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An Alternative to Cronbach's Alpha: A *L*-Moment Based Measure of Internal-consistency Reliability

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Abstract Data sets in the social and behavioral sciences are often small or heavy-tailed. Previous studies have demonstrated that small samples or leptokurtic distributions adversely affect the performance of Cronbach's coefficient alpha. To address these concerns, we propose an alternative estimator of reliability based on *L*-comoments. The empirical results of this study demonstrate that when sample sizes are small and distributions are heavy-tailed that the proposed coefficient *L*-alpha has substantial advantages over the conventional Cronbach estimator of reliability in terms of relative bias and relative standard error.

1 Introduction

Coefficient alpha [5, 7] is a commonly used index for measuring internal consistency reliability. Consider alpha (α) in terms of a model that decomposes an observed score into the sum of two independent components: a true unobservable score t_i and a random error component e_{ij}. The model can be summarized as

$$X_{ij} = t_i + e_{ij} \tag{1}$$

where X_{ij} is the observed score associated with the i-th examinee on the j-th test item, and where $i = 1,\dots,n$; $j = 1,\dots,k$; and the error terms (e_{ij}) are independent with a mean of zero. Inspection of (1) indicates that this particular model restricts the true score t_i to be the same across all k test items. The reliability measure associated

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with the test items in (1) is a function of the true score variance and cannot be computed directly. Thus, estimates of reliability such as coefficient α have been derived and will be defined herein as (e.g., [3])

$$\alpha = \frac{k}{k-1}\left(1 - \frac{\sum_j \sigma_j^2}{\sum_j \sigma_j^2 + \sum\sum_{j \neq j'} \sigma_{jj'}}\right). \tag{2}$$

A conventional estimate of α can be obtained by substituting the usual OLS sample estimates associated with σ_j^2 and $\sigma_{jj'}$ into (2) as

$$\hat{\alpha}_C = \frac{k}{k-1}\left(1 - \frac{\sum_j s_j^2}{\sum_j s_j^2 + \sum\sum_{j \neq j'} s_{jj'}}\right) \tag{3}$$

where s_j^2 and $s_{jj'}$ are the diagonal and off-diagonal elements from the variance-covariance matrix, respectively.

Although coefficient α is often used as an index for reliability, it is also well known that its use is limited when data are non-normal, in particular leptokurtic, or when sample sizes are small (e.g. [1, 3, 31, 33]). These limitations are of concern because data sets in the social and behavioral sciences can often possess heavy tails or consist of small sample sizes (e.g. [25, 34]). Specifically, it has been demonstrated that $\hat{\alpha}_C$ can substantially underestimate α when heavy-tailed distributions are encountered. For example, Sheng and Sheng [31, Table 1] sampled from a symmetric leptokurtic distribution and found the empirical estimate of α to be approximately $\hat{\alpha}_C = 0.70$ when the true population parameter was $\alpha = 0.80$. Further, it is not uncommon that data sets consist of small sample sizes e.g. $n = 10$ or 20. More specifically, small sample sizes are commonly encountered in the contexts of rehabilitation (e.g. alcohol treatment programs, group therapy, etc.) and special education as student-teacher ratios are often small. Furthermore, Monte Carlo evidence has demonstrated that $\hat{\alpha}_C$ can underestimate α - even when small samples are drawn from a normal distribution (see [31], Table 1).

L-moment estimators (e.g. [20, 22]) have demonstrated to be superior to the conventional product-moment estimators in terms of bias, efficiency, and their resistance to outliers (e.g. [10, 19, 21, 32]). Further, *L*-comoment estimators [30] such as the *L*-correlation has demonstrated to be an attractive alternative to the conventional Pearson correlation in terms of relative bias when heavy-tailed distributions are of concern [11, 12, 13, 14, 15].

In view of the above, the present aim here is to propose a *L*-comoment based coefficient L-α, and its estimator denoted as $\hat{\alpha}_L$, as an alternative to conventional alpha $\hat{\alpha}_C$ in (3). Empirical results associated with the simulation study herein indicate that $\hat{\alpha}_L$ can be substantially superior to $\hat{\alpha}_C$ in terms of relative bias and relative standard error when distributions are heavy-tailed and sample sizes are small.

The rest of the paper is organized as follows. In Section 2, summaries of univariate *L*-moments and *L*-comoments are first provided. Coefficient L-α ($\hat{\alpha}_L$) is then introduced and numerical examples are provided to illustrate the computation and sampling distribution associated with $\hat{\alpha}_L$. In Section 3, a Monte Carlo study is carried out to evaluate the performance of $\hat{\alpha}_C$ and $\hat{\alpha}_L$. The results of the study are discussed in Section 4.

2 *L*-moments, *L*-comoments, and Coefficient *L*-α

The system of univariate L-moments [20, 21, 22] can be considered in terms of the expectations of linear combinations of order statistics associated with a random variable Y. Specifically, the first four L-moments are expressed as

$$\begin{aligned}
\lambda_1 &= E[Y_{1:1}] \\
\lambda_2 &= \frac{1}{2}E[Y_{2:2} - Y_{1:2}] \\
\lambda_3 &= \frac{1}{3}E[Y_{3:3} - 2Y_{2:3} + Y_{1:3}] \\
\lambda_4 &= \frac{1}{4}E[Y_{4:4} - 3Y_{3:4} + 3Y_{2:4} - Y_{1:4}]
\end{aligned}$$

where $Y_{\ell:m}$ denotes the ℓth smallest observation from a sample of size m. As such, $Y_{1:m} \le Y_{2:m} \le \ldots \le Y_{m:m}$ are referred to as order statistics drawn from the random variable Y. The values of λ_1 and λ_2 are measures of location and scale and are the arithmetic mean and one-half the coefficient of mean difference (or Gini's index of spread), respectively. Higher order L-moments are transformed to dimensionless quantities referred to as L-moment ratios defined as $\tau_r = \lambda_r/\lambda_2$ for $r \ge 3$, and where τ_3 and τ_4 are the analogs to the conventional measures of skew and kurtosis. In general, L-moment ratios are bounded in the interval $-1 < \tau_r < 1$ as is the index of L-skew (τ_3) where a symmetric distribution implies that all L-moment ratios with odd subscripts are zero. Other smaller boundaries can be found for more specific cases. For example, the index of L-kurtosis (τ_4) has the boundary condition for continuous distributions of $(5\tau_3^2 - 1)/4 < \tau_4 < 1$.

L-comoments [26, 30] are introduced by considering two random variables Y_j and Y_k with distribution functions $F(Y_j)$ and $F(Y_k)$. The second L-moments associated with Y_j and Y_k can alternatively be expressed as

$$\begin{aligned}
\lambda_2(Y_j) &= 2\mathrm{Cov}(Y_j, F(Y_j)) \\
\lambda_2(Y_k) &= 2\mathrm{Cov}(Y_k, F(Y_k)).
\end{aligned} \tag{4}$$

The second L-comoments of Y_j toward Y_k and Y_k toward Y_j are

$$\begin{aligned}
\lambda_2(Y_j, Y_k) &= 2\mathrm{Cov}(Y_j, F(Y_k)) \\
\lambda_2(Y_k, Y_j) &= 2\mathrm{Cov}(Y_k, F(Y_j)).
\end{aligned} \tag{5}$$

The ratio $\eta_{jk} = \lambda_2(Y_j, Y_k)/\lambda_2(Y_j)$ is defined as the L-correlation of Y_j with respect to Y_k, which measures the monotonic relationship (not just linear) between two variables [13]. Note that in general, $\eta_{jk} \neq \eta_{kj}$. The estimators of (4) and (5) are U-statistics [29, 30] and their sampling distributions converge to a normal distribution when the sample size is sufficiently large.

In terms of coefficient L-α, an approach that can be taken to equate the conventional and L-moment (comoment) definitions of α is to express (2) as

$$\alpha = \frac{1}{1+(R-1)/k} = \frac{k}{k-1}\left(1 - \frac{\sum_j \sigma_j^2}{\sum_j \sigma_j^2 + \sum\sum_{j \neq j'} \sigma_{jj'}}\right) \tag{6}$$

where $R > 1$ is the common ratio between the main and off diagonal elements of the variance-covariance matrix i.e. $R = \sigma_j^2 \big/ \sigma_{jj'}$. (See the Appendix for the derivation of equation 6). As such, given a fixed value of R in (6) will allow for α to be defined in terms of the second L-moments and second L-comoments as

$$\alpha = \frac{1}{1+(R-1)/k} = \frac{k}{k-1}\left(1 - \frac{\sum_j \lambda_{2(j)}}{\sum_j \lambda_{2(j)} + \sum\sum_{j \neq j'} \lambda_{2(jj')}}\right) \tag{7}$$

where $R = \lambda_{2(j)}/\lambda_{2(jj')}$. Thus, the estimator of L-α is expressed as

$$\hat{\alpha}_L = \frac{k}{k-1}\left(1 - \frac{\sum_j \ell_{2(j)}}{\sum_j \ell_{2(j)} + \sum\sum_{j \neq j'} \ell_{2(jj')}}\right) \tag{8}$$

where $\ell_{2(j)}$ ($\ell_{2(jj')}$) denotes the sample estimate of the second L-moments (second L-comoment) in (4) and (5). An example demonstrating the computation of $\hat{\alpha}_L$ is provided below in equation (9). The computed estimate of $\hat{\alpha}_L = 0.807$ in (9) is based on the data in Table 1 and the second L-moment-comoment matrix in Table 2. The corresponding conventional estimate for the data in Table 1 is $\hat{\alpha}_C = 0.798$.

Table 1 Data (Items) for computing the second L-moment-comoment matrix in Table 2.

X_{i1}	X_{i2}	X_{i3}	$\hat{F}(X_{i1})$	$\hat{F}(X_{i2})$	$\hat{F}(X_{i3})$
2	4	3	0.15	0.45	0.15
5	7	7	0.75	0.95	1.00
3	5	5	0.35	0.65	0.40
6	6	6	0.90	0.80	0.75
7	7	6	1.00	0.95	0.75
5	2	6	0.75	0.10	0.75
2	3	3	0.15	0.25	0.15
4	3	6	0.55	0.25	0.75
3	5	5	0.35	0.65	0.40
4	4	5	0.55	0.45	0.40

The data are part of the ”Satisfaction With Life Data” from [24, p. 47].

Table 2 Second *L*-moment-comoment matrix for coefficient $\hat{\alpha}_L$ in equation (9).

Item	1	2	3
1	$\ell_{2(1)} = 0.989$	$\ell_{2(12)} = 0.500$	$\ell_{2(13)} = 0.789$
2	$\ell_{2(21)} = 0.500$	$\ell_{2(2)} = 1.022$	$\ell_{2(23)} = 0.411$
3	$\ell_{2(31)} = 0.667$	$\ell_{2(32)} = 0.333$	$\ell_{2(3)} = 0.733$

$$\hat{\alpha}_L = 0.807 = (3/2)(1 - (\ell_{2(1)} + \ell_{2(2)} + \ell_{2(3)})/(\ell_{2(1)} + \ell_{2(2)} + \ell_{2(3)} + \ell_{2(21)} + \ell_{2(31)} + \ell_{2(32)} + \ell_{2(12)} + \ell_{2(13)} + \ell_{2(23)})). \quad (9)$$

The estimator $\hat{\alpha}_L$ in (8) and (9) is a ratio of the sums of U-statistics and thus a consistent estimator of α in (7) with a sampling distribution that converges, for large samples, to the normal distribution (e.g. [26, 28, 30]). For convenience to the reader, provided in Figure 1 is the sampling distribution of $\hat{\alpha}_L$ that is approximately normal and based on $\alpha = 0.50$, $n = 100,000$, and a symmetric heavy-tailed distribution (kurtosis of 25, see Figure 2) that would be associated with t_i in (1).

3 Monte Carlo Simulation

An algorithm was written in MATLAB [23] to generate 25,000 independent sample estimates of conventional and *L*-comoment α. The estimators $\hat{\alpha}_C$ and $\hat{\alpha}_L$ were based on the parameters (α, k, R) given in Table 3 and Table 4 and the distributions in Figures 2-4. The parameters of α were selected because they represent commonly used references of various degrees of reliability i.e. 0.50 (poor); 5/7 = 0.714 (acceptable); 0.80 (good); and 0.90 (excellent). Further, for each set of parameters in Table 3 and Table 4, the empirical estimators $\hat{\alpha}_C$ and $\hat{\alpha}_L$ were generated based on sample sizes of $n = 10$, 20, 1000. For all cases in the simulation, the error term e_{ij} in (1) was normally distributed with zero mean and with the variance parameters (σ_e^2) listed in Table 3 and Table 4.

Table 3 Parameters for the Conventional covariance (*L*-comoment) matrix and distributions in Figures 2-4.

Distribution-Matrix	Diagonal	Off-Diagonal	σ_e^2
1-C	3.420	1.710	1.710
1-L	0.848	0.424	1.000
2-C	3.224	1.612	1.612
2-L	0.842	0.421	1.000
3-C	2.000	1.000	1.000
3-L	0.798	0.399	1.000

Reliability is α =0.80, 0.90; Number of Items are k =4, 9.
Ratio of Diagonal to Off-Diagonal is $R = 2$.

Table 4 Parameters for the Conventional covariance (*L*-comoment) matrix and distributions in Figures 2-4.

Distribution-Matrix	Diagonal	Off-Diagonal	σ_e^2
1-C	8.550	1.710	6.840
1-L	1.470	0.294	5.313
2-C	8.060	1.612	6.448
2-L	1.443	0.2886	5.135
3-C	5.000	1.000	4.000
3-L	1.262	0.2524	4.000

Reliability is $\alpha = 0.50, 0.714$; Number of Items are $k = 4, 10$.
Ratio of Diagonal to Off-Diagonal is $R = 5$.

The three distributions depicted in Figures 2-4 are associated with the true scores t_i in equation (1). These distributions are referred to as: Distribution 1 is symmetric and leptokurtic (skew = 0, kurtosis = 25; *L*-skew = 0, *L*-kurtosis = 0.4225); Distribution 2 is asymmetric and leptokurtic (skew = 3, kurtosis = 21; *L*-skew = 0.3130, *L*-kurtosis = 0.3335); and Distribution 3 is standard normal (skew = 0, kur-

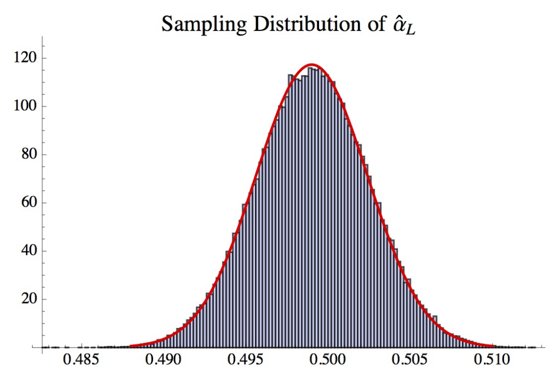

Fig. 1 Approximate normal sampling distribution of $\hat{\alpha}_L$ with $\alpha = 0.50$. The distribution consists of 25,000 statistics based on samples of size $n = 100,000$ and the heavy-tailed distribution (kurtosis of 25) in Figure 2.

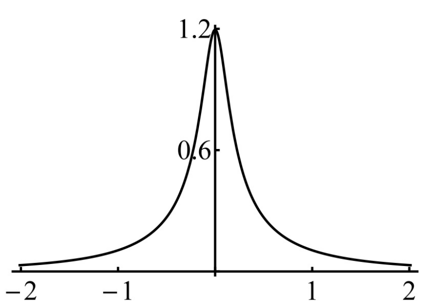

Fig. 2 Distribution 1 with skew (*L*-skew) of 0 (0) and kurtosis (*L*-kurtosis) of 25 (0.4225).

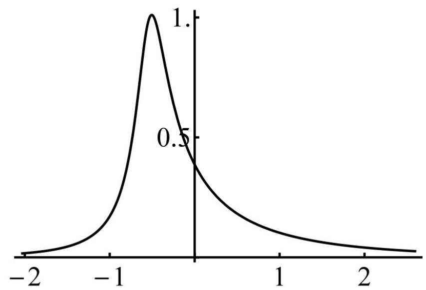

Fig. 3 Distribution 2 with skew (*L*-skew) of 3 (0.3130) and kurtosis (*L*-kurtosis) of 21 (0.3335).

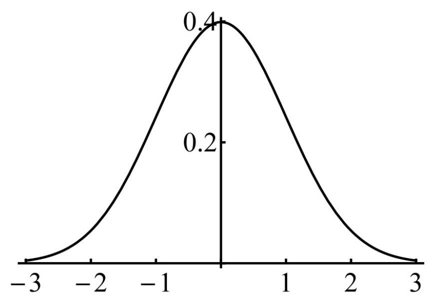

Fig. 4 Distribution 3 is standard normal with skew (*L*-skew) of 0 (0) and kurtosis (*L*-kurtosis) of 0 (0.1226).

tosis = 0; *L*-skew = 0, *L*-kurtosis = 0.1226). We would note that Distributions 1 and 2 have been used in several studies in the social and behavioral sciences (e.g. [2, 4, 8, 16, 17, 27]).

The pseudo-random deviates associated with the distributions in Figures 2-4 were generated for this study using the *L*-moment based power method transformation derived by [10]. Specifically, the true scores t_i in (1) were generated using the following Fleishman [6] type polynomial

$$t_i = c_1 + c_2 Z_i + c_3 Z_i^2 + c_4 Z_i^3 \tag{10}$$

where $Z_i \sim$ iid $N(0,1)$. The shape of the distribution of the true scores t_i in (10) is contingent on the values of the coefficients, which are computed based on Headrick's equations (2.14)-(2.17) in Headrick [10] as

$$\begin{aligned} c_1 &= -c_3 = -\tau_3 \sqrt{\frac{\pi}{3}} \\ c_2 &= \frac{-16\delta_2 + \sqrt{2}(3 + 2\tau_4)\pi}{8(5\delta_1 - 2\delta_2)} \\ c_4 &= \frac{40\delta_1 - \sqrt{2}(3 + 2\tau_4)\pi}{20(5\delta_1 - 2\delta_2)}. \end{aligned} \tag{11}$$

The three sets of coefficients for the distributions in Figures 2-4 are (respectively): (1) $c_1 = 0.0$, $c_2 = 0.3338$, $c_3 = 0.0$, $c_4 = 0.2665$; (2) $c_1 = -0.3203$, $c_2 = 0.5315$, $c_3 = 0.3203$, $c_4 = 0.1874$; and (3) $c_1 = 0.0$, $c_2 = 1.0$, $c_3 = 0.0$, $c_4 = 0.0$. The values of the three sets of coefficients are based on the values of L-skew and L-kurtosis given in Figures 2-4 and where $\delta_1 = 0.36045147$ and $\delta_2 = 1.15112868$ in (11) (see [10], Eqs. A.1, A.2). The solutions to the coefficients in (11) ensure that $\lambda_1 = 0$ and $\lambda_2 = 1/\sqrt{\pi}$, which are associated with the unit normal distribution.

The estimator $\hat{\alpha}_C$ was computed using equation (3). The estimator $\hat{\alpha}_L$ was computed using equations (4), (5), and (8) as was demonstrated in Table 1 and Table 2. The estimators were both transformed to the form of an intraclass correlation as $\bar{\rho}_{C,L} = \hat{\alpha}_{C,L}/(1-(k-1)\hat{\alpha}_{C,L})$ (e.g. [9], p. 104) and were subsequently Fisher z' transformed i.e. $z'_{\bar{\rho}_{C,L}}$. Bias-corrected accelerated bootstrapped average (mean) estimates, confidence intervals (C.I.s), and standard errors were subsequently obtained for $z'_{\bar{\rho}_{C,L}}$ using 10,000 resamples. The bootstrap results associated with the means and C.I.s were then transformed back to their original metrics (i.e. the estimators $\hat{\alpha}_C$ and $\hat{\alpha}_L$). Further, percentages of relative bias (RBias) and relative standard error (RSE) were computed for $\hat{\alpha}_{C,L}$ as: RBias $= ((\hat{\alpha}_{C,L} - \alpha)/\alpha) \times 100$ and RSE $=$ (standard error$/\hat{\alpha}_{C,L}) \times 100$. The results of the simulation are reported in Tables 5-7 and are discussed in the next section.

4 Discussion and Conclusion

One of the advantages that L-moment ratios have over conventional product-moment estimators is that they can be far less biased when sampling is from distributions with more severe departures from normality [22, 30]. And, inspection of the simulation results in Table 5 and Table 6 clearly indicates that this is the case. That is, the superiority that the L-comoment based estimator $\hat{\alpha}_L$ has over its corresponding conventional counterpart $\hat{\alpha}_C$ is obvious in the contexts of Distributions 1 and 2. For example, inspection of the first entry in Table 5 ($\alpha = 0.50$, $k = 4$, $n = 10$) indicates that the estimator $\hat{\alpha}_C$ associated with Distribution 1 was, on average, 88.32% of its associated population parameter whereas the estimator $\hat{\alpha}_L$ was 96.94% of its parameter. Further, and in the context of Distribution 1, it is also evident that $\hat{\alpha}_L$ is a more efficient estimator as its relative standard error is smaller than its corresponding conventional estimator (see Table 5, $\alpha = 0.50$, $k = 4$, $n = 10$). This demonstrates that $\hat{\alpha}_L$ has more precision because it has less variance around its estimate.

In summary, the L-comoment based $\hat{\alpha}_L$ is an attractive alternative to the traditional Cronbach alpha $\hat{\alpha}_C$ when distributions with heavy tails and small samples sizes are encountered. It is also worthy to point out that $\hat{\alpha}_L$ had a slight advantage over $\hat{\alpha}_C$ when sampling was from normal populations (see Table 5; $\alpha = 0.50$, $k = 4$, $n = 10$, 3-C, 3-L). When sample sizes were large the performance of the two estimators $\hat{\alpha}_{C,L}$ were similar (see Table 7; $n = 1000$).

Appendix

Under the assumption of parallel measures, the error term e_{ij} in equation (1) has constant variance σ_e^2, the variance-covariance matrix assumes compound-symmetry, and thus the main and off diagonal elements are $\sigma_j^2 = \sigma_X^2$ and $\sigma_{jj'} = \sigma_t^2$, respectively. Hence, equation (2) can be expressed using the true score and observed score variances as

$$\alpha = \frac{k}{k-1}\left(1 - \frac{k\sigma_X^2}{k\sigma_X^2 + k(k-1)\sigma_t^2}\right),$$

which can be simplified to

$$\begin{aligned}\alpha &= \frac{k}{k-1}\left(1 - \frac{\sigma_X^2}{\sigma_X^2 + (k-1)\sigma_t^2}\right)\\ &= \frac{k}{k-1}\left(\frac{(k-1)\sigma_t^2}{\sigma_X^2 + (k-1)\sigma_t^2}\right)\\ &= \frac{k\sigma_t^2}{\sigma_X^2 + (k-1)\sigma_t^2}.\end{aligned}$$

If we let $R = \sigma_j^2 \big/ \sigma_{jj'} = \sigma_X^2 \big/ \sigma_t^2$, then it follows that

$$\alpha = \frac{k}{R+k-1} = \frac{1}{1+(R-1)/k},$$

which is given in equation (6).

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Table 5 Simulation results for α based on the Conventional (C) and L-moment (L) procedures (Proc) based on samples of size $n = 10$.

Parameters	Dist-Proc	Estimate (α)	95% C.I.	RSE %	RBias %
$\alpha = 0.50, k = 4$	1-C	0.4416	0.4367, 0.4465	0.5661	-11.68
$\alpha = 0.50, k = 4$	1-L	0.4847	0.4801, 0.4891	0.4725	-3.06
$\alpha = 0.50, k = 4$	2-C	0.4448	0.4400, 0.4495	0.3237	-11.04
$\alpha = 0.50, k = 4$	2-L	0.4839	0.4796, 0.4883	0.2583	-3.22
$\alpha = 0.50, k = 4$	3-C	0.4888	0.4852, 0.4922	0.3621	-2.24
$\alpha = 0.50, k = 4$	3-L	0.5003	0.4968, 0.5040	0.3698	0.06
$\alpha = 0.714, k = 10$	1-C	0.6617	0.6581, 0.6652	0.2720	-7.36
$\alpha = 0.714, k = 10$	1-L	0.6960	0.6931, 0.6989	0.2155	-2.56
$\alpha = 0.714, k = 10$	2-C	0.6662	0.6628, 0.6697	0.2612	-6.73
$\alpha = 0.714, k = 10$	2-L	0.6975	0.6946, 0.7003	0.2079	-2.35
$\alpha = 0.714, k = 10$	3-C	0.7069	0.7051, 0.7086	0.1273	-1.03
$\alpha = 0.714, k = 10$	3-L	0.7131	0.7113, 0.7149	0.1290	-0.17
$\alpha = 0.80, k = 4$	1-C	0.7306	0.7275, 0.7336	0.2053	-8.67
$\alpha = 0.80, k = 4$	1-L	0.7887	0.7866, 0.7908	0.1357	-1.41
$\alpha = 0.80, k = 4$	2-C	0.7398	0.7371, 0.7426	0.1906	-7.52
$\alpha = 0.80, k = 4$	2-L	0.7924	0.7904, 0.7944	0.1287	-0.95
$\alpha = 0.80, k = 4$	3-C	0.7908	0.7893, 0.7922	0.0923	-1.15
$\alpha = 0.80, k = 4$	3-L	0.8030	0.8016, 0.8044	0.0909	0.37
$\alpha = 0.90, k = 9$	1-C	0.8591	0.8575, 0.8609	0.0989	-4.54
$\alpha = 0.90, k = 9$	1-L	0.8924	0.8914, 0.8936	0.0628	-0.84
$\alpha = 0.90, k = 9$	2-C	0.8636	0.8620, 0.8651	0.0926	-4.04
$\alpha = 0.90, k = 9$	2-L	0.8933	0.8922, 0.8944	0.0605	-0.74
$\alpha = 0.90, k = 9$	3-C	0.8934	0.8927, 0.8941	0.0381	-0.73
$\alpha = 0.90, k = 9$	3-L	0.8991	0.8985, 0.8998	0.0378	-0.10

See Tables 3 and 4 for the parameters and Figures 2-4 for the distributions (Dist).

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Table 6 Simulation results for α based on the Conventional (C) and *L*-moment (L) procedures (Proc) based on samples of size $n = 20$.

Parameters	Dist-Proc	Estimate (α)	95% C.I.	RSE %	RBias %
$\alpha = 0.50, k = 4$	1-C	0.4643	0.4606, 0.4679	0.3977	-7.15
$\alpha = 0.50, k = 4$	1-L	0.4903	0.4870, 0.4933	0.3263	-1.94
$\alpha = 0.50, k = 4$	2-C	0.4697	0.4663, 0.4732	0.3732	-6.05
$\alpha = 0.50, k = 4$	2-L	0.4938	0.4909, 0.4967	0.306	-1.24
$\alpha = 0.50, k = 4$	3-C	0.4945	0.4921, 0.4968	0.2389	-1.11
$\alpha = 0.50, k = 4$	3-L	0.4995	0.4971, 0.5019	0.2456	-0.11
$\alpha = 0.714, k = 10$	1-C	0.6852	0.6826, 0.6878	0.1926	-4.07
$\alpha = 0.714, k = 10$	1-L	0.7056	0.7036, 0.7077	0.1485	-1.22
$\alpha = 0.714, k = 10$	2-C	0.6858	0.6834, 0.6882	0.1831	-3.98
$\alpha = 0.714, k = 10$	2-L	0.7047	0.7028, 0.7066	0.1414	-1.34
$\alpha = 0.714, k = 10$	3-C	0.7098	0.7086, 0.7111	0.0881	-0.62
$\alpha = 0.714, k = 10$	3-L	0.7130	0.7117, 0.7142	0.0882	-0.19
$\alpha = 0.80, k = 4$	1-C	0.7569	0.7549, 0.7591	0.1404	-5.39
$\alpha = 0.80, k = 4$	1-L	0.7937	0.7923, 0.7952	0.0917	-0.78
$\alpha = 0.80, k = 4$	2-C	0.7612	0.7592, 0.7631	0.1330	-4.85
$\alpha = 0.80, k = 4$	2-L	0.7940	0.7926, 0.7954	0.0893	-0.75
$\alpha = 0.80, k = 4$	3-C	0.7944	0.7935, 0.7954	0.0627	-0.7
$\alpha = 0.80, k = 4$	3-L	0.8000	0.7990, 0.8010	0.0613	-0.002
$\alpha = 0.90, k = 9$	1-C	0.8750	0.8737, 0.8761	0.0690	-2.79
$\alpha = 0.90, k = 9$	1-L	0.8958	0.8950, 0.8966	0.0431	-0.47
$\alpha = 0.90, k = 9$	2-C	0.8784	0.8773, 0.8795	0.0644	-2.4
$\alpha = 0.90, k = 9$	2-L	0.8965	0.8958, 0.8972	0.0411	-0.39
$\alpha = 0.90, k = 9$	3-C	0.8969	0.8965, 0.8974	0.0247	-0.34
$\alpha = 0.90, k = 9$	3-L	0.8998	0.8994, 0.9002	0.0250	-0.02

See Tables 3 and 4 for the parameters and Figures 2-4 for the distributions (Dist).

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Table 7 Simulation results for α based on the Conventional (C) and *L*-moment (L) procedures (Proc) based on samples of size $n = 1000$.

Parameters	Dist-Proc	Estimate (α)	95% C.I.	RSE %	RBias %
$\alpha = 0.50, k = 4$	1-C	0.4988	0.4982, 0.4994	0.05814	-0.24
$\alpha = 0.50, k = 4$	1-L	0.4988	0.4984, 0.4992	0.04210	-0.24
$\alpha = 0.50, k = 4$	2-C	0.4993	0.4987, 0.4998	0.05613	-0.14
$\alpha = 0.50, k = 4$	2-L	0.5001	0.4997, 0.5005	0.04200	0.02
$\alpha = 0.50, k = 4$	3-C	0.5000	0.4997, 0.5003	0.03200	0.00
$\alpha = 0.50, k = 4$	3-L	0.5000	0.4997, 0.5004	0.03400	0.00
$\alpha = 0.714, k = 10$	1-C	0.7134	0.7129, 0.7138	0.03084	-0.12
$\alpha = 0.714, k = 10$	1-L	0.7132	0.7129, 0.7135	0.02103	-0.15
$\alpha = 0.714, k = 10$	2-C	0.7133	0.7129, 0.7137	0.02804	-0.14
$\alpha = 0.714, k = 10$	2-L	0.7140	0.7137, 0.7143	0.01961	-0.04
$\alpha = 0.714, k = 10$	3-C	0.7141	0.7140, 0.7143	0.01120	-0.03
$\alpha = 0.714, k = 10$	3-L	0.7142	0.7140, 0.7144	0.01260	-0.01
$\alpha = 0.80, k = 4$	1-C	0.7991	0.7987, 0.7994	0.02127	-0.11
$\alpha = 0.80, k = 4$	1-L	0.8017	0.8015, 0.8019	0.01247	0.21
$\alpha = 0.80, k = 4$	2-C	0.7990	0.7987, 0.7993	0.02003	-0.12
$\alpha = 0.80, k = 4$	2-L	0.8011	0.8009, 0.8013	0.01248	0.14
$\alpha = 0.80, k = 4$	3-C	0.7999	0.7998, 0.8000	0.00875	-0.01
$\alpha = 0.80, k = 4$	3-L	0.8000	0.7998, 0.8001	0.00875	0.00
$\alpha = 0.90, k = 9$	1-C	0.8992	0.8990, 0.8994	0.01001	-0.09
$\alpha = 0.90, k = 9$	1-L	0.9008	0.9007, 0.9009	0.00555	0.09
$\alpha = 0.90, k = 9$	2-C	0.8994	0.8992, 0.8995	0.01000	-0.07
$\alpha = 0.90, k = 9$	2-L	0.9005	0.9004, 0.9006	0.00556	0.06
$\alpha = 0.90, k = 9$	3-C	0.8999	0.8999, 0.9000	0.00333	-0.01
$\alpha = 0.90, k = 9$	3-L	0.9000	0.8999, 0.9000	0.00333	0.00

See Tables 3 and 4 for the parameters and Figures 2-4 for the distributions (Dist).

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