationally efficient alternative to the MOM in terms of distribution fitting and computing the GLD shape parameters. Further, it has been demonstrated that the MOP is superior to the MOM over a broad range of combinations of skew and kurtosis for fitting theoretical or empirical distributions (Karian & Dudewicz, 2003; Koran, Headrick, & Kuo, 2015; Kuo & Headrick, 2014).

In addition to obtaining more favorable unbiased estimation, the MOP is more attractive than the MOM due to information that is not available. Specifically, the conventional MOM relies on knowledge of the skew and kurtosis of the distribution, which may not be included in public reports. That is, proportions and percentiles may be more commonly included in public reports (e.g., Idaho Standards Achievement Tests). Moreover, the MOM can not generate distributions where the mean or variance do not exist (e.g, Cauchy, t distribution with 1 or 2 degrees of freedom).

In view of the above, the present aim is to obviate the problems associated with the MOM in the context of fifth-ordered PM transformation of the form in Equation (1) by characterizing these distributions through the MOP. Specifically, the purpose of this paper is to develop the methodology and a procedure for simulating distributions with specified γ_3-γ_6. In terms of simulating multivariate distributions, the Spearman correlation will be used in lieu of the Pearson correlation using the equation in Headrick (2010), p.114, Eq. 4.34. In summary, the advantages of the proposed MOP procedure are that (i) the MOP parameters (γ_3-γ_6) exist for any distribution, whether the mean and/or the variance exist or not (e.g., Dudewicz & Karian, 1999); (ii) there is less relative bias and has less relative standard error when juxtaposed with the MOM procedure; (iii) there are closed-form solutions for the c_i constants, and (iv) there is a straightforward equation for the purpose of simulating correlated non-normal distributions.

The remainder of the paper is outlined as follows. In Section 2, a summary of the univariate PM distributions based on the MOM is provided. In Section 2.1, the requisite information associated with the MOP is provided. Further, the systems of equations for determining the closed-form solutions of the c_i constants associated with Eq. (1) are subsequently derived for simulating univariate non-normal distributions with specified values of γ_1-γ_6. In section 3, a comparison of the MOM and the MOP is provided by fitting several theoretical distributions and the SPSS data from IBM Corp. (2011). In Section 4, the methodologies for simulating correlated non-normal distributions with specified Pearson correlations for the MOM and Spearman correlations for the MOP are provided. In Section 5, the steps for implementing the proposed MOP procedure are described. A numerical example and results of a simulation are also provided to confirm the derivations and compare the proposed procedure with the MOM procedure. In Section 5.2, the results of the simulation are discussed.

2. Methodology

2.1 The PM transformation based on the MOM

The requirement that $p(Z)$ in (1) be a strictly monotone increasing function implies that an inverse function (p^{-1}) exists and thus $F_{p(z)}(z) = \Phi(z)$, where $F_{q(z)}(z)$ is the general form of the CDF for both the MOM and the MOP. Differentiating both sides with respect to $p(z)$ yields $dF_{p(z)}(z)/dp(z) = f_{p(z)}(z)$, where $f_{q(z)}(z)$ is the general form of the PDF for both the MOM and the MOP. Hence, $f_{p(z)}(z) = dF_{p(z)}(z)/dp(z) = (dF_{p(z)}(z)/dz)/(dp(z)/dz) = \phi(z)/p'(z)$. Whence, the PDF integrates to one because $\phi(z)$ is the standard normal PDF and will be nonnegative for $z \in (-\infty, +\infty)$, and where $\lim_{z\to\pm\infty} \phi(z)/q'(z) = 0$ for the transformations in (1).

The constants c_1-c_6 associated with (1) that determine the shape of a distribution are computed using a moment-matching approach that involves the conventional measures of the mean (α_1), variance (α_2), skew (α_3), kurtosis (α_4), fifth-(α_5) and sixth-(α_6) ordered moments. Specifically, the values of c_1-c_6 in (1) are determined by simultaneously solving Equations (37) to (42) provided in Appendix A for specified values of α_1-α_6 (Headrick & Kowalchuk, 2007, Eqs. (A1)-(A6)). Note that $\alpha_1 - \alpha_6$ are standardized cumulants and are scaled such that the normal distribution would have values $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$. Hence, the solution values of c_1-c_6 produce a distribution with zero mean, unit variance, and the desired values of α_3-α_6.

2.2 The PM transformation based on the MOP

The percentiles (θ_p) associated with a conventional based PM PDF can be obtained by making use of the standard normal CDF, $\Phi(z)$. As such, the location, scale, and shape parameters are defined as (Karian & Dudewicz, 2011, p.172-173)

$$\gamma_1 = \theta_{0.50} \tag{2}$$
$$\gamma_2 = \theta_{0.90} - \theta_{0.10} \tag{3}$$
$$\gamma_3 = \frac{\theta_{0.50} - \theta_{0.10}}{\theta_{0.90} - \theta_{0.50}} \tag{4}$$
$$\gamma_4 = \frac{\theta_{0.75} - \theta_{0.25}}{\gamma_2}, \tag{5}$$

where (2)-(5) are the (i) median, (ii) inter-decile range, (iii) left-right tail-weight ratio (a skew function) and (iv) tail-weight factor (a kurtosis function), respectively. The parameters in (2)-(5) are defined to have the restrictions

$$-\infty < \gamma_1 < +\infty, \gamma_2 \geq 0, \gamma_3 \geq 0, 0 \leq \gamma_4 \leq 1, \tag{6}$$

and where a symmetric distribution implies that $\gamma_3 = 1$.

More recently, Headrick (2014) extended (2)-(5) to a more general fifth-ordered percentile based system:

$$\gamma_1 = \theta_{0.50} \tag{7}$$
$$\gamma_2 = \theta_{0.90} - \theta_{0.10} \tag{8}$$
$$\gamma_3 = \frac{\theta_{0.70} - \theta_{0.50}}{\theta_{0.50} - \theta_{0.30}} \tag{9}$$
$$\gamma_4 = \frac{\theta_{0.625} - \theta_{0.375}}{\theta_{0.70} - \theta_{0.30}} \tag{10}$$
$$\gamma_5 = \frac{\theta_{0.50} - \theta_{0.10}}{\theta_{0.90} - \theta_{0.50}} \tag{11}$$
$$\gamma_6 = \frac{\theta_{0.75} - \theta_{0.25}}{\gamma_2}. \tag{12}$$

The derivation of the general percentile based system PM begins by substituting the standard normal distribution percentiles (z_p) into the quantile functions:

$$\gamma_1 = q(z_{0.50}) \tag{13}$$
$$\gamma_2 = q(z_{0.90}) - q(z_{0.10}) \tag{14}$$
$$\gamma_3 = \frac{q(z_{0.70}) - q(z_{0.50})}{q(z_{0.50}) - q(z_{0.30})} \tag{15}$$
$$\gamma_4 = \frac{q(z_{0.625}) - q(z_{0.375})}{q(z_{0.70}) - q(z_{0.30})} \tag{16}$$
$$\gamma_5 = \frac{q(z_{0.50}) - q(z_{0.10})}{q(z_{0.90}) - q(z_{0.50})} \tag{17}$$

$$\gamma_6 = \frac{q(z_{0.75}) - q(z_{0.25})}{q(z_{0.90}) - q(z_{0.10})}, \tag{18}$$

where $z_{0.50} = 0$, $z_{0.625} = 0.3186\cdots$, $z_{0.70} = 0.5244\cdots$, $z_{0.75} = 0.6744\cdots$, $z_{0.90} = 1.281\cdots$ from the standard normal distribution. Note from symmetry that $z_{0.10} = -z_{0.90}$ and $z_{0.25} = -z_{0.75}$. The explicit forms of (13)-(18) are

$$\gamma_1 = q(z_{0.50}) \text{ (median)} \tag{19}$$

$$\gamma_2 = 2z_{0.90}(c_2 + z_{0.90}^2 c_4 + z_{0.90}^4 c_6) \tag{20}$$

$$\gamma_3 = 1 + \frac{2z_{0.70}(c_3 + z_{0.70}^2 c_5)}{c_2 + z_{0.70}(-z_{0.70}^3 + z_{0.70}(c_4 - z_{0.70}c_5 + z_{0.70}^2 c_6))} \tag{21}$$

$$\gamma_4 = \frac{c_2 z_{0.625} + c_4 z_{0.625}^3 + c_6 z_{0.625}^5}{z_{0.70}c_2 + z_{0.70}^3 c_4 + z_{0.70}^5 c_6} \tag{22}$$

$$\gamma_5 = 1 - \frac{2z_{0.90}(c_3 + z_{0.90}^2 c_5)}{c_2 + z_{0.90}(c_3 + z_{0.90}(c_4 + z_{0.90}(c_5 + z_{0.90}c_6)))} \tag{23}$$

$$\gamma_6 = \frac{z_{0.75}c_2 + z_{0.75}^3 c_4 + z_{0.75}^5 c_6}{z_{0.90}c_2 + z_{0.90}^3 c_4 + z_{0.90}^5 c_6}. \tag{24}$$

Simultaneously solving for the coefficients in (19)-(24) gives the convenient closed-form expressions

$$c_1 = \gamma_1 \tag{25}$$

$$\begin{aligned} c_2 =& (z_{0.75}^3\gamma_2(-z_{0.625}^5 + \gamma_4 + z_{0.75}^2(z_{0.625}^3 - z_{0.70}^3\gamma_4)) + z_{0.90}^3\gamma_2(z_{0.625}^5 - z_{0.70}^5\gamma_4 + z_{0.90}^2(-z_{0.625}^3 + z_{0.70}^3\gamma_4))\gamma_6 \\ &/(2z_{0.90}(z_{0.90} - z_{0.75})z_{0.75}(z_{0.90} + z_{0.75})(z_{0.625}(-z_{0.90}^2 + z_{0.625}^2)(-z_{0.75}^2 + z_{0.625}^2) \\ &+ (z_{0.90} - z_{0.70})z_{0.70}(z_{0.90} + z_{0.70})(-z_{0.75}^2 + z_{0.70}^2)\gamma_4) \end{aligned} \tag{26}$$

$$\begin{aligned} c_3 =& \frac{1}{2z_{0.90}^3(1+\gamma_3)}\Big(\frac{z_{0.70}\gamma_2(-1+\gamma_3)(-z_{0.625}^5 + z_{0.70}^5\gamma_4)}{-z_{0.625}^5 + z_{0.70}^5\gamma_4 + z_{0.90}^2(z_{0.625}^3 - z_{0.70}^3\gamma_4)} \\ &+ \frac{z_{0.70}^2\gamma_2(z_{0.90}(1+\gamma_3)(-1+\gamma_5) + z_{0.70}(-1+\gamma_3)(1+\gamma_5))}{(z_{0.90}^2 - z_{0.70}^2)(1+\gamma_5)} \\ &+ (z_{0.90}^2(z_{0.90} - z_{0.625})z_{0.625}(z_{0.90} + z_{0.625})(-z_{0.70}^2 + z_{0.625}^2)\gamma_2(-1+\gamma_3)(z_{0.75}^5(-z_{0.625}^3 + z_{0.70}^3\gamma_4) \\ &+ z_{0.75}^3(z_{0.625}^5 - z_{0.70}^5\gamma_4 + z_{0.90}^3(-z_{0.625}^5 + z_{0.70}^5\gamma_4 + z_{0.90}^2(z_{0.625}^3 - z_{0.70}^3\gamma_4))\gamma_6)) \\ &/(z_{0.75}(-z_{0.90}^2 + z_{0.75}^2)z_{0.70}(-z_{0.625}(-z_{0.90}^2 + z_{0.625}^2)(-z_{0.75}^2 + z_{0.625}^2) \\ &+ z_{0.70}(-z_{0.90}^2 + z_{0.70}^2)(-z_{0.75}^2 + z_{0.70}^2)\gamma_4) \end{aligned} \tag{27}$$

$$\begin{aligned} c_4 =& (\gamma_2(z_{0.75}^5(-z_{0.625} + z_{0.70}\gamma_4) + z_{0.75}(z_{0.625}^5 - z_{0.70}^5\gamma_4) + z_{0.90}(-z_{0.625}^5 + z_{0.70}^5\gamma_4 + z_{0.90}^4(z_{0.625} - z_{0.70}\gamma_4))\gamma_6)) \\ &/(2z_{0.90}(z_{0.90} - z_{0.75})z_{0.75}(z_{0.90} + z_{0.75})(z_{0.625}(-z_{0.90}^2 + z_{0.625}^2)(-z_{0.75}^2 + z_{0.625}^2) \\ &+ (z_{0.90} - z_{0.70})z_{0.70}(z_{0.90} + z_{0.70})(-z_{0.75}^2 + z_{0.70}^2)\gamma_4)) \end{aligned} \tag{28}$$

$$\begin{aligned} c_5 =& \frac{1}{2z_{0.90}^5(1+\gamma_3)}\gamma_2\Big(\frac{z_{0.70}(-1+\gamma_3)(-z_{0.625}^5 + z_{0.70}^5\gamma_4)}{z_{0.625}^5 - z_{0.70}^5\gamma_4 + z_{0.90}^2(-z_{0.625}^3 + z_{0.70}^3\gamma_4)} \\ &- \frac{z_{0.90}^3(1+\gamma_3)(-1+\gamma_5) + z_{0.70}^3(-1+\gamma_3)(1+\gamma_5)}{(z_{0.90}^2 - z_{0.70}^2)(1+\gamma_5)} \\ &- (z_{0.90}^2(z_{0.90} - z_{0.625})z_{0.625}(z_{0.90} + z_{0.625})(-z_{0.70}^2 + z_{0.625}^2)(-1+\gamma_3)(z_{0.75}^5(-z_{0.625}^3 + z_{0.70}^3\gamma_4) \end{aligned}$$

$$
\begin{aligned}
&+ z_{0.75}^3(z_{0.625}^5 - z_{0.70}^5\gamma_4) + z_{0.90}^3(-z_{0.625}^5 + z_{0.70}^5\gamma_4 + z_{0.90}^2(z_{0.625}^3 - z_{0.70}^3\gamma_4))\gamma_6)) \\
&/(z_{0.75}(-z_{0.90}^2 + z_{0.75}^2)z_{0.70}(-z_{0.625}(-z_{0.90}^2 + z_{0.625}^2)(-z_{0.75}^2 + z_{0.625}^2) \\
&+ z_{0.70}(-z_{0.90}^2 + z_{0.70}^2)(-z_{0.75}^2 + z_{0.70}^2)\gamma_4)(z_{0.625}^5 - z_{0.70}^5\gamma_4 + z_{0.90}^2(-z_{0.625}^3 + z_{0.70}^3\gamma_4))))
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
c_6 = {} & (z_{0.75}\gamma_2(-z_{0.625}^3 + z_{0.70}^3\gamma_4 + z_{0.75}^2(z_{0.625} - z_{0.70}\gamma_4)) + z_{0.90}\gamma_2(z_{0.625}^3 - z_{0.70}^3\gamma_4 + \\
& z_{0.90}^2(-z_{0.625} + z_{0.70}\gamma_4))\gamma_6)/(2z_{0.90}^2(z_{0.90} - z_{0.75})z_{0.75}(z_{0.90} + z_{0.75})(z_{0.625}(-z_{0.90}^2 + z_{0.625}^2)(-z_{0.75}^2 + z_{0.625}^2) \\
& + (z_{0.90} - z_{0.70})z_{0.70}(z_{0.90} + z_{0.70})(-z_{0.75}^2 + z_{0.70}^2)\gamma_4)).
\end{aligned} \tag{30}
$$

Estimates of γ_1-γ_6 based on the percentiles in (13)-(18) for a sample of size N can be determined by finding the j and $j+1$ integer values, and their corresponding expected values of the order statistics $E[q(Z)_{j:N}]$ and $E[q(Z)_{j+1:N}]$, by making use of the following equation (Headrick & Pant, 2012b; Johnson, Kotz, & Balakrishnan, 1994)

$$
E[q(Z)_{j:N}] = \frac{N!}{(j-1)!(N-j)!}\int_{-\infty}^{+\infty} q(z)\phi(z)\{\Phi(z)\}^{j-1}\{1-\Phi(z)\}^{N-j}dz \tag{31}
$$

such that

$$
E[q(Z)_{j:N}] \le q(z_p) \le E[q(Z)_{j+1:N}] \tag{32}
$$

and subsequently solve the equation

$$
q(z_p) = (u)E[q(Z)_{j:N}] + (1-u)E[q(Z)_{j+1:N}] \tag{33}
$$

for $0 \le u \le 1$. Thus, an empirical estimate of $q(z_p)$ can then be obtained based on the order statistics of a sample of size n as $q(z_p) \simeq q(Z_p) = (u)q(Z)_{j:N} + (1-u)q(Z)_{j+1:N}$.

3. Comparison of the MOM and the MOP on Distribution Fitting

3.1 Theoretical distributions

One of the theoretical advantages of the MOP has over the MOM is that the MOP is not limited to the existence of moments (cumulants). Some theoretical distributions, such as the Cauchy and the t distribution with 2 degrees of freedom, do not have either mean or variance. Under this circumstance, the MOM does not work since it requires the first and the second cumulants to fit a distribution. Table 1 summarizes the results of fitting various t distributions by using the MOM and the MOP transformation methods. It is noted that the t distribution needs to have degrees of freedom greater than four in order to have finite first four moments (α_1-α_4) to exist. Therefore, the MOM fails to fit the t_1 and the t_3 distributions since it can not obtain valid α_1-α_4.

On the other hand, the t_1 and the t_3 distributions can be fitted fairly well by using the MOP technique since their percentiles always exit, regardless of the existence of their moments or cumulants. In addition to the special cases of the t distributions (i.e., t_1 and t_3), Table 1 also shows the superior of the MOP over the MOM in terms of fitting the t

distributions where the moments are all finite. Specifically, the MOP approximation fits the t_7 distribution better than the MOM. In addition to the graphical illustrations, Euclidean distances (ED) are provided to compare the accuracy of the data fitting for the MOP and the MOM. The expression of ED is defined as: $D = \sqrt{\sum(O-E)^2}$, where O is the observed proportion in each interval (i.e., $\theta_{10}, \theta_{25}, \theta_{50}, \theta_{75}, \theta_{90}$) and E is the expected proportion in each interval for both the MOP and the MOM. The smaller Euclidean distances of the MOP, as shown in Table 1, also indicate the more accurate data fit of the MOP over the MOM.

Finally, in addition to the t distributions, the χ_2^2 and the F distributions were also fit by using both the MOM and the MOP approximations. The results summarized in Tables 2 and 3 show that the MOP had an overall smaller Euclidean distances than the MOM, indicating that the MOP has advantages over the MOM.

Table 1: The MOM and MOP Power Method PDF Approximations (Dashed Lines) to t Distributions

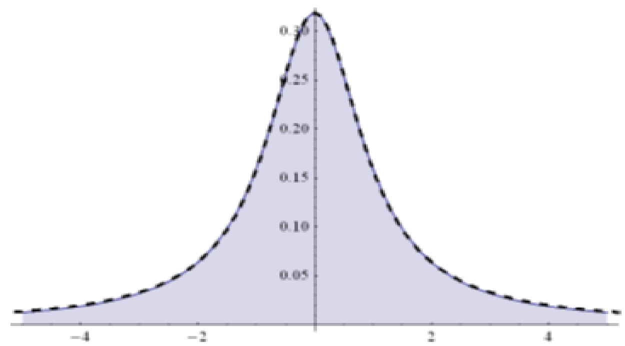
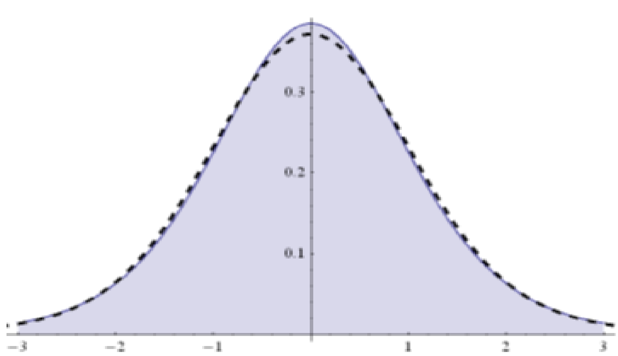
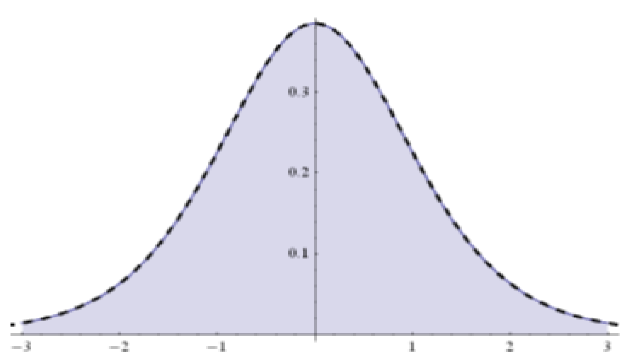
t_1 (Cauchy)	
MOM	MOP
Not applicable	
t_3	
MOM	MOP
Not applicable	
t_7	
MOM	MOP
	
ED = 0.030058243	ED $\approx$ 0

Table 2: The MOM and MOP Power Method PDF Approximations (Dashed Lines) to Chi-Squared Distributions

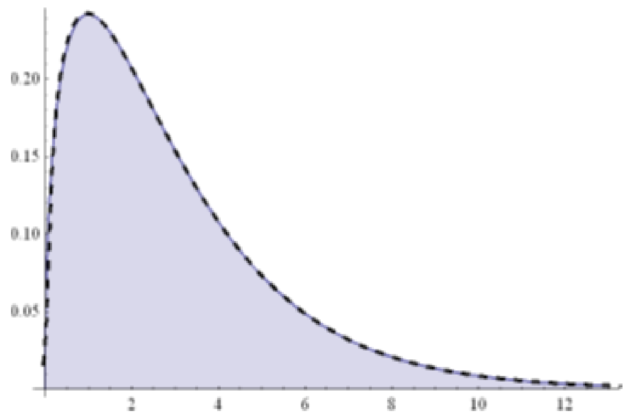
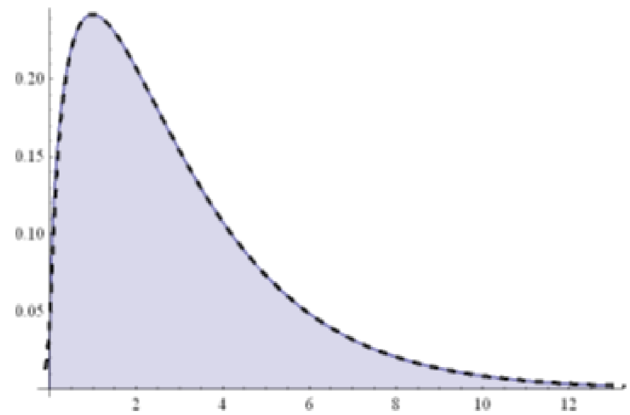
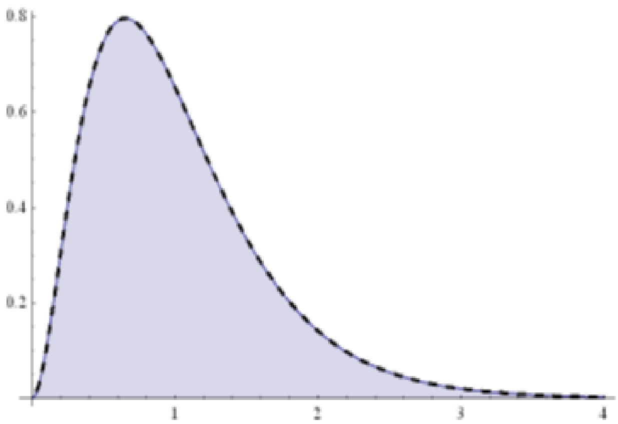
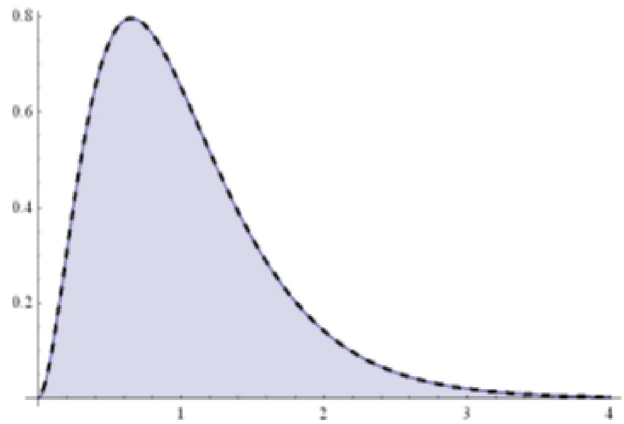
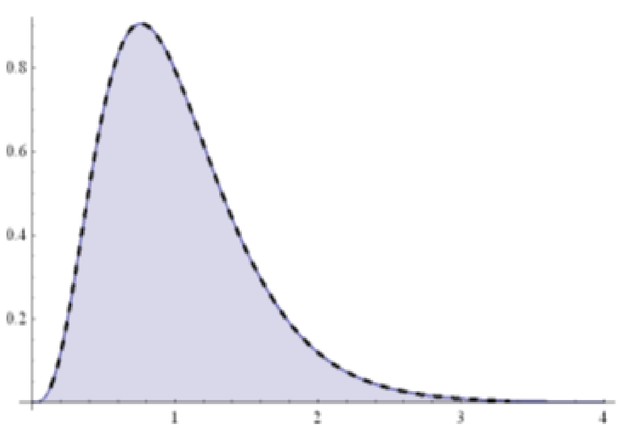
χ^2_3	
MOM	MOP
	
ED = 0.002596424	ED = 2.82843E-05
χ^2_6	
MOM	MOP
	
ED = 0.000352704	ED = 1E-05

Table 3: The MOM and MOP Power Method PDF Approximations (Dashed Lines) to F Distributions

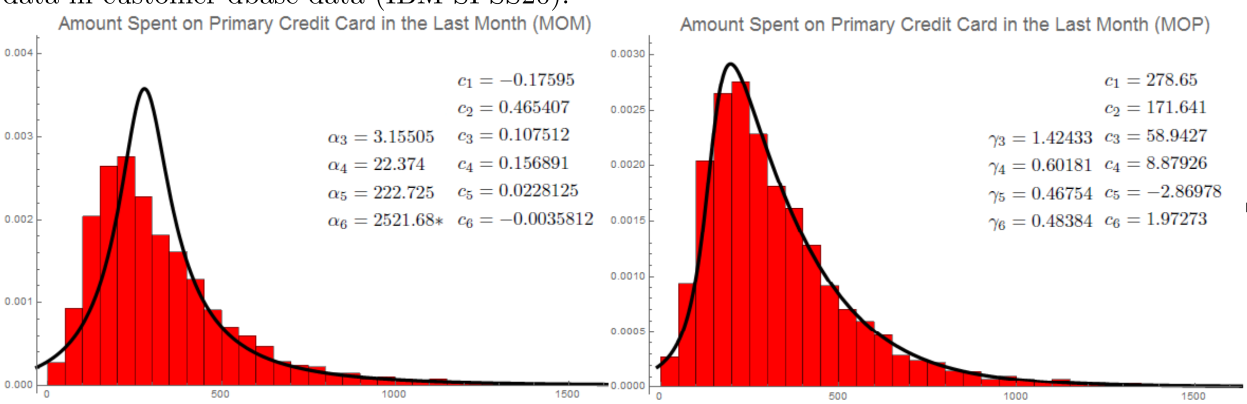
$F_{6,100}$	
MOM	MOP
	
ED = 0.000149887	ED ≈ 0
$F_{9,100}$	
MOM	MOP
	
ED = 7.27324E-05	ED = 1E-06

3.2 Empirical distribution

Presented in Figure 1 are the fifth ordered MOM and the MOP pdfs superimposed on the histogram of the SPSS *customer_dbase* data from IBM Corp. (2011). This is a data file that concerns a company's efforts to use the information in its data warehouse to make special offers to customers who are most likely to reply. Specifically, these data are the amount each customer spent on their primary credit card in the last month. The parameters (α_{3-6} and γ_{3-6}) associated with Figure 1 were based on a sample size of $N = 5,000$ participants. Note that to fit the PM distributions to the data, a linear transformation has to be imposed on $q(z) = Aq(z)+B$ for the MOM procedure. Specifically, in the context of the MOM, $A = s/\sigma$, $B = m - A\mu$. On the other hand, the MOP does not require a linear transformation to fit empirical data.

Visual inspection of the approximation in Figure 1 indicate that the MOP technique provides a more accurate fit to the actual data over the MOM. Further, Euclidean distances were calculated in each interval to compare the accuracy of the data fitting for the MOP and the MOM. The results summarized in Table 4 indicate that the MOP has a more accurate data fit because the ED of the MOP is less than that of the MOM.

Figure 1: Histograms and estimates based on the MOM and the MOP for the credit card spent data in customer dbase data (IBM SPSS20).



*This value had to be increased to 2925 to ensure a valid fifth-order power method PDF

Table 4: Percentiles, expected proportions, observed proportions and the Euclidean distances (ED) for the MOP and the MOM approximations associated with the SPSS *customer_dbase* data ($N = 5,000$) from IBM Corp. (2011).

Percentile	Expected Prop.	Obs Prop. (MOM)	Obs Prop. (MOP)
10	0.10	0.1146	0.1000
25	0.25	0.3454	0.2560
50	0.50	0.5382	0.5000
75	0.75	0.7166	0.7520
90	0.90	0.8822	0.9000
		$D = 0.110481$	$D = 0.0063246$

4. Multivariate Distributions

4.1 Pearson correlations for the system of the MOM

Suppose a T-variate distribution based on conventional PM polynomials are desired. The PM equation for solving intermediate Pearson correlations (r_{jk}) for specified Pearson correlations (ρ_{jk}) for distributions j and k is (Headrick, 2010, p.30, Eq. 2.59)

$$\begin{aligned}\rho_{jk} &= 3c_{5j}c_{1k} + 3c_{5j}c_{3k} + 9c_{5j}c_{5k} + c_{1j}(c_{1k} + c_{3k} + 3c_{5k}) + c_{2j}c_{2k}r_{jk} + 3c_{4j}c_{2k}r_{jk} + 15c_{6j}c_{2k}r_{jk} \\ &+ 3c_{2j}c_{4k}r_{jk} + 9c_{4j}c_{4k}r_{jk} + 45c_{6j}c_{4k}r_{jk} + 15c_{2j}c_{6k}r_{jk} + 45c_{4j}c_{6k}r_{jk} + 225c_{6j}c_{6k}r_{jk} + 12c_{5j}c_{3k}r_{jk}^2 \\ &+ 72c_{5j}c_{5k}r_{jk}^2 + 6c_{4j}c_{4k}r_{jk}^3 + 60c_{6j}c_{4k}r_{jk}^3 + 60c_{4j}c_{6k}r_j^3k + 600c_{6j}c_{6k}r_{jk}^3 + 24c_{5j}c_{5k}r_{jk}^4 \\ &+ 120c_{6j}c_{6k}r_{jk}^5 + c_{3j}(c_{1k} + c_{3k} + 3c_{5k} + 2c_{3k}r_{jk}^2 + 12c_{5k}r_{jk}^2). \end{aligned} \quad (34)$$

Note that the purpose of the intermediate Pearson correlations (r_{jk}) in Equation (34) is to adjust for the effect of the transformation, such that the transformed T-variate distribution will have the specified Pearson correlations (ρ_{jk}). $c'_{ji}s$ and $c'_{ki}s$ $(i = 1, \ldots, 6)$ coefficients are obtained from specifying α_{ji} and α_{ki} for distributions j and k, respectively.

4.2 Spearman correlations for the system of the MOP

We assume that the variates $Y_j = q(Z_j)$ and $Y_k = q(Z_k)$ in (1) produce valid pdfs and are thus increasing monotonic transformations in Z_j and Z_k. This implies that the rank orders of Y_j ($R(Y_j)$) and Z_k ($R(Z_k)$) are identical and thus will have rank correlations of $\rho_{R(Y_j),R(Z_j)} = \rho_{R(Y_k),R(Z_k)} = 1$.

Given these assumptions, suppose it is desired to simulate a T-variate distribution from the quantile functions in Equation (1) with a specified $T \times T$ Spearman correlation matrix $(\epsilon_{jk}, j, k = 1, \ldots, T)$ and where each distribution has specified γ_3-γ_6. The Spearman correlation between distributions j and k, ϵ_{jk}, can be obtained from the derivation of $\rho_{R(Z_j),R(Z_k)}$ given in Moran (1948). That is, because (1) is a strictly increasing monotonic transformation, $\epsilon_{jk} = \rho_{R(Z_j),R(Z_k)}$ and thus the intermediate correlations r_{jk} can be obtained by numerically solving the equation (Headrick, 2010, Eq. 4.34)

$$\epsilon_{jk} = \frac{6}{\pi}\left[(\frac{N-2}{N+1})\sin^{-1}(\frac{r_{j,k}}{2}) + \frac{1}{N+1}\sin^{-1}(r_{j,k})\right]. \quad (35)$$

5. The Procedure for Simulation and Monte Carlo Study

5.1 Simulation data

To implement the method for simulating normal and nonnormal distributions with specified γ_3-γ_6 and Spearman correlations, we suggest the following steps:

1. Specify the values of γ_3-γ_6 for the T transformations of the forms in (1) i.e. $p_1(Z_1), \ldots, p_T(Z_T)$ and obtain the constants of c_1-c_6 for each polynomial by solving equations (25)-(30) using the specified values of γ_1-γ_6 for each distribution. Specify a $T \times T$ matrix of Spearman correlations between $p_j(Z_j)$ and $p_k(Z_k)$, where $j < k \in \{1, 2, \ldots, T\}$.

2. Compute the intermediate correlations (IC) $r_{j,k}$ by substituting the solutions of the constants from Step 1 into (35) and then solve for $r_{j,k}$. Repeat this step separately for all $T(T-1)/2$ pairwise combinations of correlations.

3. Assemble the ICs into a $T \times T$ matrix and decompose this matrix using a Cholesky factorization. Note that this step requires the IC matrix to be positive definite.

4. Use the results of the Cholesky factorization from Step 3 to generate T standard normal variables $(Z_1, \ldots, Z_T)$ correlated at the intermediate levels as follows

$$\begin{aligned} Z_1 &= a_{11}V_1 \\ Z_2 &= a_{12}V_1 + a_{22}V_2 \\ &\vdots \\ Z_j &= a_{1j}V_1 + a_{2j}V_2 + \ldots + a_{ij}V_i + \ldots + a_{jj}V_j \\ &\vdots \\ Z_T &= a_{1T}V_1 + a_{2T}V_2 + \ldots + a_{iT}V_i + \ldots + a_{jT}V_j + \ldots + a_{TT}V_T, \end{aligned} \tag{36}$$

   where $V_1, \ldots, V_T$ are independent standard normal random variables and where a_{ij} represents the element in the i-th row and the j-th column of the matrix associated with the Cholesky factorization performed in Step 3.

5. Substitute $Z_1, \ldots, Z_T$ from step 4 into T equations of the form in (1), as noted in step 1, to generate the PM distributions with the specified values of γ_3-γ_6 and Spearman correlations.

To demonstrate the steps above and evaluate the proposed procedure, a comparison between the MOP and the MOM procedures is subsequently described. Specifically, the distributions in Figure 2 are used as a basis for a comparison using the specified correlation matrices in Table 5. Tables 6-8 give the solved IC matrices for the MOM and the MOP procedures with samples of sizes 25 and 750, respectively.

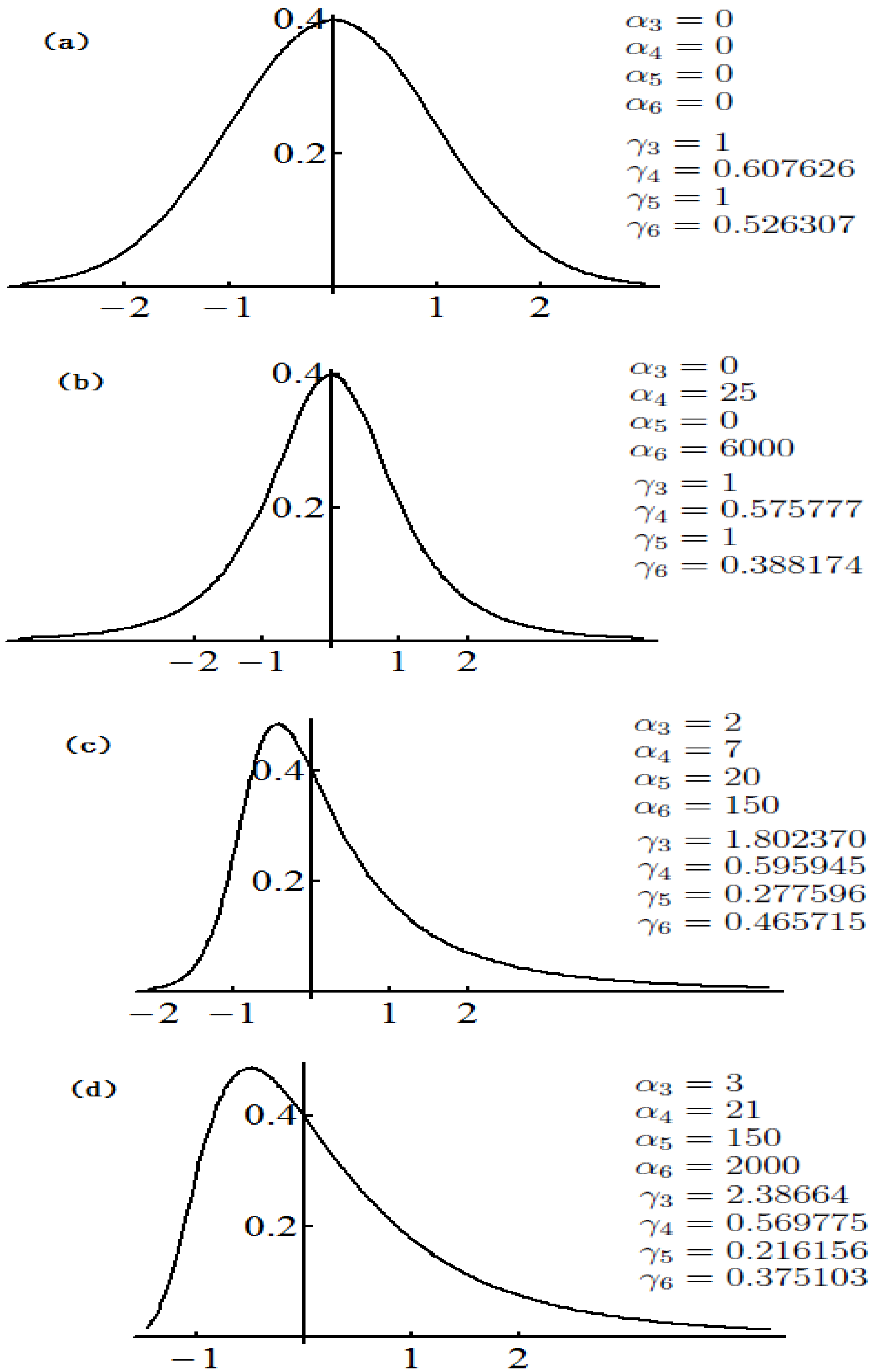


Figure 2: Two symmetric (a)-(b) and two asymmetric distributions (c)-(d) with their MOM and MOP parameters.

In terms of the simulation, a Fortran algorithm was written for both methods to generate 25,000 independent sample estimates for the specified parameters of: (i) conventional moments (α_3-α_6) and Pearson correlation; and (ii) moments for MOP (γ_3-γ_6) and Spearman correlation. All estimates were based on sample sizes of $N = 25$ and $N = 750$. The formulae used for computing estimates of α_3-α_6 were based on Fisher's k-statistics i.e. the formulae currently used by most commercial software packages such as SAS, SPSS, Minitab, and so forth. The formulae used for computing estimates of γ_3-γ_6 were based on (13)-(18). Note that the estimates of percentiles were based on (31). The estimates for the Pearson and Spearman correlations were both transformed using Fisher's z transformation. Bias-corrected accelerated bootstrapped average (median) estimates, confidence intervals (CIs), and standard errors were subsequently obtained for the estimates associated with the parameters using 10,000 resamples via the commercial software package Spotfire S+ TIBCO Software (2008). The bootstrap results for the estimates of the medians and CIs associated with the Pearson and Spearman correlations were transformed back to their original matrices (i.e. estimates for the Pearson and Spearman correlations). Further, if a parameter (P) was outside its associated bootstrap CI, then an index of relative bias (RB) was computed for the estimate (E) as: RB= $(((E - P))/P) \times 100$. Note that the small amount of bias associated with any bootstrap CI containing a parameter was considered negligible and thus not reported. The results of the simulation are reported in Tables 12-15 and are discussed in the next section.

Table 5: Specified correlation matrix for the distributions in Figure 2 (a)-(d).

	1	2	3	4
1	1			
2	0.75	1		
3	0.70	0.60	1	
4	0.55	0.40	0.65	1

Table 6: Intermediate correlations for the MOM procedure.

	1	2	3	4
1	1			
2	0.8329812	1		
3	0.777728	0.7039272	1	
4	0.657191	0.502905	0.722099	1

Table 7: Intermediate correlations for the MOP procedure ($N = 25$).

	1	2	3	4
1	1			
2	0.787157463	1		
3	0.738500867	0.638650356	1	
4	0.587658483	0.431321177	0.688961108	1

Table 8: Intermediate correlations for the MOP procedure ($N = 750$).

	1	2	3	4
1	1			
2	0.766121007	1		
3	0.717483143	0.618734137	1	
4	0.568694702	0.41634356	0.668342174	1

Table 9: Cholesky decompositions for the MOM procedure.

$a_{11} = 1$	$a_{12} = 0.832981$	$a_{13} = 0.777728$	$a_{14} = 0.657191$
0	$a_{22} = 0.553301$	$a_{23} = 0.101381$	$a_{24} = -0.080467$
0	0	$a_{33} = 0.620372$	$a_{34} = 0.353242$
0	0	0	$a_{44} = 0.660943$

Table 10: Cholesky decompositions for the MOP procedure ($N = 25$).

$a_{11} = 1$	$a_{12} = 0.787157$	$a_{13} = 0.738501$	$a_{14} = 0.587658$
0	$a_{22} = 0.616752$	$a_{23} = 0.092961$	$a_{24} = -0.050683$
0	0	$a_{33} = 0.667813$	$a_{34} = 0.388861$
0	0	0	$a_{44} = 0.707726$

Table 11: Cholesky decompositions for the MOP procedure ($N = 750$).

$a_{11} = 1$	$a_{12} = 0.766121$	$a_{13} = 0.717483$	$a_{14} = 0.568695$
0	$a_{22} = 0.642696$	$a_{23} = 0.107446$	$a_{24} = -0.0301004$
0	0	$a_{33} = 0.6882394$	$a_{34} = 0.382930$
0	0	0	$a_{44} = 0.727355$

Table 12: Skew (α_3), Kurtosis (α_4), fifth order (α_5), sixth order (α_6) results for the MOM procedure ($N = 25$).

Dist	Parameter		Estimate	Standard Error	95% Bootstrap CI		Relative Bias %	Relative SE %
1	α_3	0	0.0044	0.0030	−0.0010	0.0104	–	–
	α_4	0	0.0018	0.0058	−0.0089	0.0139	–	–
	α_5	0	−0.0087	0.0132	−0.0340	0.0173	–	–
	α_6	0	−0.0415	0.0353	−0.1107	0.0270	–	–
2	α_3	0	0.0110	0.0101	−0.0071	0.0329	–	–
	α_4	25	4.4700	0.0273	4.4137	4.5210	−82.1200	0.6096
	α_5	0	0.0173	0.1441	−0.1030	0.4631	–	–
	α_6	6000	38.1600	0.5528	37.0926	39.2513	−99.3640	1.4486
3	α_3	2	1.5440	0.0052	1.5337	1.5541	−22.8000	0.3381
	α_4	7	3.3490	0.0224	3.3029	3.3905	−52.1571	0.6677
	α_5	20	5.7420	0.0962	5.5600	5.9334	−71.2900	1.6759
	α_6	150	10.9600	0.3819	10.2336	11.7427	−92.6933	3.4845
4	α_3	3	1.7390	0.0080	1.7232	1.7547	−42.0333	0.4585
	α_4	21	5.6180	0.0303	5.5613	5.6809	−73.2476	0.5399
	α_5	150	12.2400	0.1583	11.8814	12.5125	−91.8400	1.2933
	α_6	2000	42.3800	0.6683	41.1088	43.7196	−97.8810	1.5769

Table 13: Skew (α_3), Kurtosis (α_4), fifth order (α_5), sixth order (α_6) results for the MOM procedure ($N = 750$).

Dist	Parameter		Estimate	Standard Error	95% Bootstrap CI		Relative Bias %	Relative SE %
1	α_3	0	0.0000	0.0006	−0.0012	0.0010	–	–
	α_4	0	−0.0005	0.0011	−0.0029	0.0016	–	–
	α_5	0	0.0013	0.0025	−0.0035	0.0063	–	–
	α_6	0	−0.0074	0.0061	−0.0189	0.0052	–	–
2	α_3	0	0.0090	0.0100	−0.0092	0.0295	–	–
	α_4	25	18.61	0.1145	18.3819	18.8304	−25.5600	0.6153
	α_5	0	1.4340	2.1510	−2.7690	5.5213	–	–
	α_6	6000	1709	37.54	1638.34	1785.575	−71.5167	2.1966
3	α_3	2	1.981	0.0023	1.9769	1.9857	−0.9500	0.1143
	α_4	7	6.697	0.0176	0.0176	6.7340	−4.3286	0.2631
	α_5	20	19.97	0.1942	19.5671	20.3388	–	–
	α_6	150	109.6	2.3150	105.9363	115.3542	−26.9333	2.1122
4	α_3	3	2.868	0.0059	2.8567	2.88	−4.4000	0.2048
	α_4	21	18.22	0.0763	18.0683	18.3680	−13.2381	0.4190
	α_5	150	110.5	1.2480	108.5032	113.0405	−26.3333	1.1294
	α_6	2000	1140	20.5200	1101.571	1181.889	−43.0000	1.8000

Table 14: Left-right tail-weight ratio (γ_3) and tail-weight factor (γ_4), fifth order (γ_5), sixth order (γ_6) results for the MOP procedure ($N = 25$).

Dist	Parameter		Estimate	SE	95% Bootstrap CI		Relative Bias %	Relative SE %
1	γ_3	1	0.9962	0.0057	0.9846	1.0075	–	–
	γ_4	0.607626	0.6155	0.0012	0.6132	0.6180	1.2959	0.1955
	γ_5	1	0.9978	0.0034	0.9911	1.0044	–	–
	γ_6	0.526307	0.5294	0.0008	0.5281	0.5311	0.5877	0.1500
2	γ_3	1	1.0000	0.0050	0.9903	1.0097	–	–
	γ_4	0.575777	0.5837	0.0012	0.5815	0.5864	1.3761	0.2118
	γ_5	1	0.9913	0.0046	0.9822	1.0006	–	–
	γ_6	0.388174	0.3981	0.0009	0.3964	0.3999	2.5571	0.2301
3	γ_3	1.802370	1.8120	0.0093	1.7930	1.8301	–	–
	γ_4	0.595945	0.6051	0.0012	0.6075	0.6075	1.5362	0.1991
	γ_5	0.277596	0.2781	0.0010	0.2762	0.2800	–	–
	γ_6	0.465715	0.4745	0.0011	0.4723	0.4767	1.8863	0.2310
4	γ_3	2.38664	2.3680	0.0121	2.3440	2.3914	–	–
	γ_4	0.569775	0.5830	0.0012	0.5807	0.5807	2.3211	0.2063
	γ_5	0.216156	0.2159	0.0009	0.2139	0.2176	–	–
	γ_6	0.375103	0.3828	0.0011	0.3809	0.3850	2.0520	0.2861

Table 15: Left-right tail-weight ratio (γ_3) and tail-weight factor (γ_4), fifth order (γ_5), sixth order (γ_6) results for the MOP procedure ($N = 750$).

Dist	Parameter		Estimate	SE	95% Bootstrap CI		Relative Bias %	Relative SE %
1	γ_3	1	0.9998	0.0009	0.0009	1.0016	–	–
	γ_4	0.607626	0.6078	0.0002	0.6074	0.6082	–	–
	γ_5	1	1.0000	0.0005	0.9992	1.0014	–	–
	γ_6	0.526307	0.5264	0.0002	0.5261	0.5267	–	–
2	γ_3	1	1.0000	0.0010	0.9982	1.0019	–	–
	γ_4	0.575777	0.5764	0.0002	0.5759	0.5769	0.1082	0.0404
	γ_5	1	1.0000	0.0009	0.9982	1.0018	–	–
	γ_6	0.388174	0.3886	0.0002	0.3882	0.3890	0.1097	0.0456
3	γ_3	1.802370	1.8050	0.0018	1.8016	1.8085	–	–
	γ_4	0.595945	0.5963	0.0002	0.5958	0.5968	–	–
	γ_5	0.277596	0.2776	0.0002	0.2771	0.2780	–	–
	γ_6	0.465715	0.4659	0.0002	0.4655	0.4663	–	–
4	γ_3	2.38664	2.3870	0.0027	2.3824	2.3926	–	–
	γ_4	0.569775	0.5701	0.0002	0.5696	0.5705	–	–
	γ_5	0.216156	0.2162	0.0002	0.2158	0.2165	–	–
	γ_6	0.375103	0.3758	0.0002	0.3755	0.3763	0.1858	0.0550

Table 16: Pearson correlation (Corr) results for the MOM procedure ($N = 25$).

Corr Parameter	Estimate	SE	95% Bootstrap CI		Relative Bias %	Relative SE
0.75	0.7954	0.0012	0.7944	0.7961	6.0551	0.1507
0.7	0.7222	0.0012	0.7210	0.7233	3.1695	0.1666
0.55	0.5911	0.0013	0.5894	0.5926	7.4662	0.2118
0.6	0.6411	0.0016	0.6392	0.6429	6.8462	0.2469
0.4	0.4469	0.0016	0.4444	0.4494	11.7210	0.3565
0.65	0.6876	0.0020	0.6855	0.6896	5.7774	0.2870

Table 17: Pearson correlation (Corr) results for the MOM procedure ($N = 750$).

Corr Parameter	Estimate	SE	95% Bootstrap CI		Relative Bias %	Relative SE
0.75	0.7544	0.0003	0.7542	0.7548	0.5930	0.0446
0.7	0.7007	0.0002	0.7005	0.7009	0.1019	0.0307
0.55	0.5520	0.0002	0.5517	0.5523	0.3569	0.0420
0.6	0.6028	0.0003	0.6024	0.6032	0.4631	0.0547
0.4	0.4024	0.0003	0.4019	0.4029	0.5980	0.0819
0.65	0.6518	0.0004	0.6514	0.6523	0.2838	0.0572

Table 18: Spearman correlation (Corr) results for the MOP procedure ($N = 25$).

Corr Parameter	Estimate	SE	95% Bootstrap CI		Relative Bias %	Relative SE
0.75	0.7653	0.0018	0.7646	0.7677	2.0464	0.2398
0.7	0.7143	0.0020	0.7131	0.7169	2.0422	0.2794
0.55	0.5652	0.0018	0.5638	0.5685	2.7585	0.3153
0.6	0.6152	0.0018	0.6138	0.6177	2.5287	0.2898
0.4	0.4121	0.0017	0.4100	0.4162	3.0171	0.4208
0.65	0.6655	0.0016	0.6632	0.6669	2.3913	0.2469

Table 19: Spearman correlation (Corr) results for the MOP procedure ($N = 750$).

Corr Parameter	Estimate	SE	95% Bootstrap CI		Relative Bias %	Relative SE
0.75	0.7502	0.0003	0.7500	0.7505	–	–
0.7	0.7004	0.0003	0.7000	0.7007	0.0509	0.0460
0.55	0.5503	0.0003	0.5498	0.5508	–	–
0.6	0.6002	0.0003	0.5998	0.6006	–	–
0.4	0.3996	0.0003	0.3991	0.4001	–	–
0.65	0.6503	0.0003	0.6500	0.6507	–	–

5.2 Discussion and conclusion

One of the advantages that the MOP has over the MOM is that it is far less biased when sampling is from distributions with more severe departures from normality. Moreover, inspection of the simulation results in Tables 12-15 clearly indicate this conclusion. Specifically, the superiority that the MOP estimates (γ_3-γ_6) have over their corresponding MOM counterparts (α_3-α_6). For example, with samples size of $N = 25$, the estimates of skew, kurtosis, fifth and sixth moments for Distribution 4 (heavy-skewed and heavy-tailed) were, on average, 42.03% and 73.425%, 91.84%, and 97.88% below their associated population parameters, whereas the estimates of γ_2 and γ_4 were 2.32% and 2.05% over their respective parameters. Tables 12-15 also show that γ_3-γ_6 are more efficient estimators as their relative standard errors RSE = (standard error/estimate)$\times$100 are considerably smaller than the MOM estimators of skew and kurtosis. Presented in Tables 16-19 are the results associated with the conventional Pearson and the Spearman correlations. Overall inspection of these tables indicates that the Spearman correlation is superior to the Pearson correlation in terms of RB.

In summary, the proposed MOP procedure is an attractive alternative to the traditional MOM procedure due to its distinct advantages when distributions with large departures from normality are used. In addition, it can generate distributions where the mean and/or the variance does not exist (e.g., Cauchy, t distribution with 1 or 2 degrees of freedom). Furthermore, in practice situations where only proportions or percentiles are provided to public, the MOP is more preferable than the MOM since the required moments for MOM are not available.

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The system of equations of $\alpha_1 - \alpha_6$ given by Headrick and Kowalchuk (2007), Eqs. (A9)-(A12)

$$\alpha_1 = c_1 + c_3 + 3c_5 \tag{37}$$

$$\alpha_2 = c_1^2 + c_2^2 + 2c_1(c_3 + 3c_5) + 6c_2(c_4 + 5c_6) + 3(c_3^2 + 10c_3c_5 + 5(c_4^2 + 7c_5^2 + 14c_4c_6 + 63c_6^2)) \tag{38}$$

$$\begin{aligned}\alpha_3 = {} & c_1^3 + 3c_1^2(c_3 + 3c_5) + 3c_1(c_2^2 + 6c_2(c_4 + 5c_6) + 3(c_3^2 + 10c_3c_5 + 5(c_4^2 + 7c_5^2 + 14c_4c_6 + 63c_6^2))) \\ & + 3(3c_2^2(c_3 + 5c_5) + 30c_2(c_3(c_4 + 7c_6) + 7c_5(c_4 + 9c_6)) \\ & + 5(c_3^3 + 21c_3^2c_5 + 21c_3(c_4^2 + 9c_5^2 + 18c_4c_6 + 99c_6^2) + 63c_5(3c_4^2 + 11c_5^2 + 66c_4c_6 + 429c_6^2)))\end{aligned} \tag{39}$$

$$\begin{aligned}\alpha_4 = {} & c_1^4 + 4c_1^3(c_3 + 3c_5) + 6c_1^2(c_2^2 + 6c_2(c_4 + 5c_6) + 3(c_3^2 + 10c_3c_5 + 5(c_4^2 + 7c_5^2 + 14c_4c_6 + 63c_6^2))) \\ & + 12c_1(3c_2^2(c_3 + 5c_5) + 30c_2(c_3(c_4 + 7c_6) + 7c_5(c_4 + 9c_6)) \\ & + 5(c_3^3 + 21c_3^2c_5 + 21c_3(c_4^2 + 9c_5^2 + 18c_4c_6 + 99c_6^2) + 63c_5(3c_4^2 + 11c_5^2 + 66c_4c_6 + 429c_6^2))) \\ & + 3(c_2^4 + 20c_2^3(c_4 + 7c_6) + 30c_2^2(c_3^2 + 14c_3c_5 + 7(c_4^2 + 9c_5^2 + 18c_4c_6 + 99c_6^2)) \\ & + 420c_2(c_3^2(c_4 + 9c_6) + 18c_3c_5(c_4 + 11c_6) + 3(c_4^3 + 33c_4^2c_6 + 429c_6(c_5^2 + 5c_6^2) + 33c_4(c_5^2 + 13c_6^2))) \\ & + 35(c_3^4 + 36c_3^3c_5 + 54c_3^2(c_4^2 + 22c_4c_6 + 11(c_5^2 + 13c_6^2)) + 396c_3c_5(3c_4^2 + 78c_4c_6 + 13(c_5^2 + 45c_6^2)) \\ & + 99(c_4^4 + 52c_4^3c_6 + 78c_4^2(c_5^2 + 15c_6^2) + 780c_4(3c_5^2c_6 + 17c_6^3) + 195(c_5^4 + 102c_5^2c_6^2 + 323c_6^4)))) - 3\end{aligned} \tag{40}$$

$$\begin{aligned}\alpha_5 = {} & c_1^5 + 5c_1^4(c_3 + 3c_5) + 10c_1^3(c_2^2 + 6c_2(c_4 + 5c_6) + 3(c_3^2 + 10c_3c_5 + 5(c_4^2 + 7c_5^2 + 14c_4c_6 + 63c_6^2))) \\ & + 30c_1^2(3c_2^2(c_3 + 5c_5) + 30c_2(c_3(c_4 + 7c_6) + 7c_5(c_4 + 9c_6)) \\ & + 5(c_3^3 + 21c_3^2c_5 + 21c_3(c_4^2 + 9c_5^2 + 18c_4c_6 + 99c_6^2) + 63c_5(3c_4^2 + 11c_5^2 + 66c_4c_6 + 429c_6^2))) \\ & + 15c_1(c_2^4 + 20c_2^3(c_4 + 7c_6) + 30c_2^2(c_3^2 + 14c_3c_5 + 7(c_4^2 + 9c_5^2 + 18c_4c_6 + 99c_6^2)) \\ & + 420c_2(c_3^2(c_4 + 9c_6) + 18c_3c_5(c_4 + 11c_6) + 3(c_4^3 + 33c_4^2c_6 + 429c_6(c_5^2 + 5c_6^2) + 33c_4(c_5^2 + 13c_6^2))) \\ & + 35(c_3^4 + 36c_3^3c_5 + 54c_3^2(c_4^2 + 22c_4c_6 + 11(c_5^2 + 13c_6^2)) + 396c_3c_5(3c_4^2 + 78c_4c_6 + 13(c_5^2 + 45c_6^2)) \\ & + 99(c_4^4 + 52c_4^3c_6 + 78c_4^2(c_5^2 + 15c_6^2) + 780c_4(3c_5^2c_6 + 17c_6^3) + 195(c_5^4 + 102c_5^2c_6^2 + 323c_6^4)))) \\ & + 15(5c_2^4(c_3 + 7c_5) + 140c_2^3(c_3(c_4 + 9c_6) + 9c_5(c_4 + 11c_6)) \\ & + 70c_2^2(c_3^3 + 27c_3^2c_5 + 27c_3(c_4^2 + 11c_5^2 + 22c_4c_6 + 143c_6^2) + 99c_5(3c_4^2 + 13c_5^2 + 78c_4c_6 + 585c_6^2)) \\ & + 1260c_2(c_3^3(c_4 + 11c_6) + 33c_3^2c_5(c_4 + 13c_6) + 11c_3(c_4^3 + 39c_4^2c_6 + 39c_4(c_5^2 + 15c_6^2) \\ & + 195c_6(3c_5^2 + 17c_6^2)) + 143c_5(c_4^3 + 45c_4^2c_6 + 255c_6(c_5^2 + 19c_6^2) + 15c_4(c_5^2 + 51c_6^2))) \\ & + 63(c_3^5 + 55c_3^4c_5 + 110c_3^3(c_4^2 + 26c_4c_6 + 13(c_5^2 + 15c_6^2)) + 4290c_3^2c_5(c_4^2 + 30c_4c_6 + 5(c_5^2 + 51c_6^2)) \\ & + 715c_3(c_4^4 + 60c_4^3c_6 + 90c_4^2(c_5^2 + 17c_6^2) + 1020c_4(3c_5^2c_6 + 19c_6^3) + 255(c_5^4 + 114c_5^2c_6^2 + 399c_6^4)) \\ & + 2145c_5(5c_4^4 + 340c_4^3c_6 + 6460c_4c_6(c_5^2 + 21c_6^2) + 170c_4^2(c_5^2 + 57c_6^2) \\ & + 323(c_5^4 + 210c_5^2c_6^2 + 2415c_6^4))))) - 10 * \alpha_3\end{aligned} \tag{41}$$

$$\begin{aligned}\alpha_6 = {} & c_1^6 + 6c_1^5(c_3 + 3c_5) + 15c_1^4(c_2^2 + 6c_2(c_4 + 5c_6) + 3(c_3^2 + 10c_3c_5 + 5(c_4^2 + 7c_5^2 + 14c_4c_6 + 63c_6^2))) \\ & + 60c_1^3(3c_2^2(c_3 + 5c_5) + 30c_2(c_3(c_4 + 7c_6) + 7c_5(c_4 + 9c_6)) \\ & + 5(c_3^3 + 21c_3^2c_5 + 21c_3(c_4^2 + 9c_5^2 + 18c_4c_6 + 99c_6^2) + 63c_5(3c_4^2 + 11c_5^2 + 66c_4c_6 + 429c_6^2))) \\ & + 45c_1^2(c_2^4 + 20c_2^3(c_4 + 7c_6) + 30c_2^2(c_3^2 + 14c_3c_5 + 7(c_4^2 + 9c_5^2 + 18c_4c_6 + 99c_6^2)) \\ & + 420c_2(c_3^2(c_4 + 9c_6) + 18c_3c_5(c_4 + 11c_6) + 3(c_4^3 + 33c_4^2c_6 + 429c_6(c_5^2 + 5c_6^2) + 33c_4(c_5^2 + 13c_6^2))) \\ & + 35(c_3^4 + 36c_3^3c_5 + 54c_3^2(c_4^2 + 22c_4c_6 + 11(c_5^2 + 13c_6^2)) + 396c_3c_5(3c_4^2 + 78c_4c_6 + 13(c_5^2 + 45c_6^2))\end{aligned}$$

$$
\begin{aligned}
&+99(c_4^4+52c_4^3c_6+78c_4^2(c_5^2+15c_6^2)+780c_4(3c_5^2c_6+17c_6^3)+195(c_5^4+102c_5^2c_6^2+323c_6^4))))\\
&+90c_1(5c_2^4(c_3+7c_5)+140c_2^3(c_3(c_4+9c_6)+9c_5(c_4+11c_6))\\
&+70c_2^2(c_3^3+27c_3^2c_5+27c_3(c_4^2+11c_5^2+22c_4c_6+143c_6^2)+99c_5(3c_4^2+13c_5^2+78c_4c_6+585c_6^2))\\
&+1260c_2(c_3^3(c_4+11c_6)+33c_3^2c_5(c_4+13c_6)+11c_3(c_4^3+39c_4^2c_6+39c_4(c_5^2+15c_6^2)\\
&+195c_6(3c_5^2+17c_6^2))+143c_5(c_4^3+45c_4^2c_6+255c_6(c_5^2+19c_6^2)+15c_4(c_5^2+51c_6^2)))\\
&+63(c_3^5+55c_3^4c_5+110c_3^3(c_4^2+26c_4c_6+13(c_5^2+15c_6^2))+4290c_3^2c_5(c_4^2+30c_4c_6+5(c_5^2+51c_6^2))\\
&+715c_3(c_4^4+60c_4^3c_6+90c_4^2(c_5^2+17c_6^2)+1020c_4(3c_5^2c_6+19c_6^3)+255(c_5^4+114c_5^2c_6^2+399c_6^4))\\
&+2145c_5(5c_4^4+340c_4^3c_6+6460c_4c_6(c_5^2+21c_6^2)+170c_4^2(c_5^2+57c_6^2)+323(c_5^4+210c_5^2c_6^2+2415c_6^4))))\\
&+15(c_2^6+42c_2^5(c_4+9c_6)+105c_2^4(c_3^2+18c_3c_5+9(c_4^2+11c_5^2+22c_4c_6+143c_6^2))\\
&+1260c_2^3(3c_3^2(c_4+11c_6)+66c_3c_5(c_4+13c_6)+11(c_4^3+39c_4c_5^2+39c_4^2c_6+585c_5^2c_6+585c_4c_6^2+3315c_6^3))\\
&+945c_2^2(c_3^4+44c_3^3c_5+66c_3^2(c_4^2+26c_4c_6+13(c_5^2+15c_6^2))+1716c_3c_5(c_4^2+30c_4c_6+5(c_5^2+51c_6^2))\\
&+143(c_4^4+60c_4^3c_6+90c_4^2(c_5^2+17c_6^2)+1020c_4(3c_5^2c_6+19c_6^3)+255(c_5^4+114c_5^2c_6^2+399c_6^4)))\\
&+20790c_2(c_3^4(c_4+13c_6)+52c_3^3c_5(c_4+15c_6)+26c_3^2(c_4^3+45c_4^2c_6+45c_4(c_5^2+17c_6^2)+255c_6(3c_5^2+19c_6^2))\\
&+780c_3c_5(c_4^3+51c_4^2c_6+323c_6(c_5^2+21c_6^2)+17c_4(c_5^2+57c_6^2))\\
&+39(c_4^5+85c_4^4c_6+9690c_4^2c_6(c_5^2+7c_6^2)+170c_4^3(c_5^2+19c_6^2)+33915c_6(c_5^4+46c_5^2c_6^2+115c_6^4)\\
&+1615c_4(c_5^4+126c_5^2c_6^2+483c_6^4)))+693(c_3^6+78c_3^5c_5+195c_3^4(c_4^2+30c_4c_6+15(c_5^2+17c_6^2))\\
&+3900c_3^3c_5(3c_4^2+102c_4c_6+17(c_5^2+57c_6^2))+2925c_3^2(c_4^4+68c_4^3c_6+3876c_4c_6(c_5^2+7c_6^2)\\
&+102c_4^2(c_5^2+19c_6^2)+323(c_5^4+126c_5^2c_6^2+483c_6^4))\\
&+19890c_3c_5(5c_4^4+380c_4^3c_6+7980c_4c_6(c_5^2+23c_6^2)+190c_4^2(c_5^2+63c_6^2)+399(c_5^4+230c_5^2c_6^2+2875c_6^4))\\
&+3315(c_4^6+114c_4^5c_6+285c_4^4(c_5^2+21c_6^2)+7980c_4^3(3c_5^2c_6+23c_6^3)+275310c_4c_6(c_5^4+50c_5^2c_6^2+135c_6^4)\\
&+5985c_4^2(c_5^4+138c_5^2c_6^2+575c_6^4)+9177(c_5^6+375c_5^4c_6^2+10125c_5^2c_6^4+19575c_6^6)))))-15*\alpha_4-10*\alpha_3^2-15
\end{aligned}
\tag{42}
$$