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Random Dynamics and Memory: Structure within Chaos? (MAA Invited Address / David Blackwell Lecture)

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David Blackwell Lecture

MAA MathFest 2008

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Madison, Wisconsin, USA

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- High school days in Atbara (Sudan), “motivated" by free lunches! from school principal.

The Plan

- A probabilist’s glossary:

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- Examples of random systems with memory: from feedback control to stock market fluctuations.

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- Evolution of random systems with memory.
- Encoding of the memory via "slicing" the random evolution path.

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- Concept of “random flow” to describe pathwise dynamics in state space.

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- Existence of non-linear stable/unstable smooth submanifolds of the state space near equilibria.

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$\mathcal{F}_t :=$ (completed) sub-σ-field of $\mathcal{F}$ generated by the evaluations $\omega \mapsto \omega(u), \;\; u \leq t, \quad t \in \mathbf{R}$.

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- Each increment $W(t_2) - W(t_1)$ is normal with mean zero and variance $t_2 - t_1$.

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Work by David Blackwell on Markov chains (discrete case).

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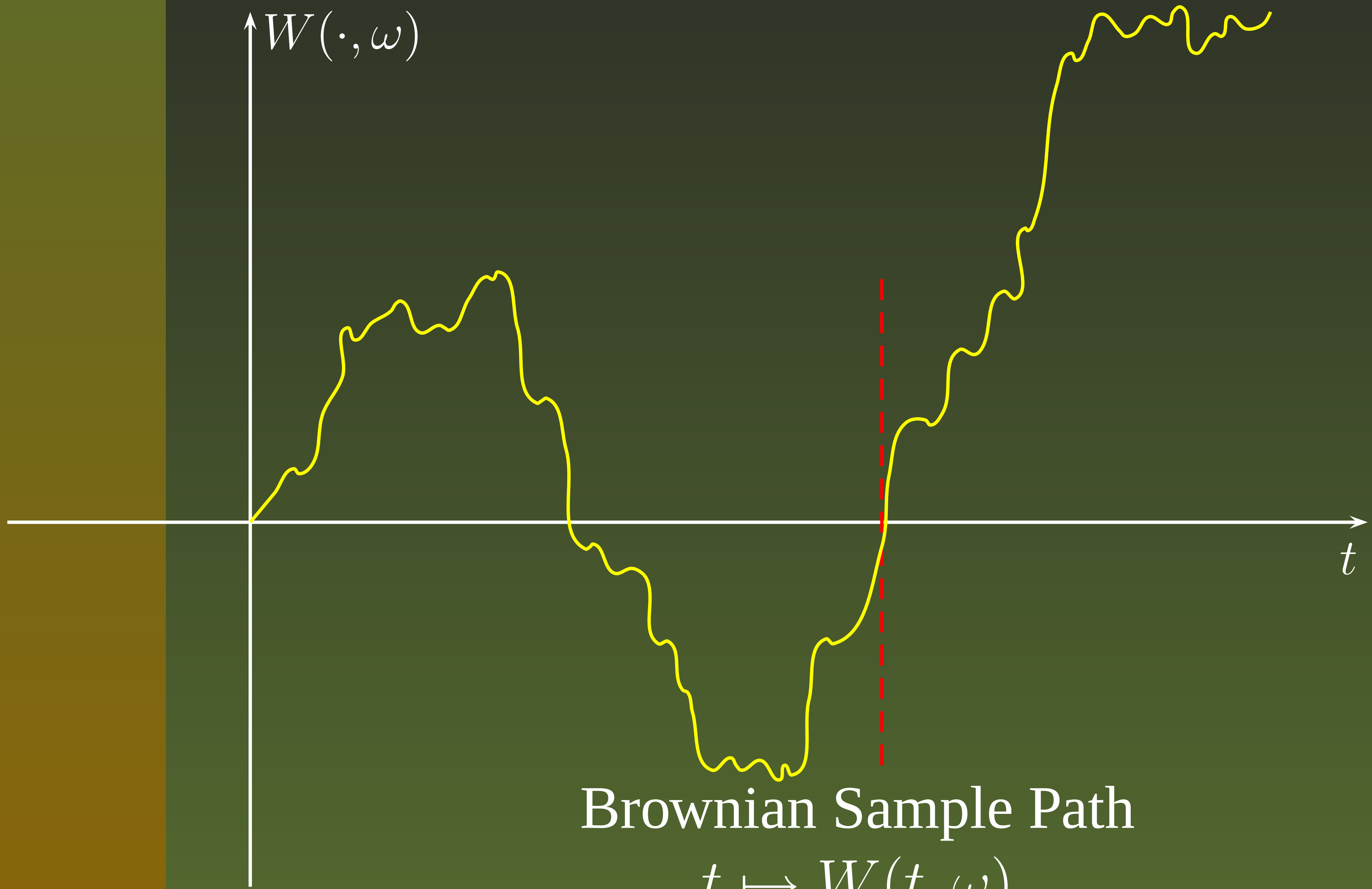
Nevertheless, I will go ahead and show you one!

Brownian Sample Path



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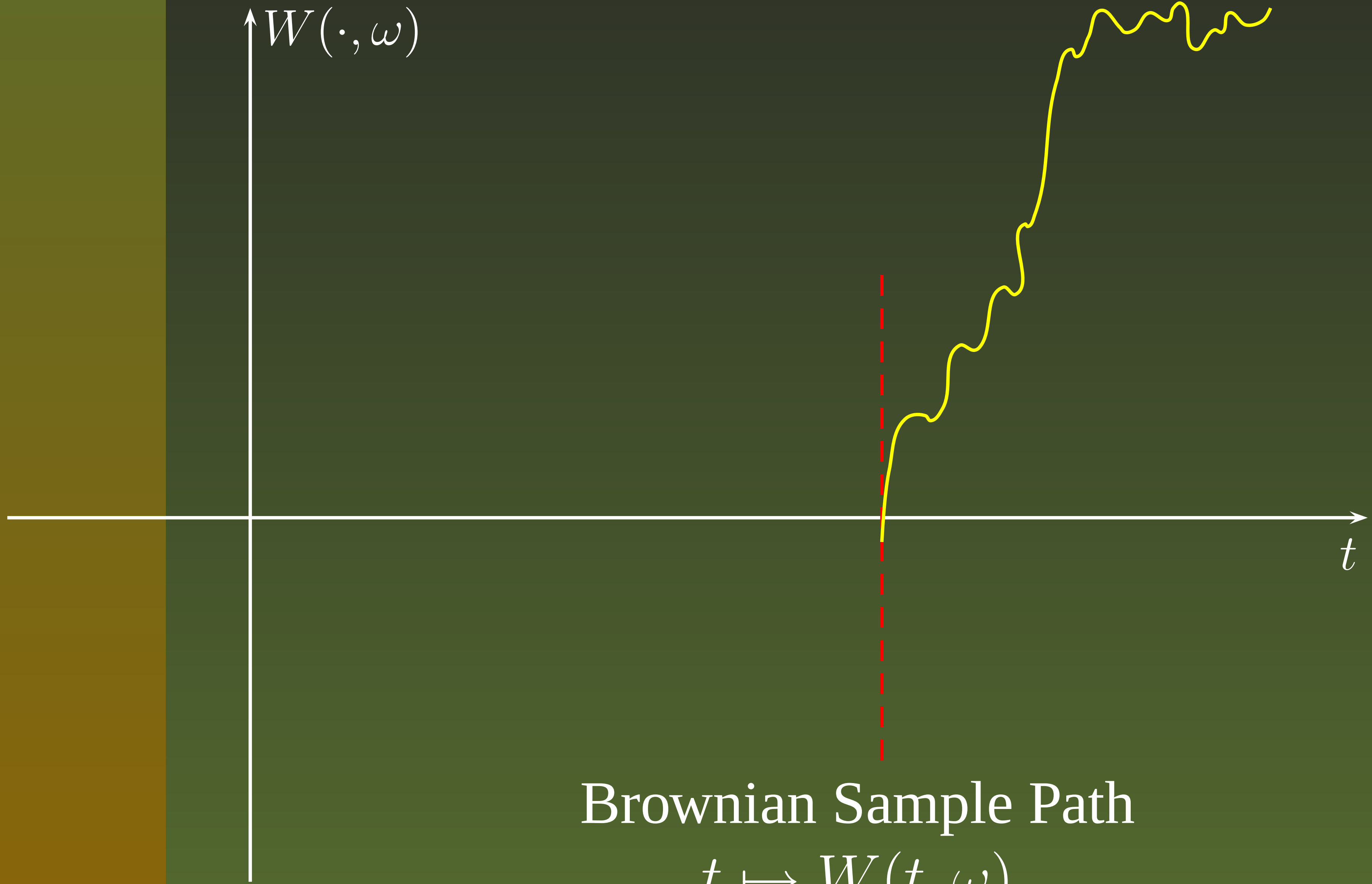

Brownian Sample Path

$$t \mapsto W(t,\omega)$$

Glossary-Contd

Each Brownian shift

$$\theta(t, \cdot) : \Omega \to \Omega, \quad t \in \mathbf{R}$$

$$\theta(t, \omega)(s) := W(t + s, \omega) - W(t, \omega), \quad s \in \mathbf{R}, \omega \in \mathbf{\Omega},$$

transforms the probability space Ω into itself (by moving the sample points ω around) while preserving the probabilities of all events.

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transforms the probability space Ω into itself (by moving the sample points ω around) while preserving the probabilities of all events.

Theorem:

The probability space Ω is perfectly mixed by the Brownian shift $\theta(t)$: The only events that are unchanged are either sure or impossible. (alias "ergodicity")

Noisy Feedback Loop

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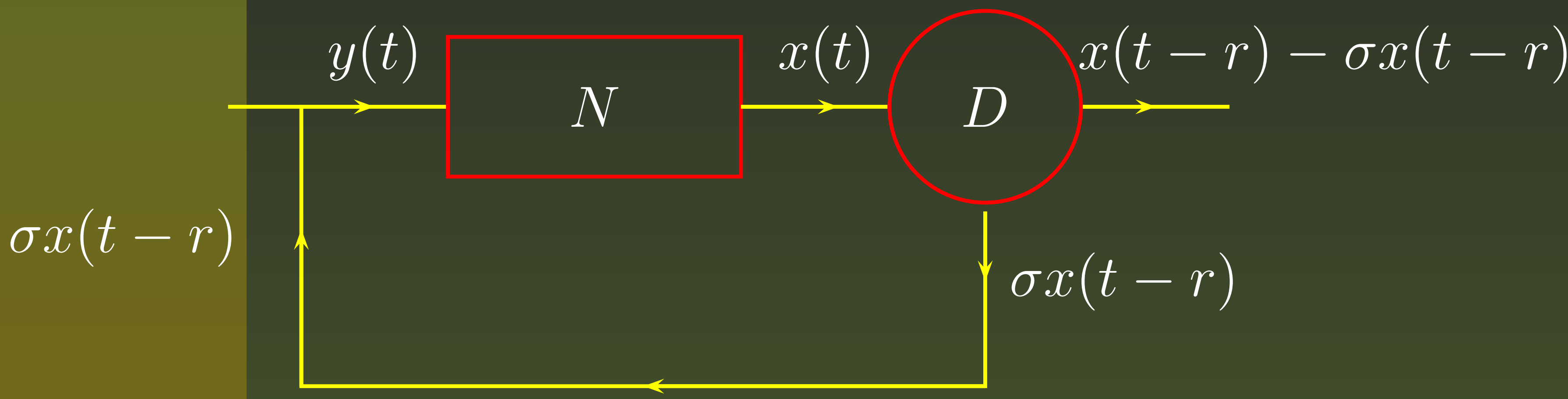

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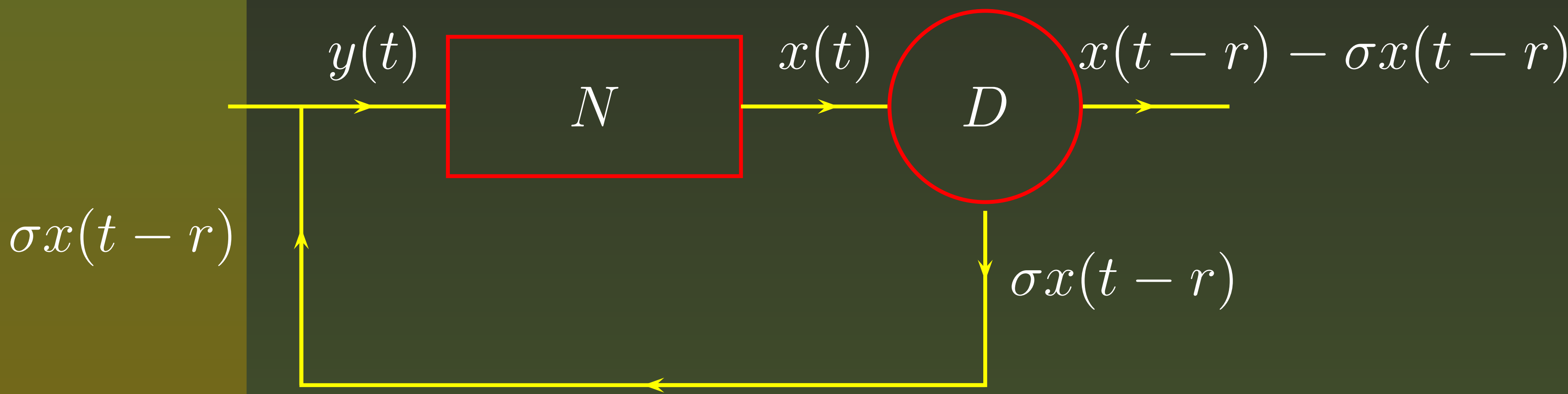

Box N: Input signal $= y(t)$, output $= x(t)$ at time $t > 0$ related by

$$\frac{dx(t)}{dt} = y(t)\,\frac{dW(t)}{dt}$$

where $W(t)$ is Brownian motion "white noise" in EE.

Noisy Feedback– Cont'd

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Call (I) a stochastic differential equation with delay (memory). Use differentials:

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To solve (I), need an initial process $\eta(t)$, $-r \leq t \leq 0$:

$$x(t) = \eta(t) \qquad -r \leq t \leq 0$$

Noisy Feedback-Contd

View (I) as a stochastic integral equation

$$x(t) = \eta(0) + \int_0^t \sigma x(u-r)\, dW(u), \qquad t > 0$$

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Use idea of stochastic integration with respect to Brownian motion (K. Itô):

Partition time interval $[0, t]$ by points

$$0 = u_0 < u_1 < u_2 < \cdots u_i < u_{i+1} < \cdots u_n = t$$

which get closer and closer to each other as n gets infinitely large.

Partition of $[0, t]$

Noisy Feedback-Contd

The corresponding sums:

$$\sum_{i=0}^{n-1} \sigma x(u_i - r)[W(u_{i+1}) - W(u_i)]$$

will approach the Itô stochastic integral:

$$\int_0^t \sigma x(u - r)\, dW(u)$$

as the number of partition points n gets larger and larger.

Noisy Feedback-contd

To solve

$$dx(t) = \sigma x(t-r)\,dW(t), \qquad t > 0 \qquad \text{(I)}$$

proceed by successive forward (stochastic) integrations:

$$0 \le t \le r,\ \ r \le t \le 2r,\ \ 2r \le t \le 3r,\ \ \cdots\ ,$$

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The current value $x(t)$ of the solution x of (I) is non-Markov.

Segment Process

Segment Process

The segment x_t is a path $[-r, 0] \to \mathbf{R}$ defined by

$$x_t(s) := x(t+s), \qquad -r \leq s \leq 0$$

Segment Process-Contd

Although the solution $x(t)$ of the stochastic delay equation

$$dx(t) = \sigma x(t-r)\, dW(t), \qquad t > 0$$

is non-Markov, yet the segment process x_t is Markov within the state space of all paths η.

Segment Process-Contd

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is non-Markov, yet the segment process x_t is Markov within the state space of all paths η.

In order to capture the true dynamics of the stochastic delay equation, we observe the random evolution of the segment x_t rather than the current value $x(t)$.

Immediate Feedback-No memory

Conside the case $\mathbf{r} = \mathbf{0}$: (I) becomes a linear stochastic differential equation (without memory)

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and has closed form solution

$$x(t) = x(0) \exp\left\{ \sigma W(t) - \frac{\sigma^2 t}{2} \right\}, \qquad t \geq 0.$$

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$x(t)$ is Markov (no delay= no memory).

Simple Population Dynamics

- Consider a large population $x(t)$ at time t evolving with a constant birth rate $\beta > 0$ and a constant death rate α per capita.

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- Assume immediate removal of the dead from the population.
- Let $r > 0$ (fixed, non-random$= 9$ months!) be the development period of each individual.
- Assume there is migration whose overall rate is distributed like white noise $\sigma \dot{W}$ (mean zero and variance $\sigma > 0$), where W is one-dimensional Brownian motion.

Simple Population – Cont'd

The change in population $\Delta x(t)$ over a small time interval $(t, t + \Delta t)$ is

$$\Delta x(t) = -\alpha x(t)\Delta t + \beta x(t - r)\Delta t + \sigma \dot{W} \Delta t$$

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Letting $\Delta t \to 0$ and using Itô stochastic differentials,

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Associate with the above stochastic delay equation the initial path η

$$x(s) = \eta(s), \quad -r \leq s \leq 0.$$

Logistic Population Growth

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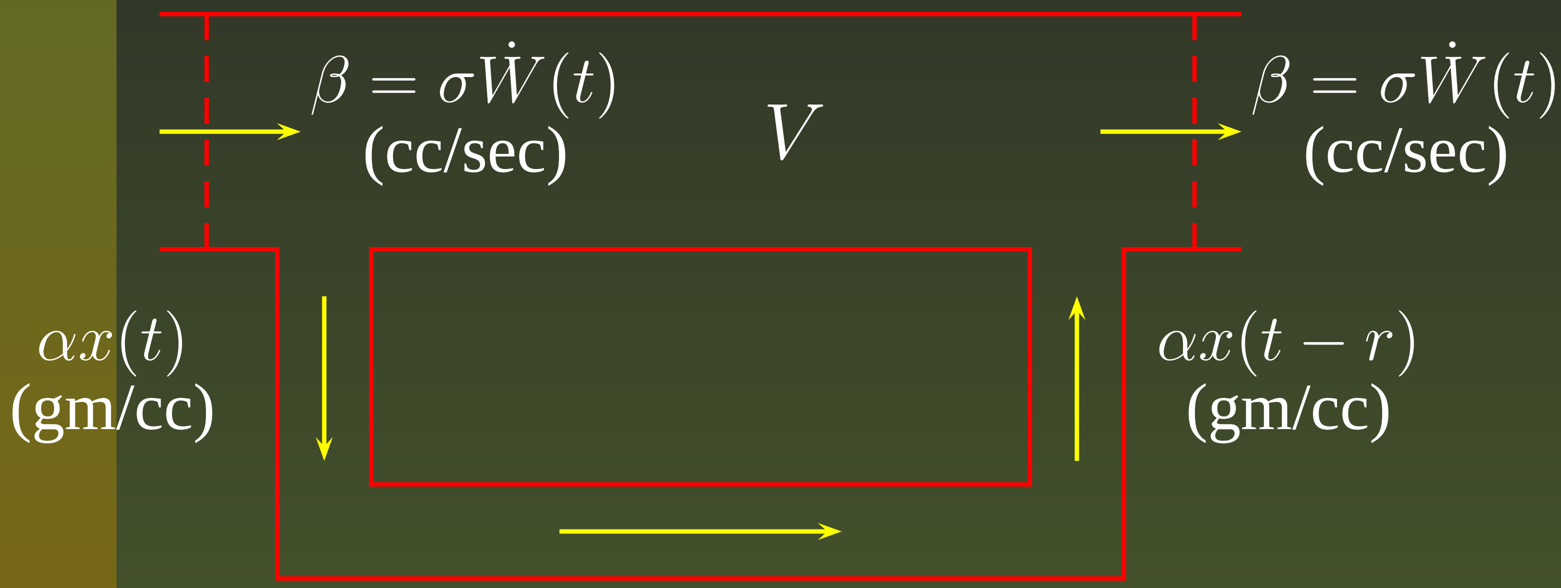
with initial condition

$$x(t) = \eta(t) \qquad -r \le t \le 0.$$

Fluid Flow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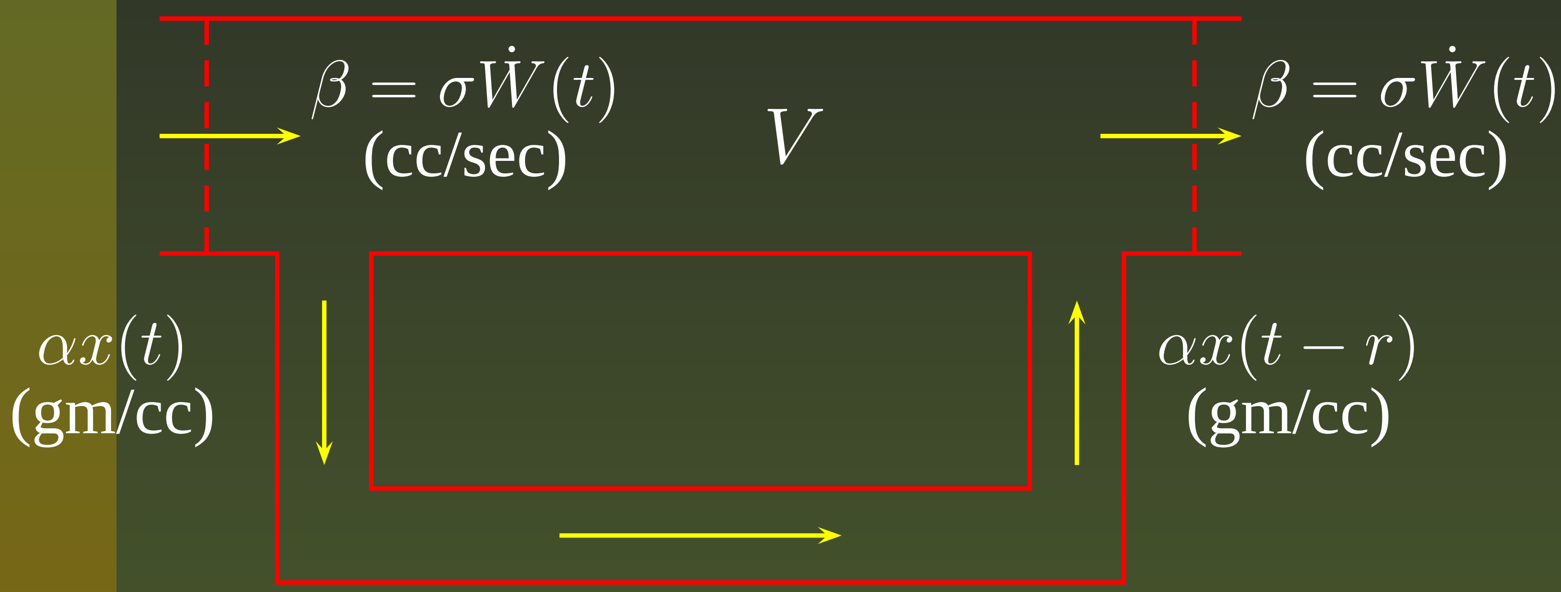

Main canal has dye (pollutant) with concentration $x(t)$ (gm/cc) at time t.

A fixed proportion α of fluid in the main canal is pumped into the side canal(s).

Fluid Flow– Cont'd

The fluid takes $r > 0$ seconds to traverse the side canal. Assume flow rate (cc/sec) in the main canal is Gaussian with constant mean and variance σ.

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Write equation for rate of dye transfer through a fixed part V of the main canal.

Then get the following stochastic delay equation:

$$\left.\begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r)\}\, dt + \sigma x(t)\, dW(t), \;\; t > 0 \\ x(s) &= \eta(s), \quad -r \le s \le 0 \end{aligned}\right\}$$

where η is a path $[-r, 0] \to \mathbf{R}$, ν and μ are real constants.

Delayed Stock Model

Consider a stock whose price $S(t)$ at any time t satisfies the following stochastic delay differential equation (sdde):

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\left.\begin{aligned}
dS(t) &= h(S(t-a))S(t)\,dt + g(S(t-b))S(t)\,dW(t), \\
&\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T] \\
S(t) &= \eta(t), \quad t \in [-L,0]
\end{aligned}\right\}
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$$\left.\begin{aligned} dS(t) &= h(S(t-a))S(t)\,dt + g(S(t-b))S(t)\,dW(t), \\ &\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T] \\ S(t) &= \eta(t), \quad t \in [-L,0] \end{aligned}\right\}$$

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Trading Strategy: $\pi_S(t)$ shares of stock $S(t)$ and $\pi_B(t)$ of bond $B(t)$.

Delayed Stock Model-contd

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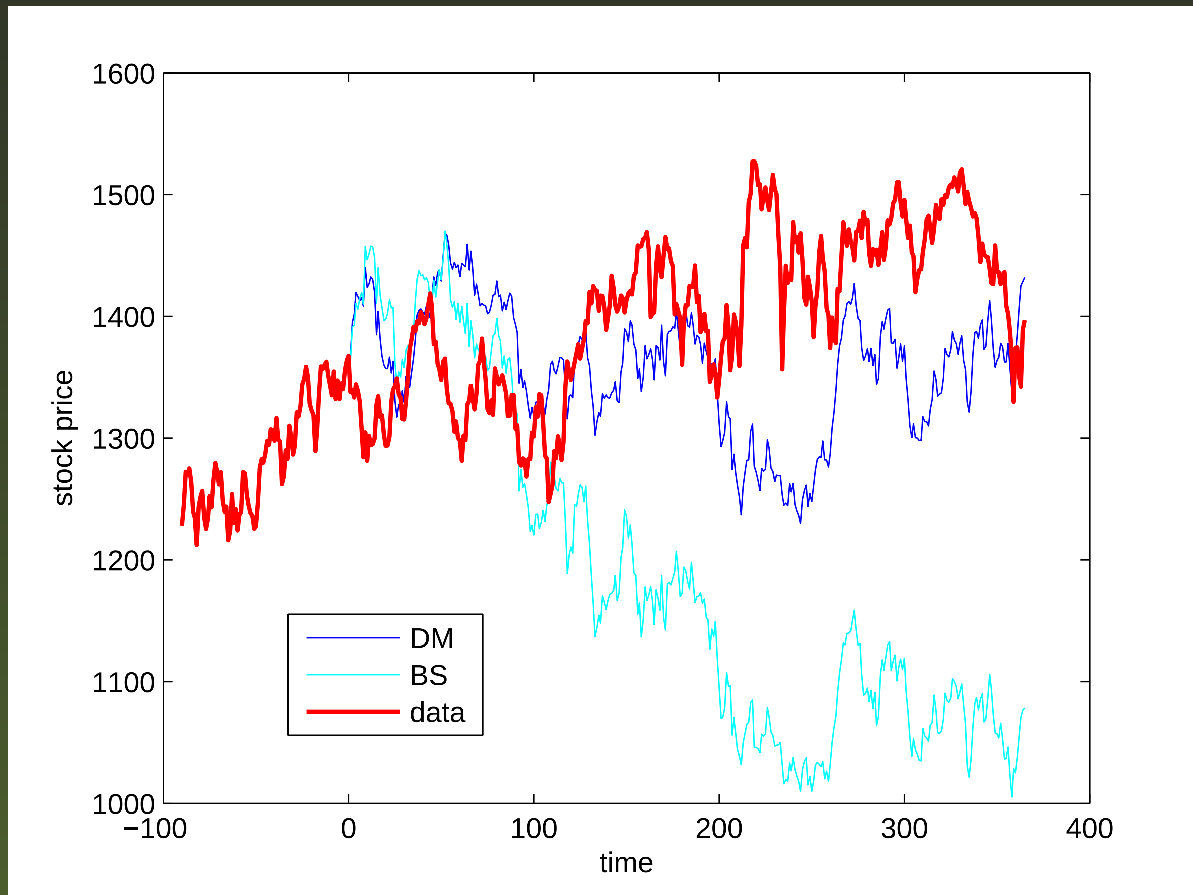
Constant volatility g and h corresponds to Black-Scholes model.

Stock Dynamics

Stock Dynamics



Stock prices when h = constant, $b = 2$, $T = 365$, $L = 100$.

Stock data: DJX Index at CBOE.

Delayed BS Formula

(–>)

“Now let’s do the math”!

Stochastic Systems with Memory

Combine all dynamic models encountered so far in a single stochastic differential equation of the form

$$\left.\begin{aligned} dx(t) &= h(x(t), x_t)\, dt + g(x(t), x_t) dW(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in \mathbf{R} \times \mathbf{L^2}([-\mathbf{r}, \mathbf{0}], \mathbf{R}). \end{aligned}\right\}$$

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W is Brownian motion; x_t is the segment process (encoding the memory of the solution process x); η is a given initial path $[-r, 0] \to \mathbf{R}$ (starting process for x); $v \in \mathbf{R}$ is a given initial point.

State Space

Collect all possible initial conditions (v, η) in a state space, denoted by H, and defined by

$$H := \{(v, \eta) : v \in \mathbf{R}, \eta \in \mathbf{L^2([-r, 0], R)}\}.$$

The state space H is a Hilbert space under the norm

$$\|(v, \eta)\|^2 := |v|^2 + \int_{-r}^{0} |\eta(s)|^2 \, ds$$

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The state space H is BIG: infinite-dimensional.

Existence

A stochastic differential system with memory is a relation between the current rate of change of the system and its past random states.

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Theorem:

Under appropriate (fairly general) conditions on the coefficients h, g*, the stochastic equation with memory has a unique solution* x *for each choice of the initial state* (v, η) *in the state space* H.

Random Dynamics with Memory

- Exploit idea of the segment as paradigm for encoding the memory as an infinite-dimensional object that evolves randomly in infinite-dimensional space (even if the original stochastic signal is one-dimensional).

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- Within this setting the mathematics is harder but doable: No free lunch! For example, the Itô calculus fails for the encoded process, although it works for the original signal.
- Random dynamics is described via the flow.

Average Dynamics: Hypotheses

The coefficients $h, \; g$ in the SDE are globally Lipschitz:

$$|h(v_1,\eta_1) - h(v_2,\eta_2)| + \|g(v_1,\eta_1) - g(v_2,\eta_2)\|$$

$$\leq L\|(v_1,\eta_1) - (v_2,\eta_2)\|_H$$

for all $(v_1,\eta_1),(v_2,\eta_2) \in H$.

Markov Property

${}^{(v,\eta)}x^{t_1}$:= solution starting off at $(v,\eta) \in L^2(\Omega, H; \mathcal{F}_{t_1})$ at $t = t_1$ for the stochastic differential equation with memory:

$$
{}^{\eta}x^{t_1}(t) = \begin{cases} v + \int_{t_1}^{t} h(x^{t_1}(u), x_u^{t_1})\, du \\ \qquad\qquad + \int_{t_1}^{t} g(x^{t_1}(u), x_u^{t_1})\, dW(u), \;\; t > t_1, \\ \eta(t - t_1), \qquad\qquad\qquad t_1 - r \le t \le t_1. \end{cases}
$$

Markov Property – Cont'd

This gives a two-parameter family of mappings:

$$T_{t_2}^{t_1} : L^2(\Omega, H; \mathcal{F}_{t_1}) \to L^2(\Omega, H; \mathcal{F}_{t_2}),\ t_1 \leq t_2,$$

$$T_{t_2}^{t_1}(v,\eta) := \left({}^{(v,\eta)}x^{t_1}(t_2), {}^{(v,\eta)}x_{t_2}^{t_1}\right),\ (v,\eta) \in L^2(\Omega, H; \mathcal{F}_{t_1}).$$

Uniqueness of solutions gives the *two-parameter* semigroup property:

$$T_{t_2}^{t_1} \circ T_{t_1}^{0} = T_{t_2}^{0}, \quad t_1 \leq t_2.$$

Markov Property–contd

In the SDE with memory, the trajectory field $\{({}^{(v,\eta)}x(t), {}^{(v,\eta)}x_t) : t \geq 0, (v,\eta) \in H\}$ *is a time-homogeneous Feller process on* H *with transition probabilities*

$$p(t_1, (v,\eta), t_2, B) := P\Big(({}^{(v,\eta)}x^{t_1}(t_2), {}^{(v,\eta)}x^{t_1}_{t_2}) \in B\Big),$$

for $t_1 \leq t_2$, $(v,\eta) \in H$ *and* $B \in$ *Borel* H. That is:

$$\begin{aligned} P\Big((x(t_2), x_{t_2}) \in B \Big| \mathcal{F}_{t_1}\Big) &= p(t_1, (x(t_1)(\cdot), x_{t_1}(\cdot)), t_2, B) \\ &= P\Big((x(t_2), x_{t_2}) \in B \Big| (x(t_1), x_{t_1})\Big) \end{aligned}$$

almost surely.

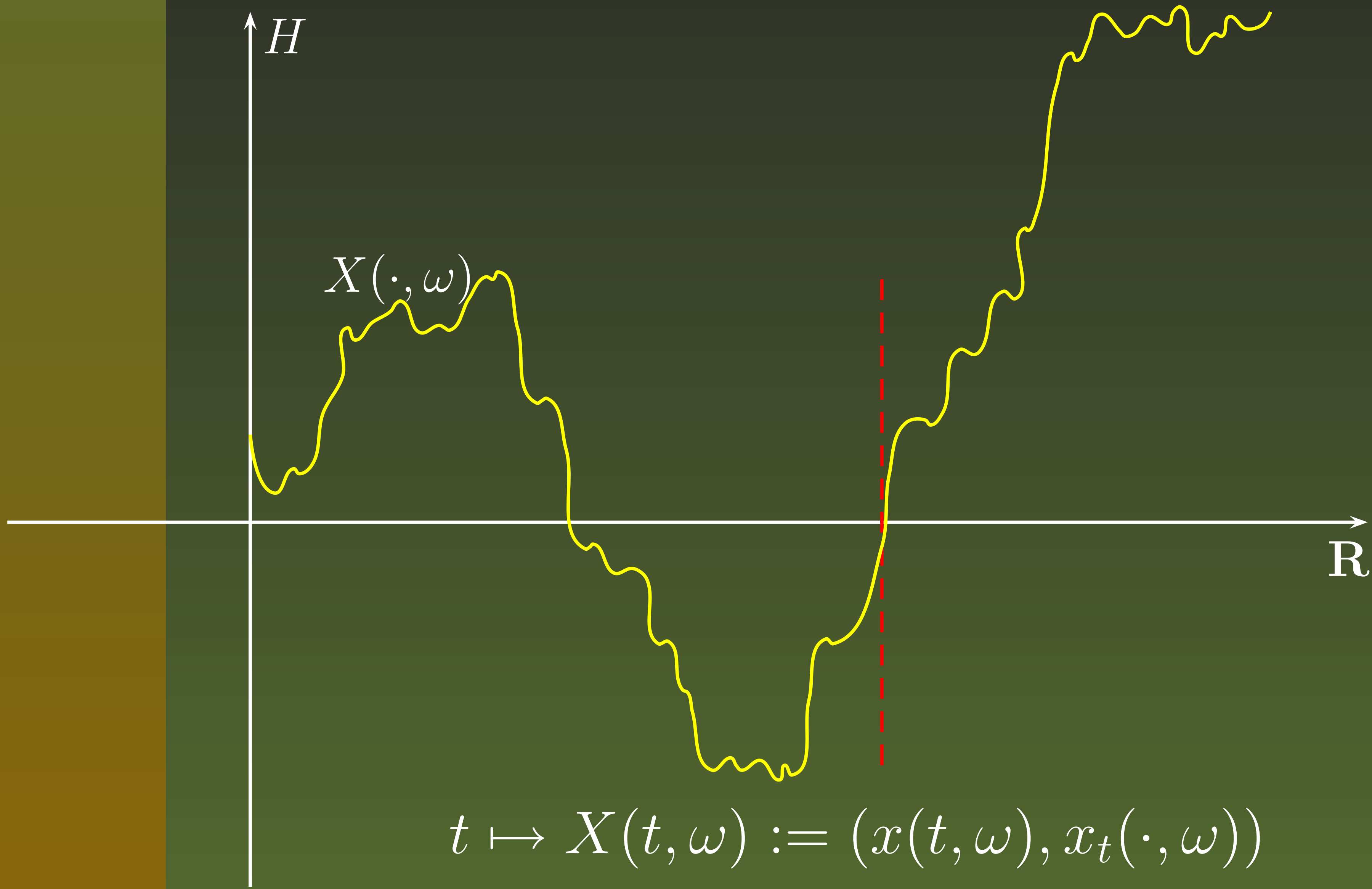
Markov Property – Cont'd

Further, the trajectory is time-homogeneous:

$$p(t_1, (v, \eta), t_2, \cdot) = p(0, (v, \eta), t_2 - t_1, \cdot), \;\; 0 \leq t_1 \leq t_2$$

for $(v, \eta) \in H$.

Trajectory Sample Path



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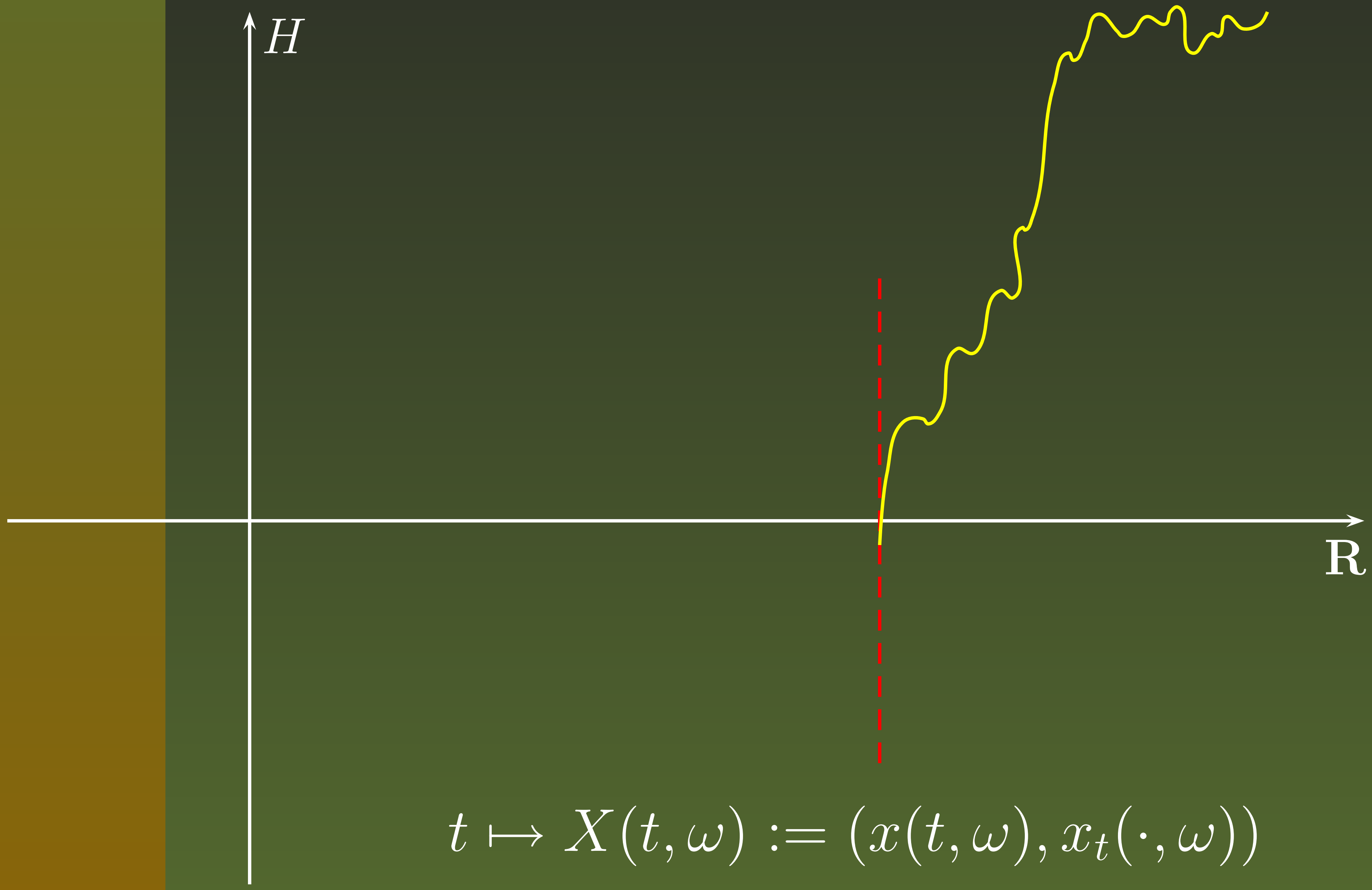

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The Semigroup

In the autonomous SDE with memory

$$\left.\begin{array}{r} dx(t) = h(x(t), x_t)\, dt + g(x(t), x_t)\, dW(t), \;\; t > 0 \\ (x(0), x_0) = (v, \eta) \in H, \end{array}\right\}$$

assume the coefficients $h : H \longrightarrow \mathbf{R^d}$, and $g : H \longrightarrow \mathbf{R^{d \times m}}$ are *globally bounded* and globally Lipschitz.

$C_b :=$ Banach space of all bounded uniformly continuous functions $\phi : H \longrightarrow \mathbf{R}$, with the sup norm

$$\|\phi\|_{C_b} := \sup_{(v,\eta) \in H} |\phi(v, \eta)|, \quad \phi \in C_b.$$

The Semigroup – Cont'd

Define the linear operators $P_t : C_b \hookrightarrow C_b, t \geq 0$, on C_b by

$$P_t(\phi)(v,\eta) := E\phi\left({}^{(v,\eta)}x(t), {}^{(v,\eta)}x_t\right), \ t \geq 0, \ (v,\eta) \in H,$$

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for all $\phi \in C_b$.

An (average) equilibrium is an invariant probability measure μ_0 on H:

$$\int_H P_t\phi \, d\mu_0 = \int_H \phi \, d\mu_0$$

for all $\phi \in C_b$ and all $t \geq 0$.

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- *Weak derivative of* $\{P_t\}_{t\geq 0}$ *at* $t = 0$ *gives its infi nitesimal generator* A*, a partial differential operator on* H*: Formally,* $P_t = \exp(tA)$*. [Mo.1]*

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- The stable manifolds have infinite dimension (and finite non-random codimension).

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Theorem:

For each sample point ω, we can observe the whole state space H as it mixes under the random smooth flow.

The Random Flow-contd

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The solution of the regular SDE with memory can be viewed as a function

$$X(t,(v,\eta),\omega)$$

of three variables: time t, state (v,η) and chance ω, continuous in t, smooth in (v,η) and satisfying:

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- $X(0,(v,\eta),\omega) = (v,\eta)$ for all initial paths $(v,\eta) \in H$, and all $\omega \in \Omega$.

The Flow Property

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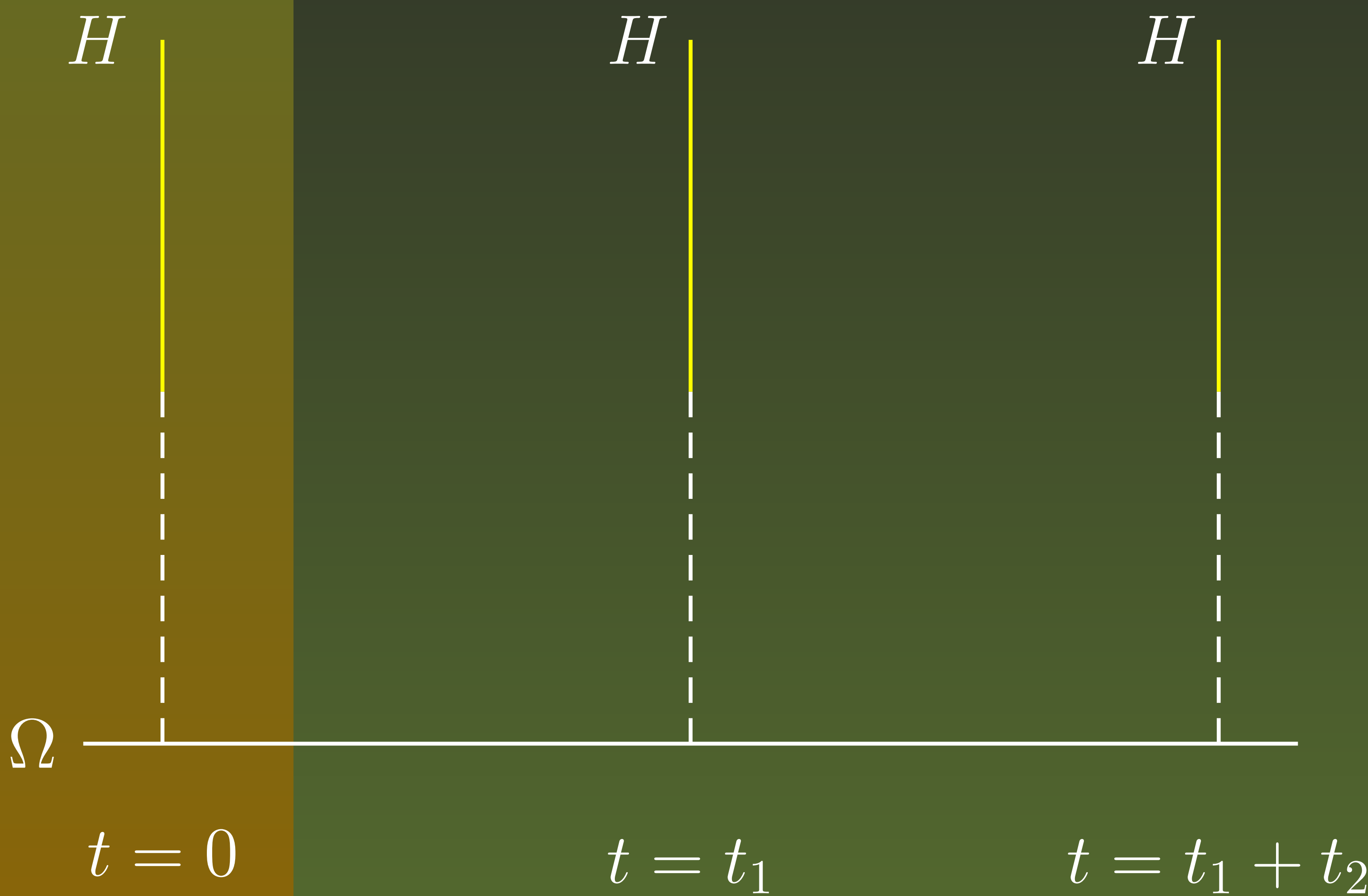

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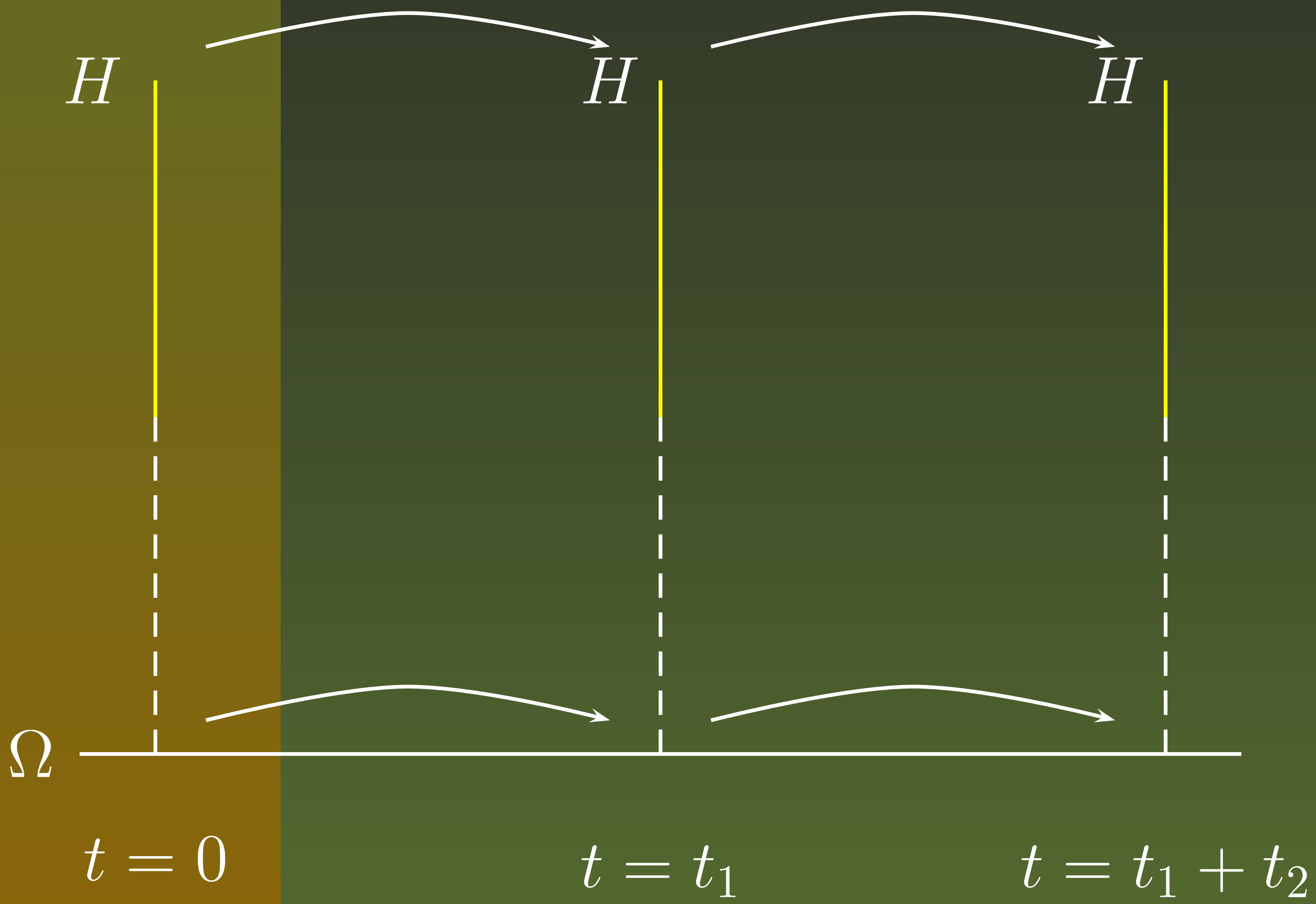

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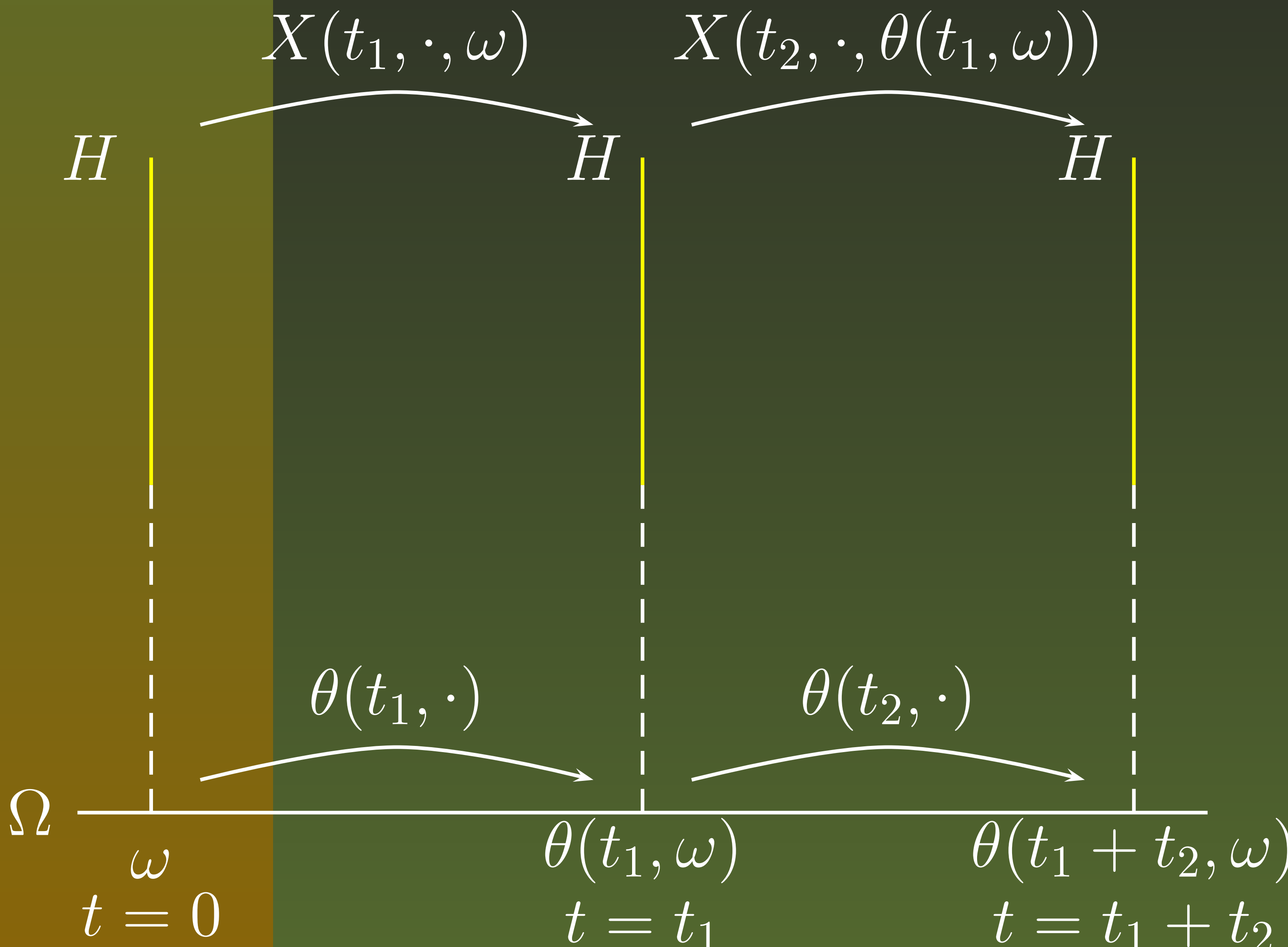

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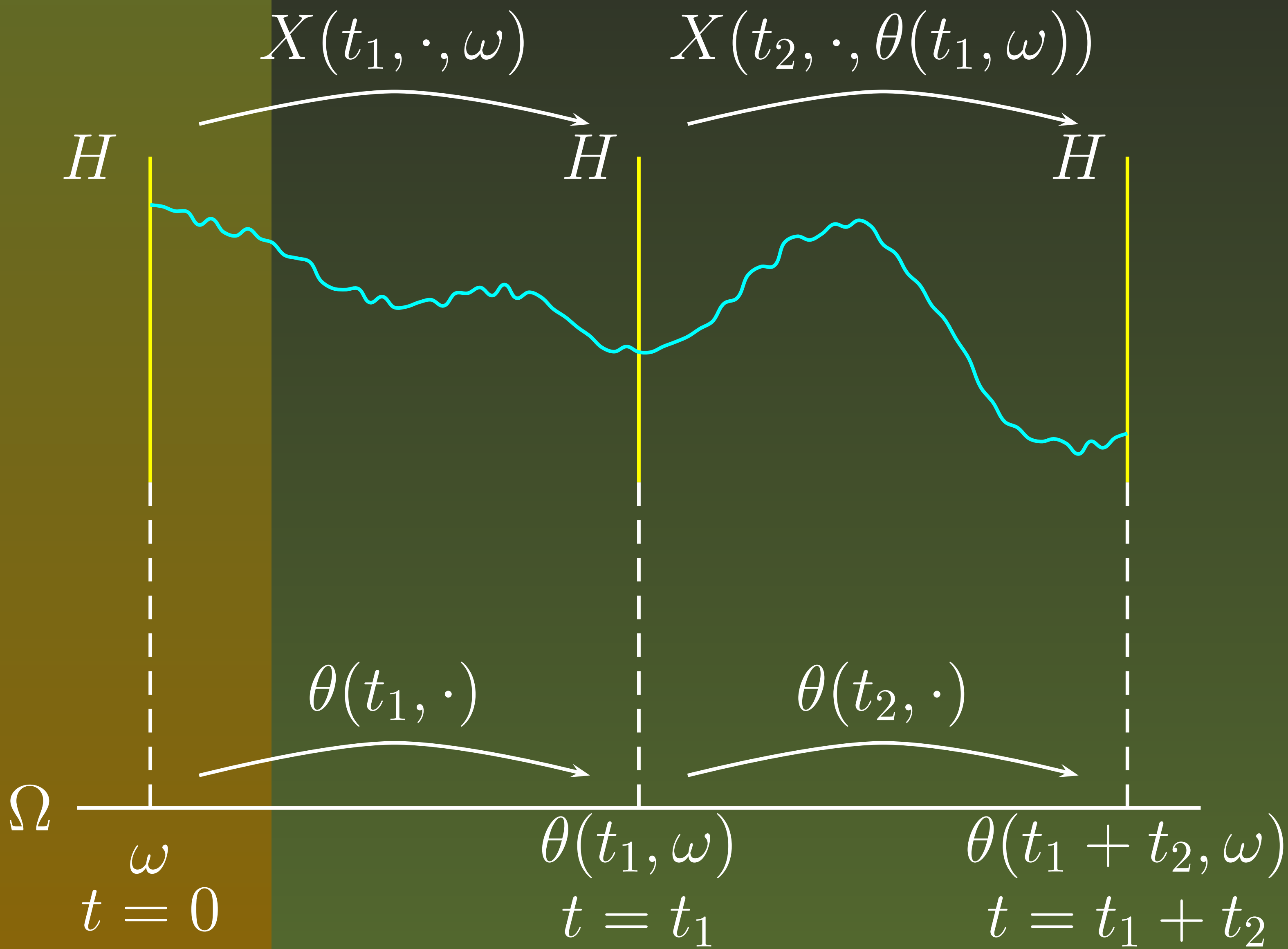

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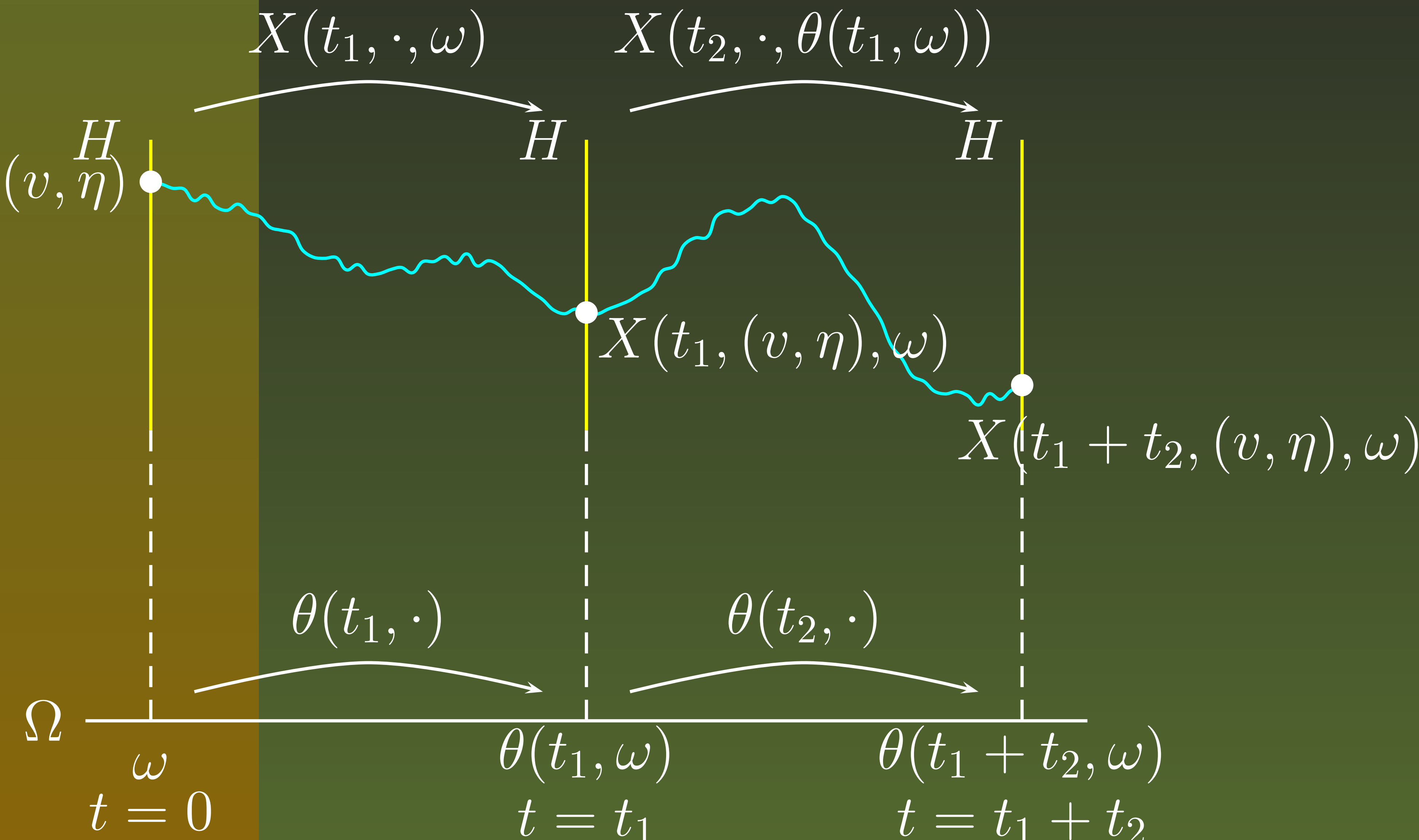

Stationary Point-Equilibrium

A random variable $Y : \Omega \to H$ is a *stationary point* for the flow (X, θ) if

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The distribution $\mu_0 := P \circ Y^{-1}$ of Y is an invariant measure (or average equilibrium) for the semigroup $\{P_t\}_{t \geq 0}$ (if Y is independent of W).

Exponential Growth

Theorem:

Within the state space H*, each stationary point* $Y(\omega)$ *has a ball* $B(Y(\omega), \rho(\omega))$ *center* $Y(\omega)$ *and radius* $\rho(\omega)$ *with the property that for any* $(v, \eta) \in B(Y(\omega), \rho(\omega))$ *the distance between* $X(t, (v, \eta), \omega)$ *and* $Y(\theta(t, \omega))$ *grows like* $e^{\lambda_i t}$ *for large* t *where*

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$$\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$$

are fixed countable and non-random. These represent exponential growth rates of the random flow near its equilibrium.

A Random Tube

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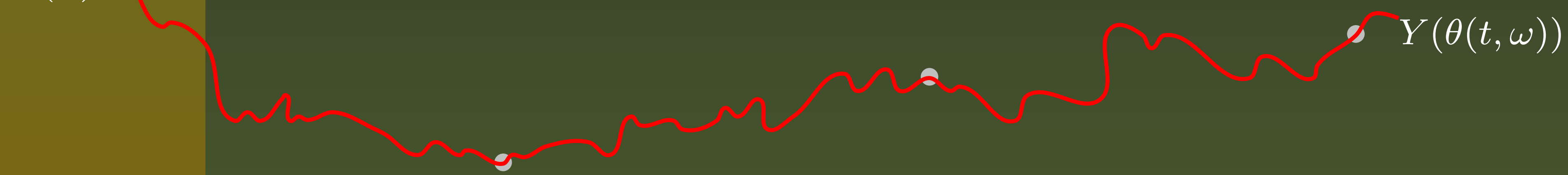

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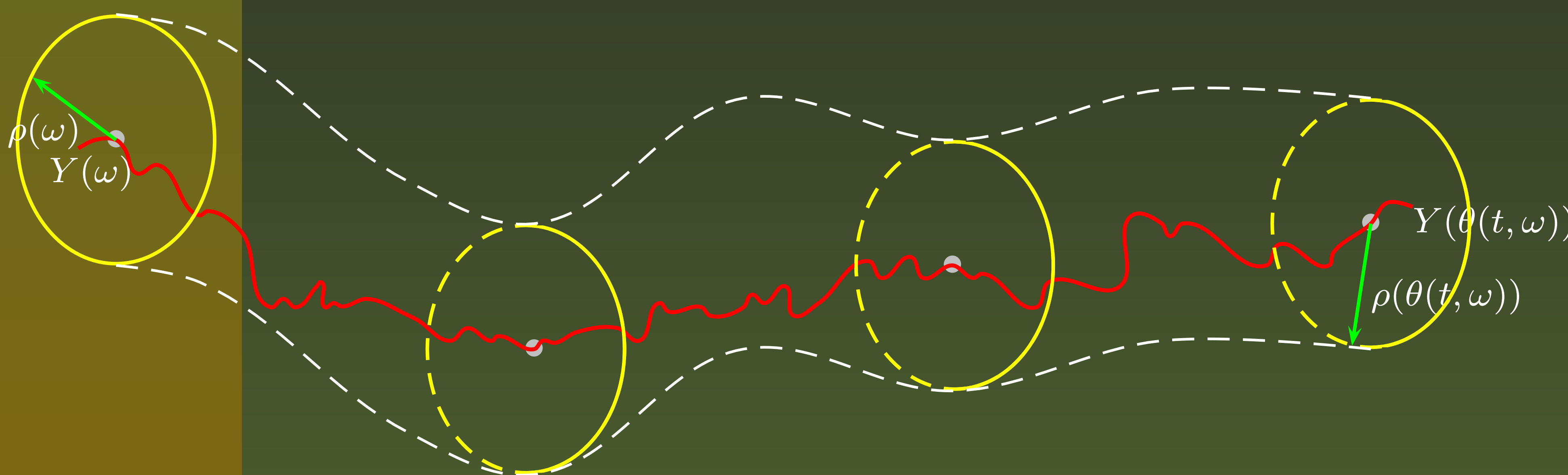

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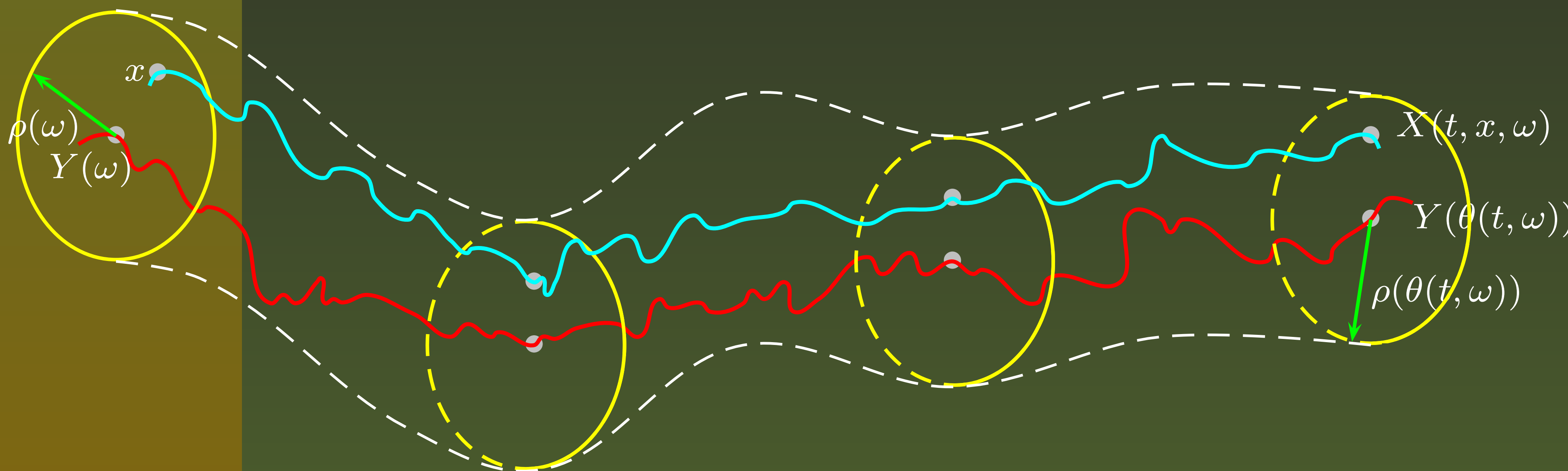

$$x := (v, \eta) \in H$$

Hyperbolicity

An equilibrium $Y(\omega)$ is hyperbolic if all exponential growth rates λ_i are non-zero:

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Cf. Classical work by S. Smale and his school on hyperbolicity in the deterministic case.

Stable Manifold Theorem

Let Y *be a hyperbolic equilibrium of the SDE with memory. Then there is a random tube* $B(Y(\omega), \rho(\omega))$ *around* Y, *a smooth stable manifold* $\mathcal{S}(\omega)$, *and unstable one* $\mathcal{U}(\omega)$ *in* $B(Y(\omega), \rho(\omega))$ *with the following properties:*

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The stable manifold $\mathcal{S}(\omega)$ is the set of all states (v, η) in $B(Y(\omega), \rho(\omega))$ such that the distance between $X(t, (v, \eta), \omega)$ and $Y(\theta(t, \omega))$ decays like $e^{\lambda_{i_0} t}$ for large t.

Theorem-contd

(Flow-invariance of the stable manifolds): *The stable manifold* $\mathcal{S}(\omega)$ *is eventually transported into* $\mathcal{S}(\theta(t,\omega))$*: That is*

$X(t,\cdot,\omega)(\mathcal{S}(\omega))$ *is a subset of* $\mathcal{S}(\theta(t,\omega))$ *for all large* t.

Theorem-contd

The unstable manifold $\mathcal{U}(\omega)$ *is the set of all states* (v,η) *in* $B(Y(\omega),\rho(\omega))$ *such that there is a unique continuous-time history process also denoted by* $y(\cdot,\omega):(-\infty,0]\to H$ *such that* $y(0,\omega)=(v,\eta)$, $X(t,y(s,\omega),\theta(s,\omega))=y(t+s,\omega)$ *for all* $s\leq 0$, $0\leq t\leq -s$, *and the distance between* $y(-t,\omega)$ *and* $Y(\theta(-t,\omega))$ *decays like* $e^{-\lambda_{i_0-1}t}$ *for large* t.

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The dimension of the unstable manifold $\mathcal{U}(\omega)$ *is finite and non-random.*

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$\mathcal{U}(\omega)$ *and* $\mathcal{S}(\omega)$ *intersect transversally at* $Y(\omega)$.

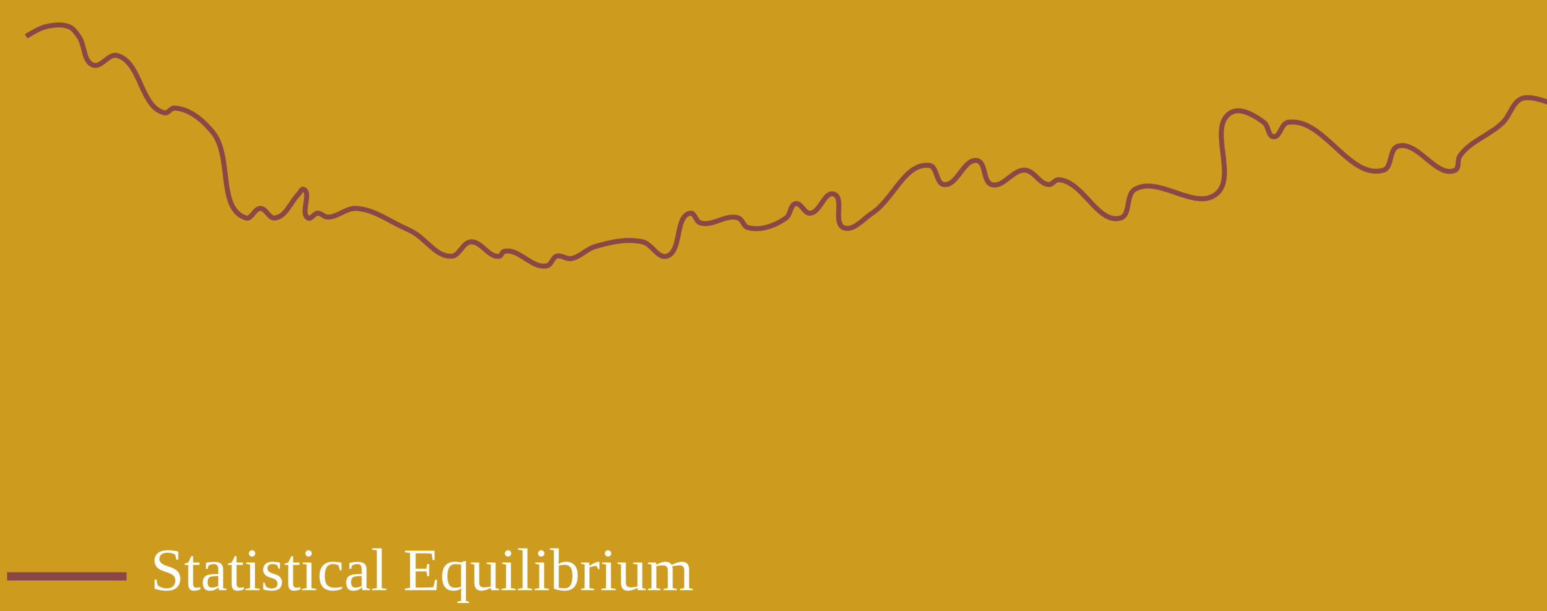
Theorem-contd

(Flow-invariance of the unstable manifolds):

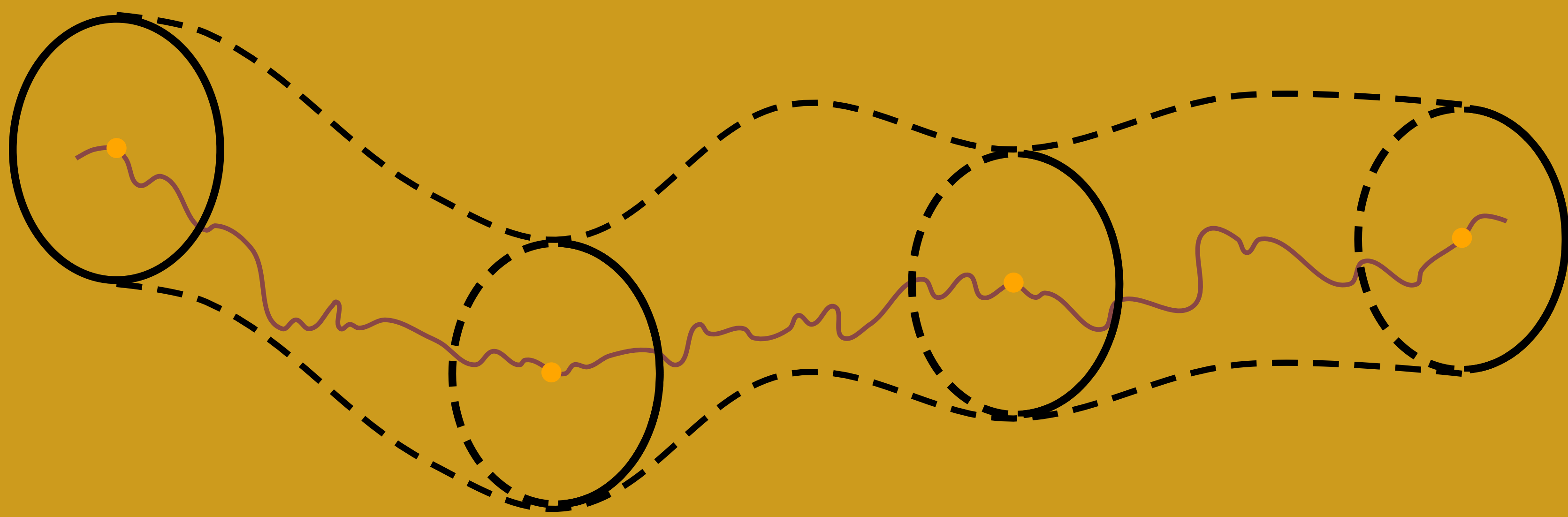
The remote history of the unstable manifold $\mathcal{U}(\omega)$ *may be traced back to* $\mathcal{U}(\theta(-t,\omega))$*: That is* $\mathcal{U}(\omega)$ *is a subset of* $X(t,\cdot,\theta(-t,\omega))(\mathcal{U}(\theta(-t,\omega)))$ *for sufficiently large* t.

$$\mathcal{U}(\omega) \subseteq X(t,\cdot,\theta(-t,\omega))(\mathcal{U}(\theta(-t,\omega)))$$

Random Saddles

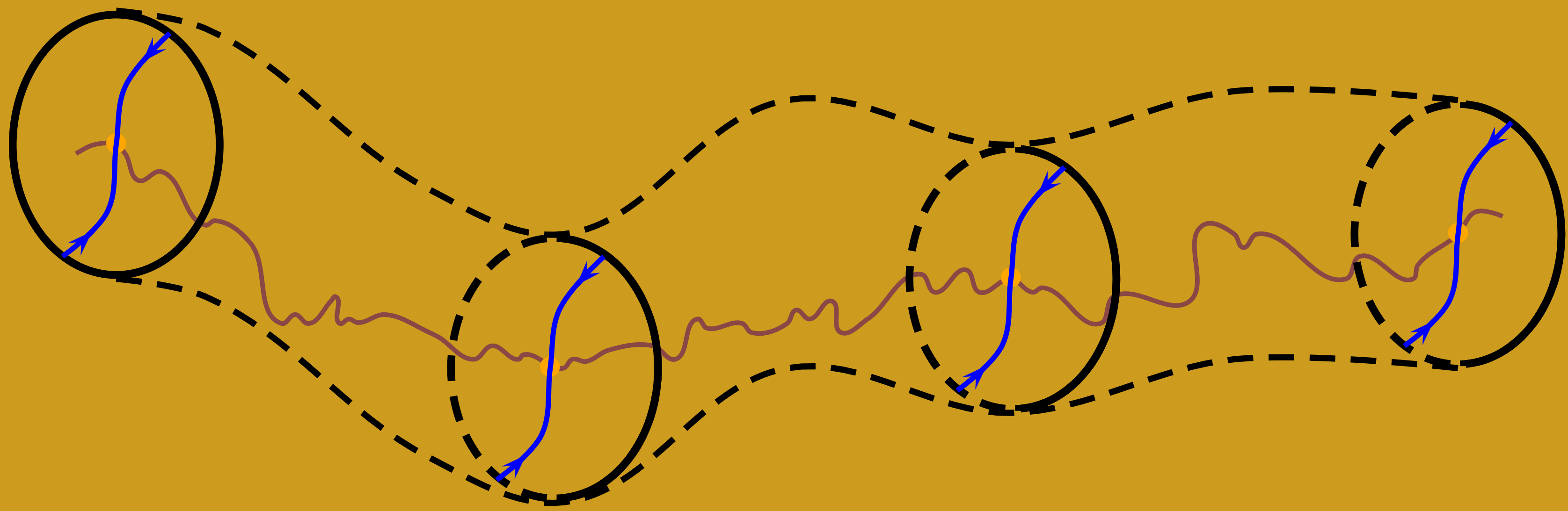


Random Saddles



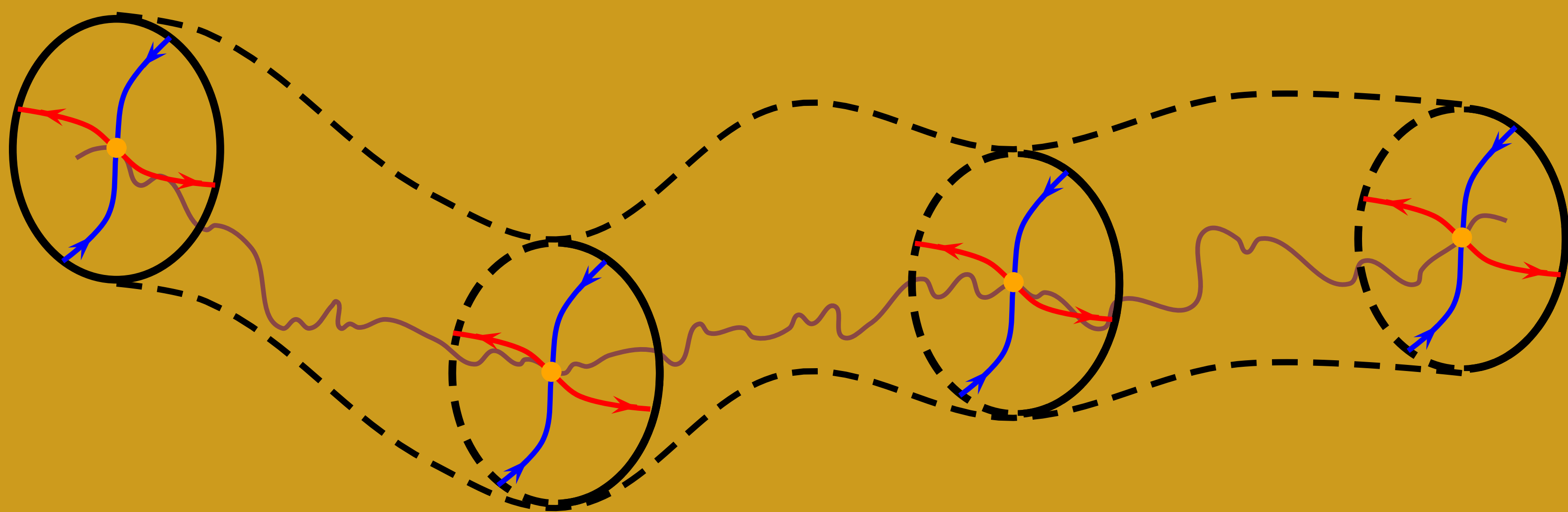
Statistical Equilibrium

Random Saddles



Statistical Equilibrium
Stable Manifold

Random Saddles

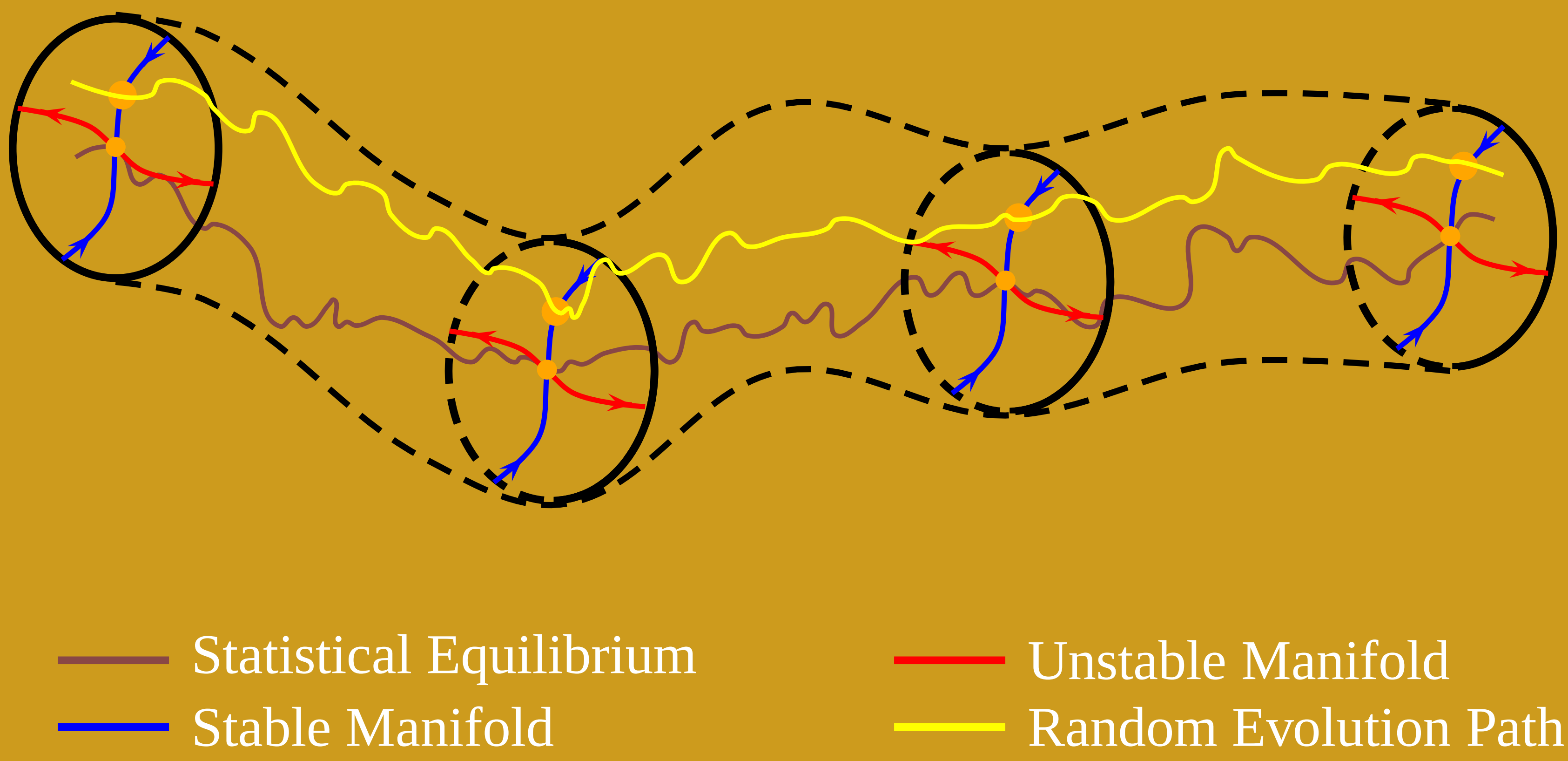


Statistical Equilibrium

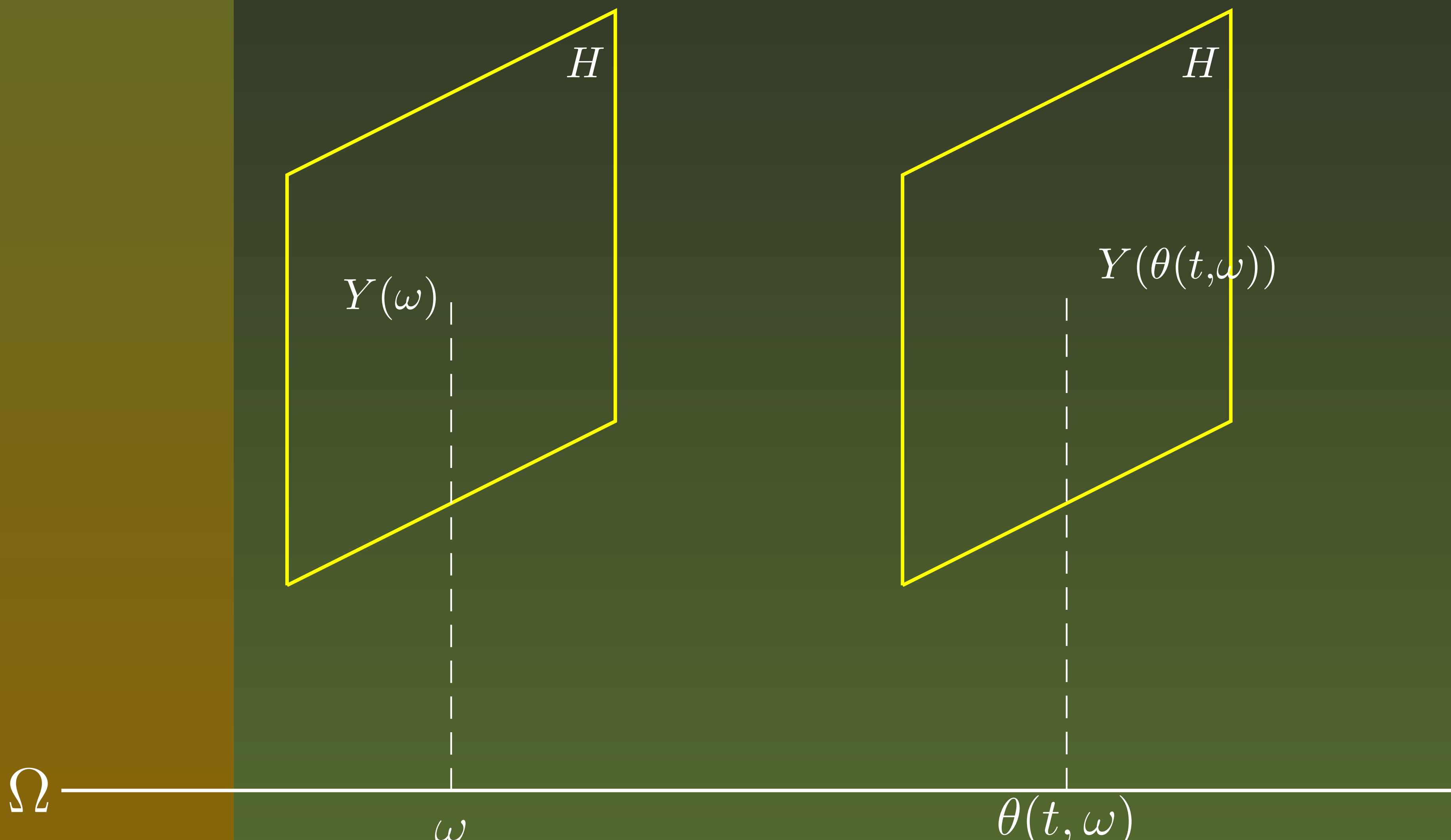
Stable Manifold

Unstable Manifold

Random Saddles



Stable/Unstable Manifolds



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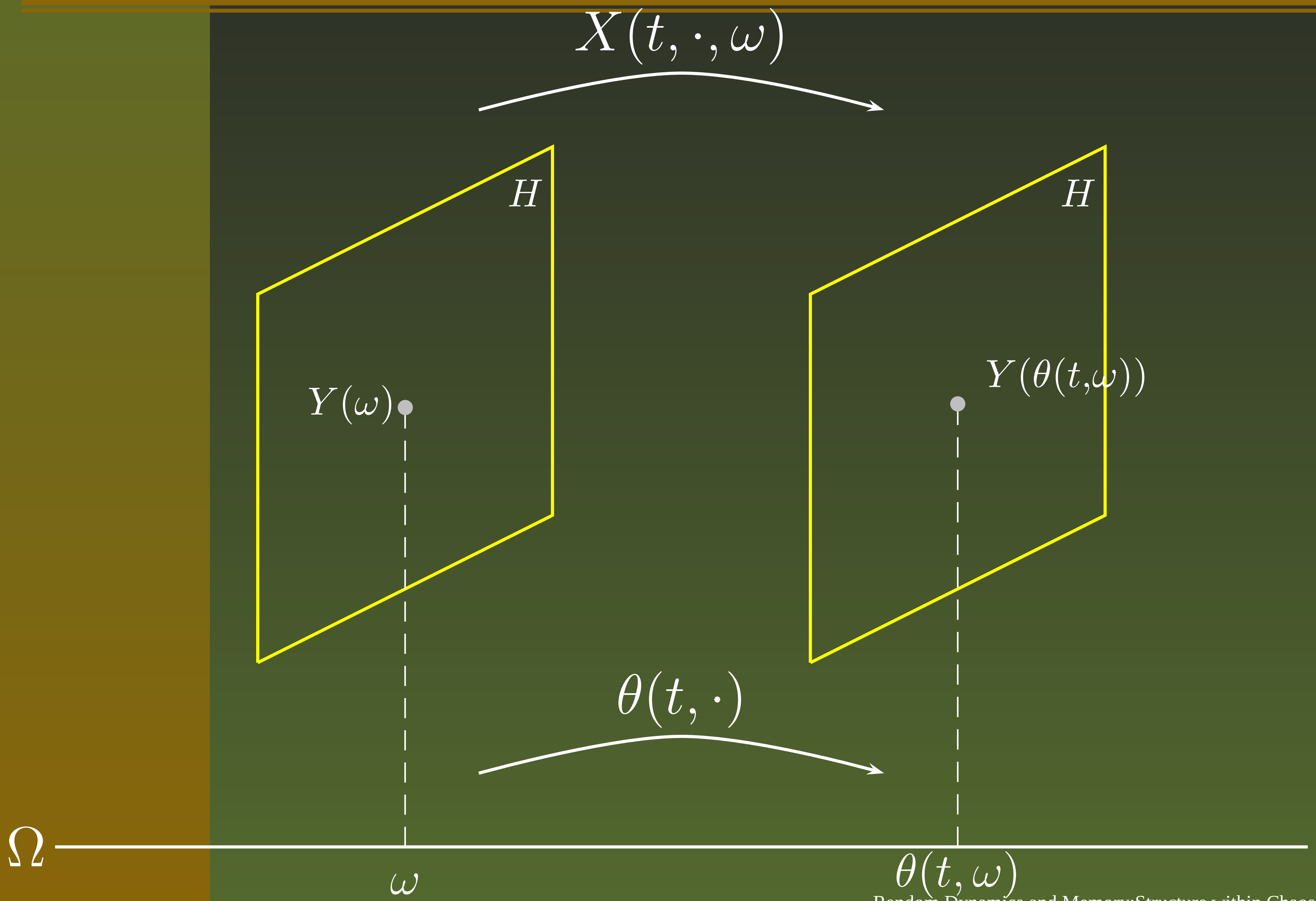

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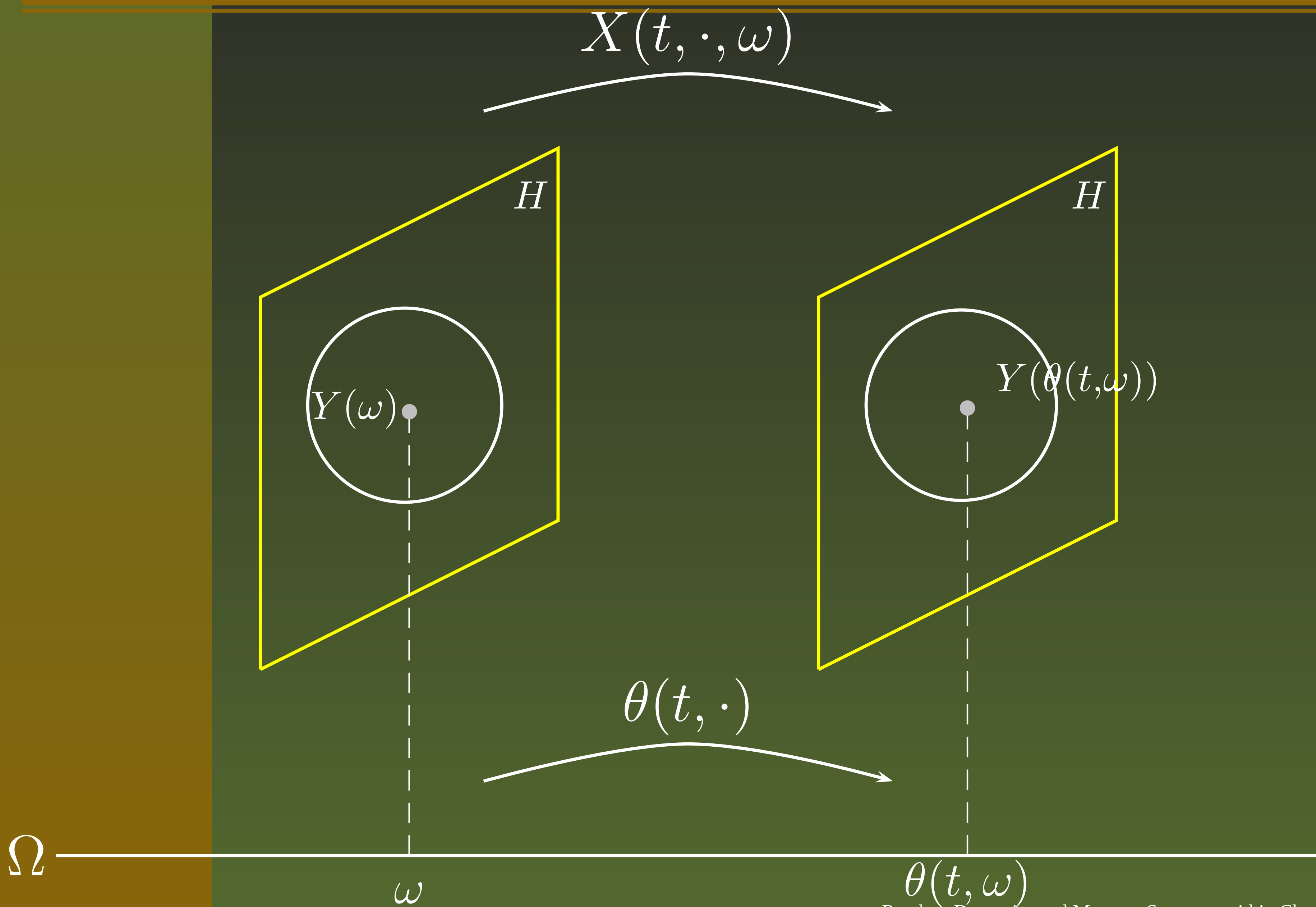

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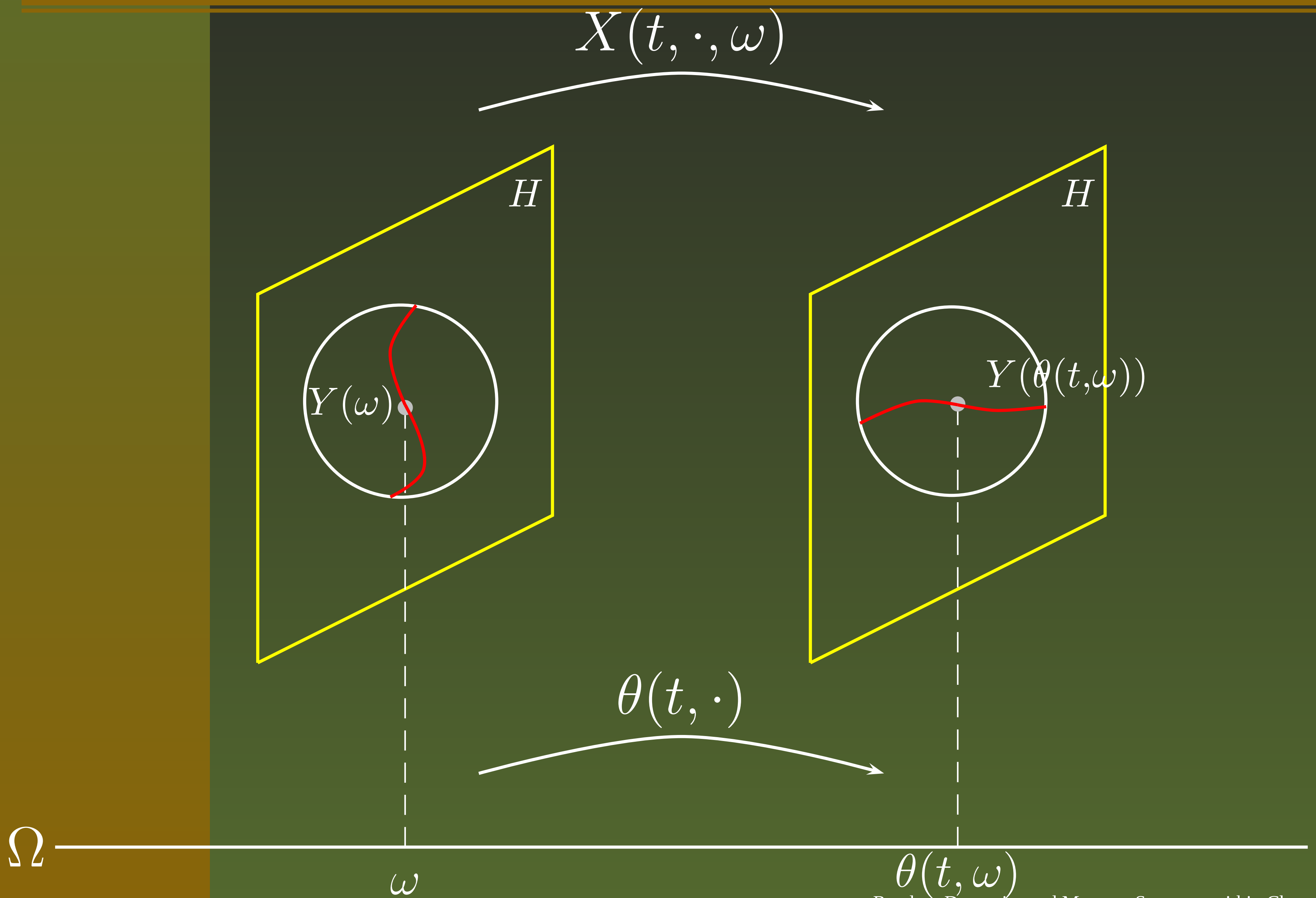

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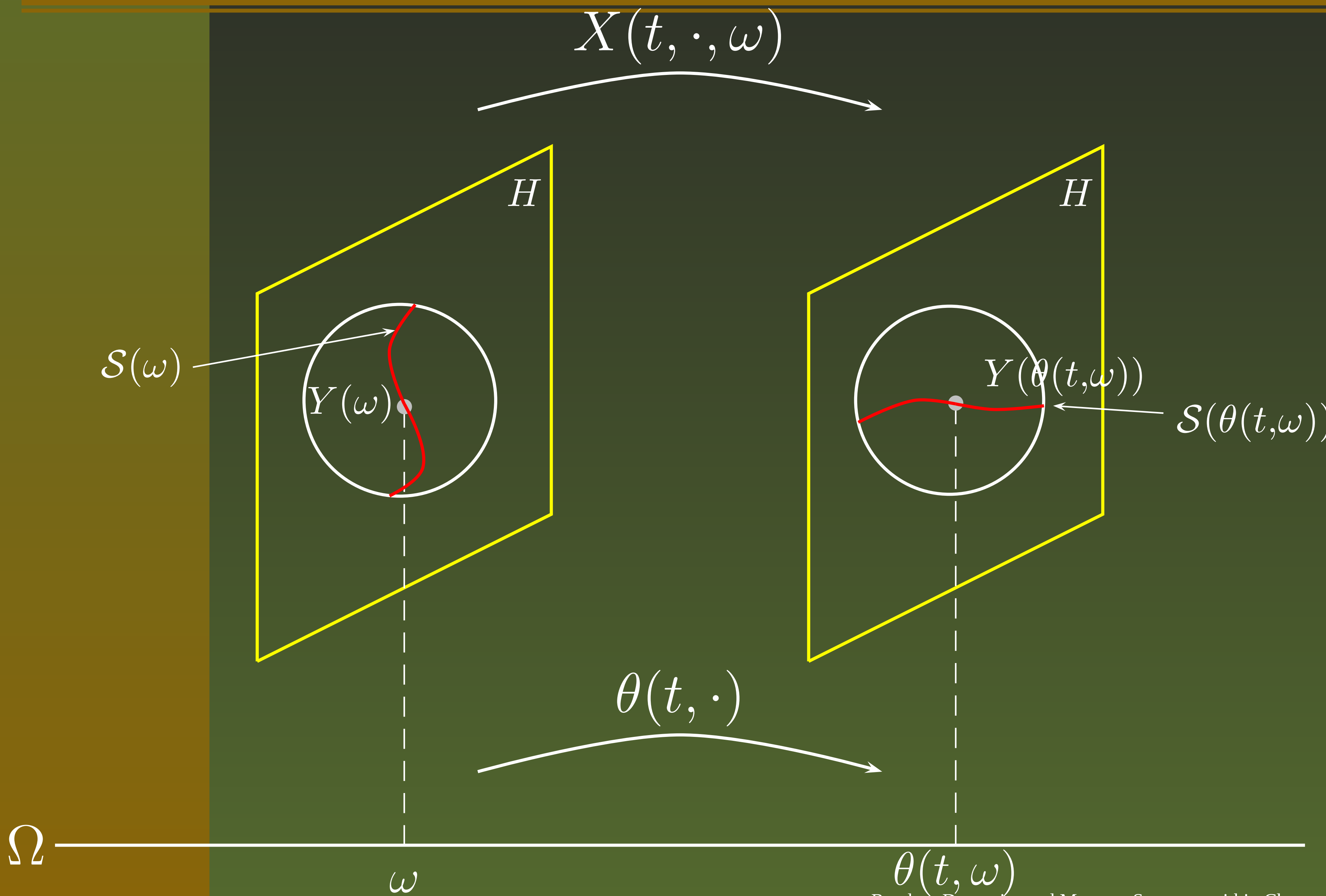

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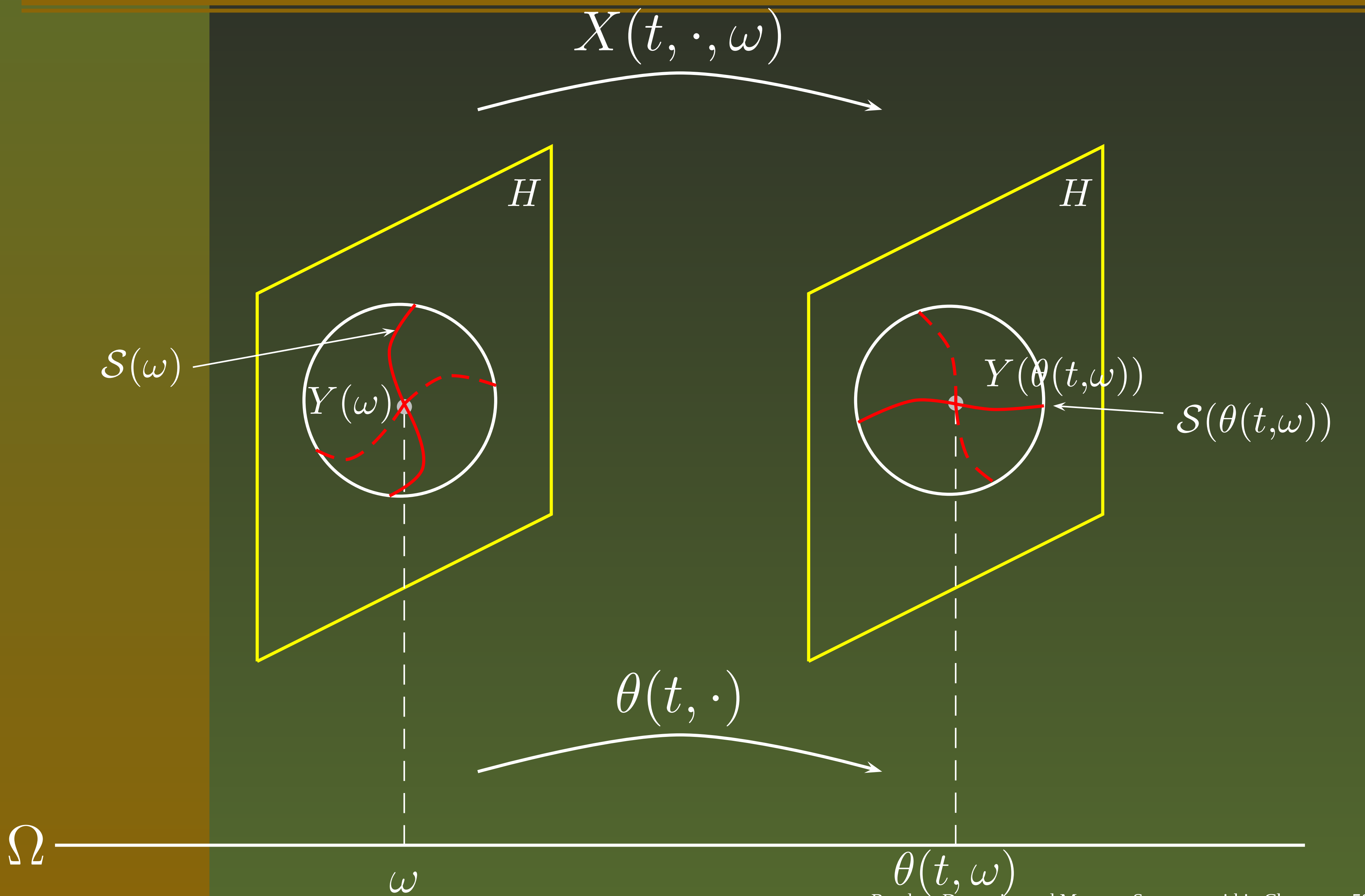

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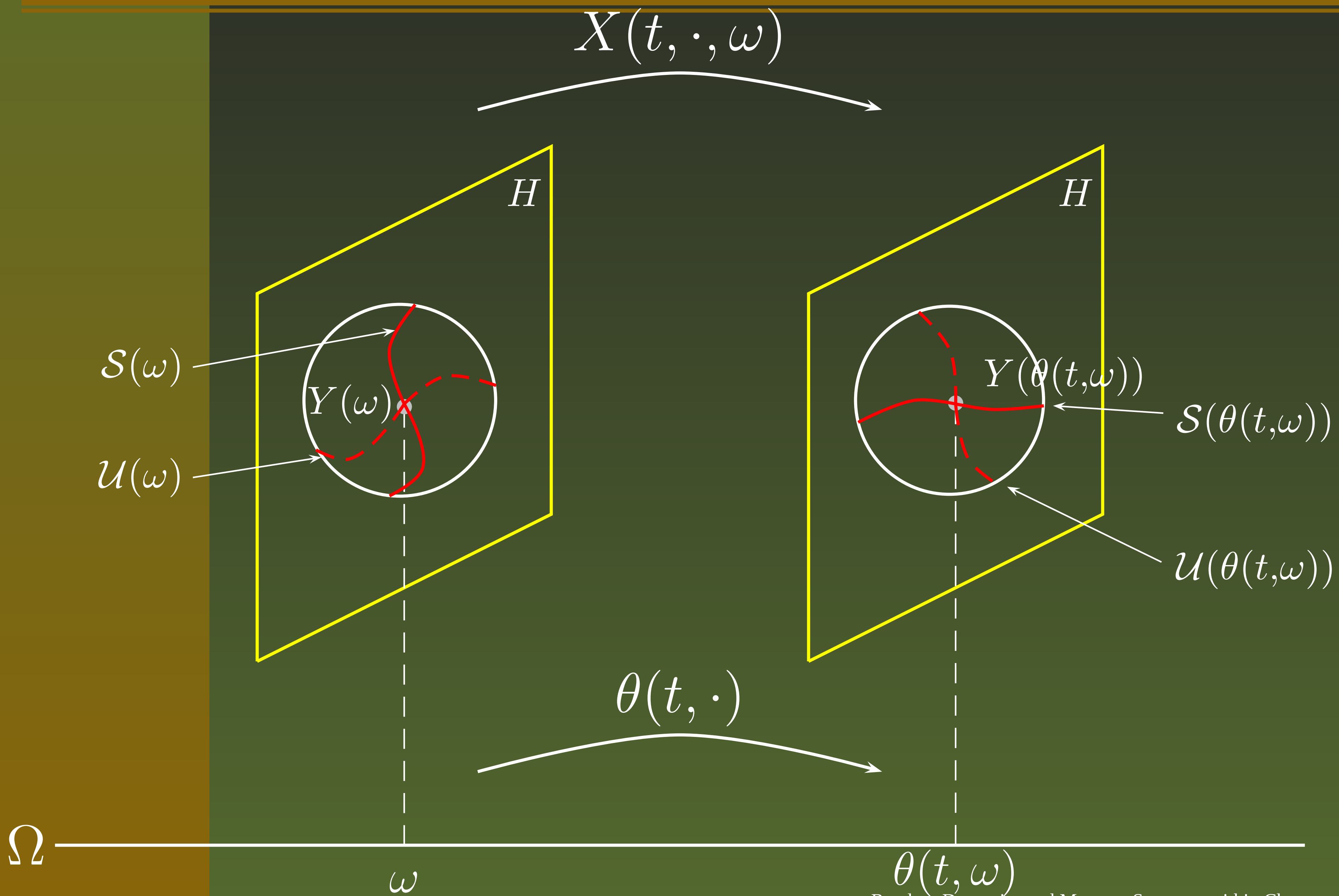

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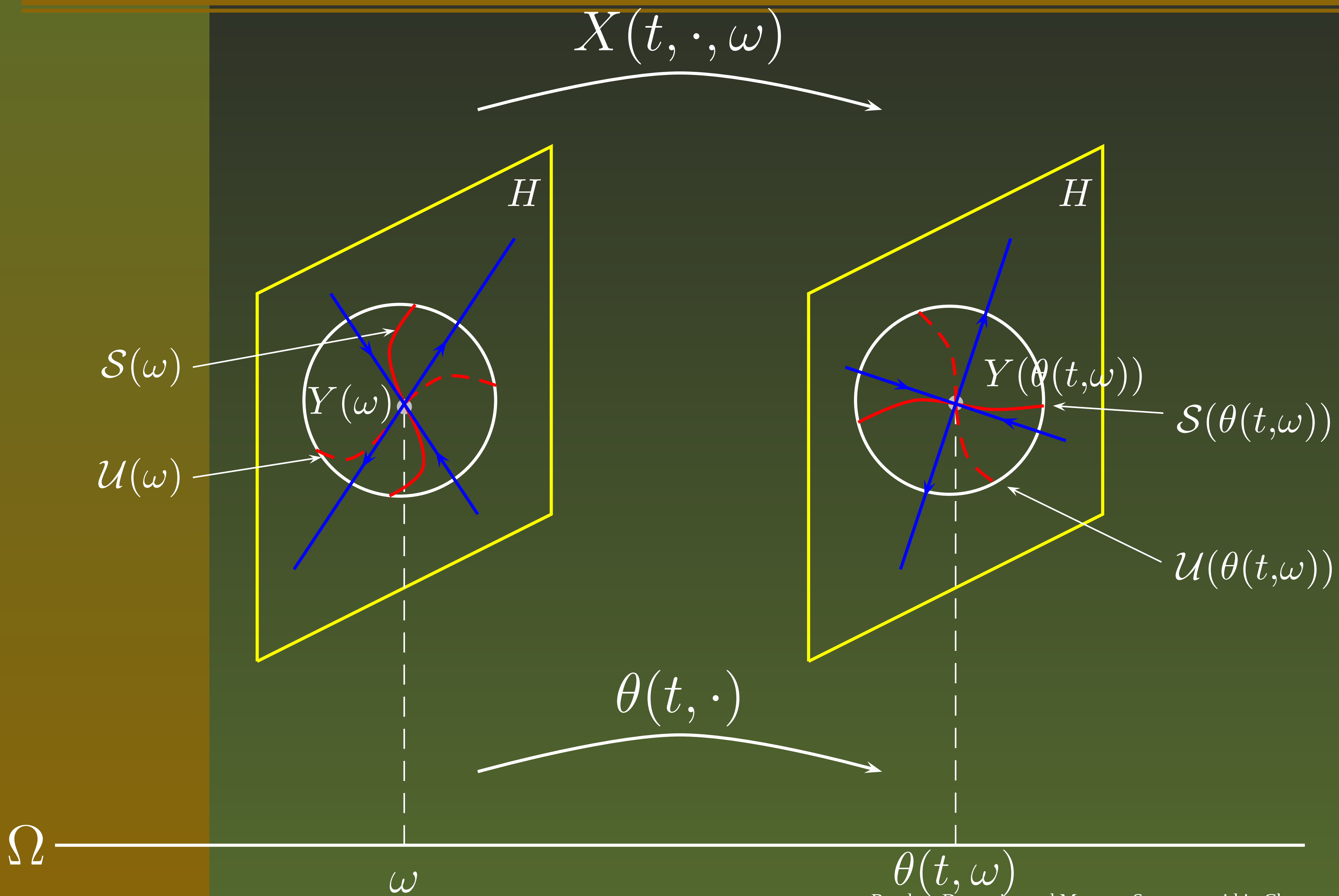

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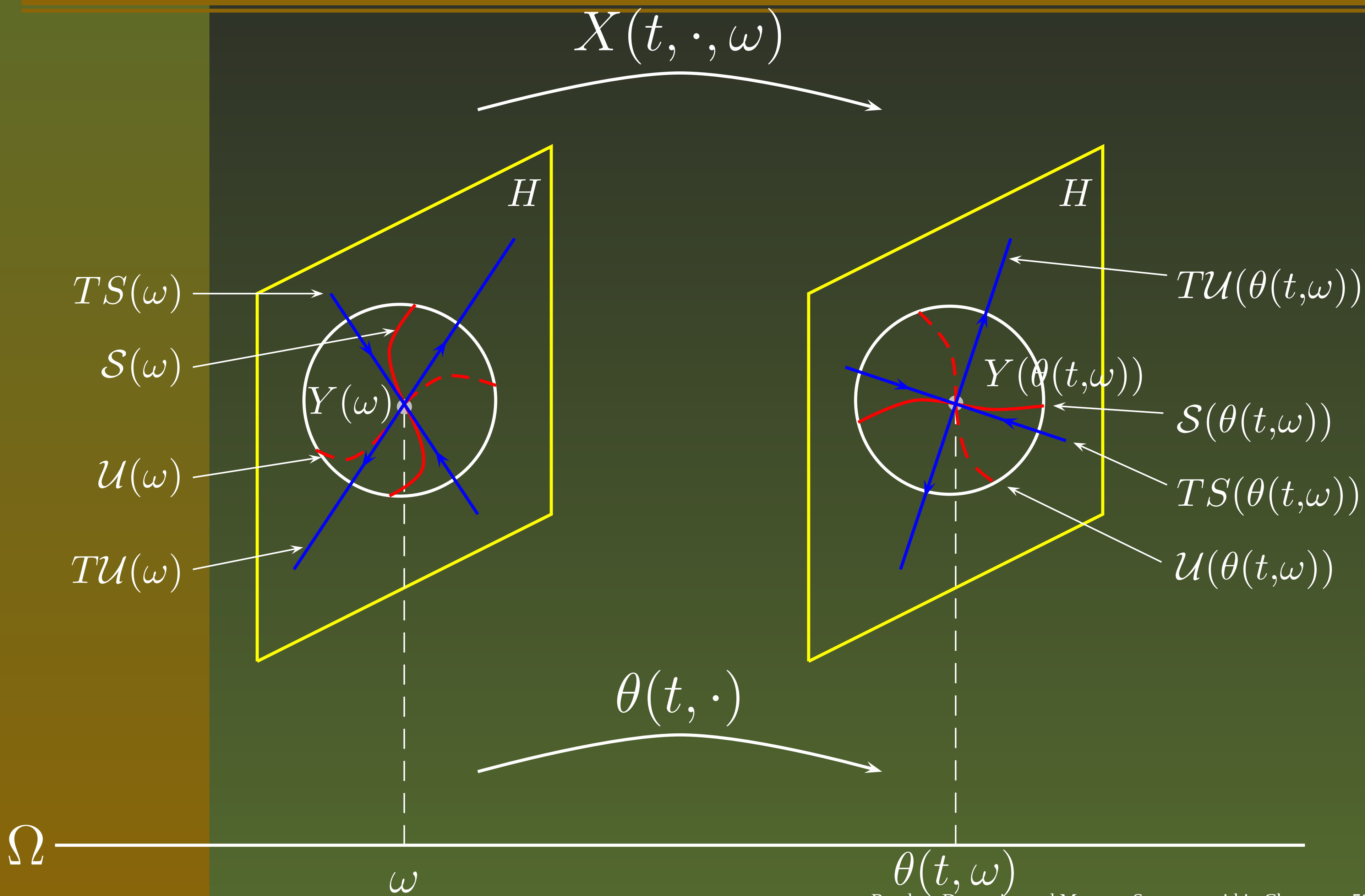

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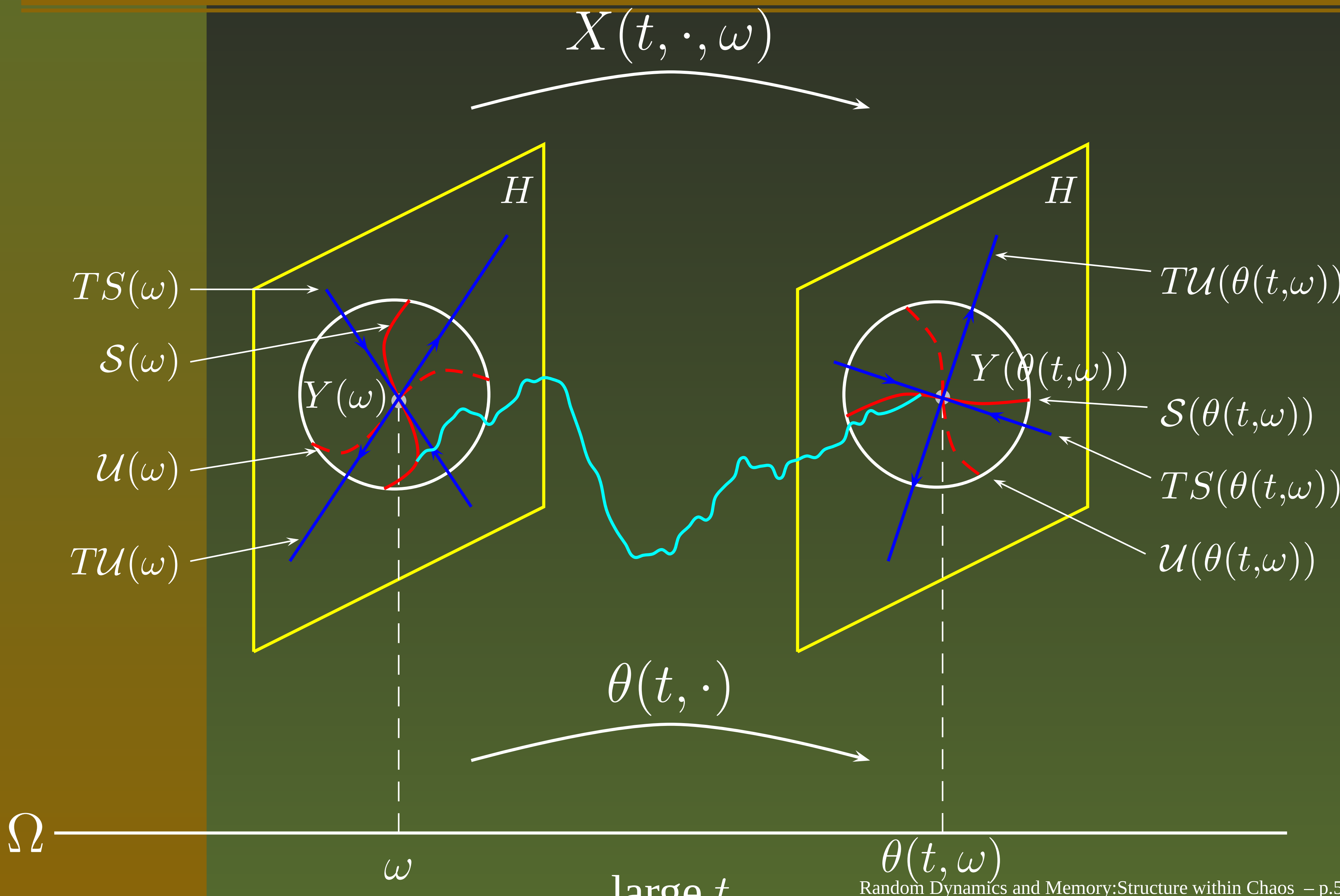

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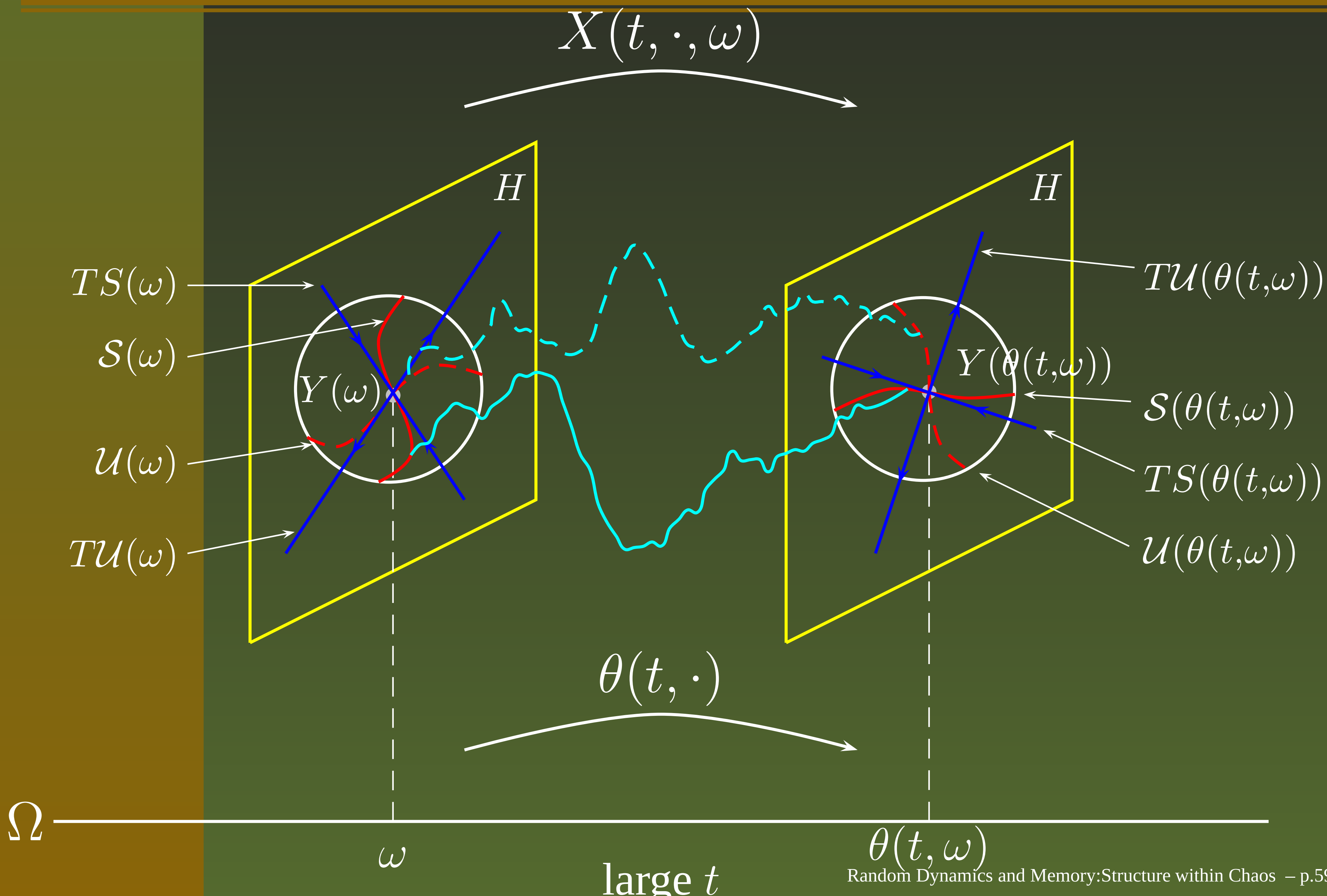

Proof

Details in [M.S]

Sketch of Strategy

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An idea whose time has come!

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An idea whose time has come!

Please contact *salah@sfde.math.siu.edu* with suggestions and/or ideas.

THE END!

THANK YOU!

SKETCH OF PROOF

OF STABLE MANIFOLD

THEOREM

(<–)

Strategy

- By definition, a *stationary* random point $Y(\omega) \in H$ is invariant under the random flow X; viz $X(t, Y) = Y(\theta(t, \cdot))$ for all times t.

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- Linearize the random flow X along the stationary point $Y(\omega)$ in H. By stationarity of Y and the random flow property of X, this gives a random linear flow $(DX(t, Y), \theta(t, \cdot))$ in $L(H)$, the space of all continuous linear operators on H.

Strategy-contd

- Ergodicity of θ allows for the notion of hyperbolicity of a stationary point Y of the random flow X via Oseledec-Ruelle theorem:

Strategy-contd

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$$\lim_{t\to\infty}\left\{\left[DX(t,Y(\omega),\omega)\right]^*\circ\left[DX(t,Y(\omega),\omega)\right]\right\}^{1/2t}$$

is a compact, symmetric, non-negative operator with discrete non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_i} > \cdots$$

with Lyapunov exponents $\{\lambda_i, i \geq 1\}$.

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with Lyapunov exponents $\{\lambda_i, i \geq 1\}$.

Y is *hyperbolic* if $\lambda_i \neq 0$ for every i.

Strategy-contd

- Assume that $\|Y\|^{\epsilon_0}$ is integrable (for small ϵ_0). Variational method of construction of the random flow shows that the linearized flow satisfies hypotheses of refined versions of ergodic theorem and Kingman's subadditive ergodic theorem. These refined versions give invariance of the Oseledec spaces under the continuous-time linearized flow. Thus the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear random flow X.

Strategy-contd

- Establish continuous-time integrability estimates on the spatial derivatives of the non-linear flow X in a neighborhood of the stationary point Y. Estimates follow from the variational construction of the random flow.

Strategy-contd

- Introduce the auxiliary random flow

$$Z(t,\cdot,\omega) := X(t,(\cdot)+Y(\omega),\omega) - Y(\theta(t,\omega)),$$
$$t \in \mathbf{R}^+, \omega \in \Omega.$$

Refine arguments in [Ru.2] to construct local stable/ unstable manifolds for the discrete random flow $(Z(n,\cdot,\omega),\theta(n,\omega))$ near 0 and hence (by translation) for $X(n,\cdot,\omega)$ near $Y(\theta(n,\omega))$ for all ω sampled from a $\theta(t,\cdot)$-invariant sure event in Ω.

Strategy-contd

- This is possible via continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between discrete times and further refining the arguments in [Ru.2], show that the above manifolds also serve as local stable/unstable manifolds for the *continuous-time* random flow X near Y.

Strategy-contd

- Final key step:

  Establish the asymptotic invariance of the local stable manifolds under the random flow X. Combine arguments in [Ru.2] with some difficult estimates using the continuous-time integrability properties, and the perfect subadditive ergodic theorem. Asymptotic invariance of the local unstable manifolds follows by employing the concept of a *stochastic history process* for X coupled with similar arguments to the above. Existence of history process compensates for the lack of invertibility of the random flow. $\square$ (<–)

Some Finance-Definitions

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European call options can only be exercised at the maturity date.

Delayed Black-Scholes Formula

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Assume delayed stock dynamics with portfolio consisting of a stock S *and a bond* $B(t) = e^{rt}$*. Let* $V(t)$ *be the fair price of a European call option written on the stock* S *with exercise price* K *and maturity time* T*.*

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Then for all $t \in [T-l, T]$ (where $l := \min\{a, b\}$), the option price $V(t)$ is given by

$$V(t) = S(t)\varphi(\beta_+(t)) - Ke^{-r(T-t)}\varphi(\beta_-(t)),$$

Delayed Formula – Cont'd

where

$$\beta_{\pm}(t) := \frac{\log \frac{S(t)}{K} + \int_t^T \left(r \pm \frac{1}{2} g(S(u-b))^2\right) du}{\sqrt{\int_t^T g(S(u-b))^2 du}}.$$

The hedging strategy is given by

$$\pi_S(t) = \varphi(\beta_+(t)),$$

$$\pi_B(t) = -Ke^{-rT}\varphi(\beta_-(t)),$$

for $t \in [T-\ell, T]$.